GSR Riffle Shuffle

$$M\left\{\begin{array}{c}1\\2\\3\end{array}\right. \qquad \begin{array}{cc}1 & 4\\2 & \\3 & 5\end{array} \Longrightarrow \begin{array}{cc}4 & 1\\1 & 0\\2 & 0\\5 & 1\\3 & 0\end{array}$$

$$n-M \quad \begin{array}{c}4\\5\end{array}$$

$M \sim \mathrm{Bin}(n, \frac{1}{2})$

M cards from left stack

Occupy uniform choice

among $\binom{n}{M}$ possibilities

Aldous (1983)

Bayer-Diaconis (1982)

After riffle shuffle, label locations with 0 left stack

1 right stack

$$P\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix} = \binom{n}{m}2^{-n}\frac{1}{\binom{n}{m}} = 2^{-n}$$

$m = \#0$

in $b_1, \dots, b_n$

Reverse riffle shuffle

Given n cards $w_1, \dots, w_n$ mark location by uniformly chosen random bits

then move cards marked 0 to top

move " " 1 to bottom

1	0	1
2	1	4
3	1	5
4	0	2
5	0	3

Theorem: For transitive Markov chain (in particular for RW on a group) the chain and its time-reversal have same $d(t)$ function and same $t_{mix}(\varepsilon)$

1	1	5	5
2	4	3	3
3	5	1	2
4	2	4	1
5	3	2	4
	0	1	0
	1	1	0
	1	0	1
	0	1	1
	0	0	0

1

Claim: The reverse riffle shuffle has a strong stationary time

First time τ where all rows of bit array are distinct

Given $\tau = t$, the rows are n bit vectors, in $\{0,1\}^t$ all distinct,

any such config. have same probab.

Dist. of τ:

$t = \theta \log_2 n + C$ Take $\theta = 2$

$\mathbb{P}(\tau > t) \leq \binom{n}{2} 2^{-t} \leq \frac{n^2}{2} n^{-\theta} 2^{-c} = 2^{-1-c}$

$t_{mix}(\varepsilon) \leq 2\log_2(n) + \log_2\left(\frac{1}{\varepsilon}\right)$

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$\Delta = 2^n$ $|V| = n!$

$t_{mix}(\varepsilon) \geq \dfrac{\log(n!(1-\varepsilon))}{\log(2^n)}$

1
2
3
4

$\sim \log_2 n$

$t_{mix}(\varepsilon) \sim \frac{3}{2}\log_2 n$

1 1
2 2

0
1

RW and electrical Networks

Setting: irreducible reversible Markov chain $\Longleftrightarrow$

Connected Weighted graph, weights $c_{\{xy\}} > 0$

called edge-conductance.

$\Longleftarrow$ Given weights $c_{\{xy\}}$ define $p(x,y) = \frac{c_{xy}}{c_x}$

where $c_x = \sum_{\{x,y\}} c_{xy}$ Let $\pi(x) = \frac{c_x}{\sum_{z \in V} c_z} = \frac{c_x}{c_{tot}}$

$$\boxed{c_{tot} = 2 \sum_{\{xy\}} c_{xy}}$$

$$\pi(x) p(x,y) = \frac{c_{xy}}{c_{tot}} = \pi(y) p(y,x)$$

Conversely $\Longrightarrow$ Given π, p let $c_{xy} = Q(x,y) = \pi(x) p(x,y) \Rightarrow p(x,y) = \frac{c_{xy}}{c_x}$

Given nodes a, z in V call a source
z sink

voltage function W is harmonic at all nodes $\neq a, z$

$\forall x \notin \{a, z\}$

$$\left.\begin{aligned} \sum P(x,y) W(y) &= W(x) \\ \sum c_{xy} W(y) &= W(x) c_x \end{aligned}\right\}$$

We also require $W(a) \geq W(z)$.

Given such W we define a flow

$$I(xy) = \frac{W(x) - W(y)}{r_{xy}} = c_{xy} \left(W(x) - W(y) \right)$$

Note $r_{xy} = \frac{1}{c_{xy}}$

I current flow that corresponds to W

More generally a <u>flow</u> from a to z

is a function $\theta : \vec{E} \longrightarrow \mathbb{R}$

antisymmetric $\theta(\vec{xy}) = -\theta(yx)$

$$(\operatorname{div}\theta)(x) = \sum_{\vec{xy} \in \vec{E}} \theta(\vec{xy}) = \begin{cases} 0 & x \notin \{a,z\} \\ \geq 0 & x = a \\ \leq 0 & x = z \end{cases}$$

$$\boxed{\sum_{x \in V} (\operatorname{div}\theta)(x) = \sum_{x} \sum_{\substack{y \\ \vec{xy}\ \text{edge}}} \theta(\vec{xy}) = 0}$$

$$(\operatorname{div}\theta)(z) = -(\operatorname{div}\theta)(a)$$

$$\boxed{\|\theta\| = \operatorname{div}\theta(a)}$$

$$\sum_{\substack{y \\ \vec{xy} \in \vec{E}}} I_{xy} = \sum_{y} \frac{W(x) - W(y)}{r_{xy}} = \sum_{y \sim x} c_{xy} W(y) - c_x W(x) = 0 \quad x \neq a, z$$

Current flow I satisfies cycle law:

$$0 = \sum_{\substack{\vec{xy} \\ \text{in cycle}}} I_{xy} \cdot r_{xy} = \sum_{j=1}^{\ell} I_{x_{j-1},x_j} \cdot r_{x_{j-1},x_j} = \sum_{j=1}^{\ell} \frac{W(x_{j-1}) - W(x_j)}{r_{x_{j-1},x_j}} r_{x_{j-1},x_j} = 0.$$

cycle $x_0, x_1, \dots x_{\ell-1}, x_0 = x_\ell$

<u>Fact:</u> If $\|\theta\| = \|I\|$ both θ, I $\boxed{\text{flows}}$ that satisfy cycle law, then $\theta = I$.

$\tilde{\theta} = \theta - I$ has $\operatorname{div} \tilde{\theta} = 0$ everywhere and cycle law holds.

Then $\tilde{\theta} = 0$. Why? Define $h(x)$ where $\dfrac{h(x) - h(y)}{r_{xy}} = \tilde{\theta}(xy)$

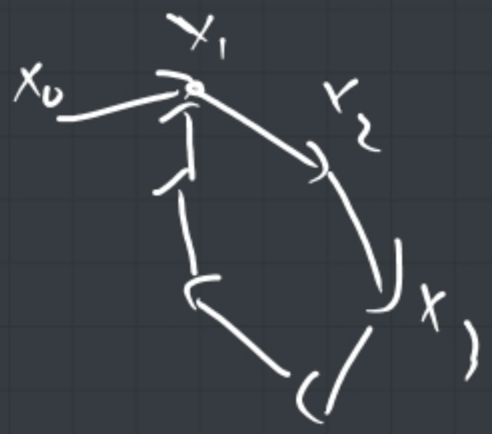
$$h(x) = \sum \tilde{\Theta}(x_{j-1}, x_j) \cdot r_{x_{j-1} x_j}$$

h is well defined (indep ~~eff~~ choice of path $x_0 \ldots x$)

because $\tilde{\Theta}$ satisfies cycle law.

h is harmonic at all x because $\operatorname{div} \tilde{\Theta} = 0$.

<u>More direct:</u> Given $\tilde{\Theta}$, suppose $\tilde{\Theta}(\overrightarrow{xy}) > 0$



$\tilde{\Theta}(\overrightarrow{x_0 x_1}) > 0$

$\tilde{\Theta}(\overrightarrow{x_1 x_2}) > 0$

$\tilde{\Theta}(x_2 x_3) > 0$

Def: Given $G=(V,E)$ connected with conductances $\{C_{xy}\}$

let W be a voltage function from a to z

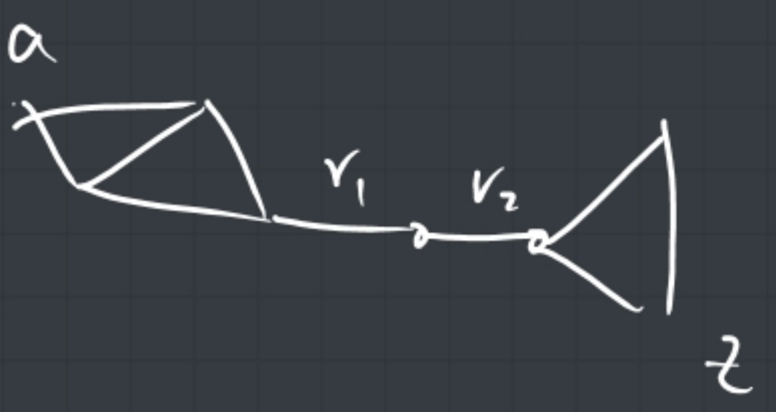
(Example $W(x) = \mathbb{P}_x(\tau_a < \tau_z)$)

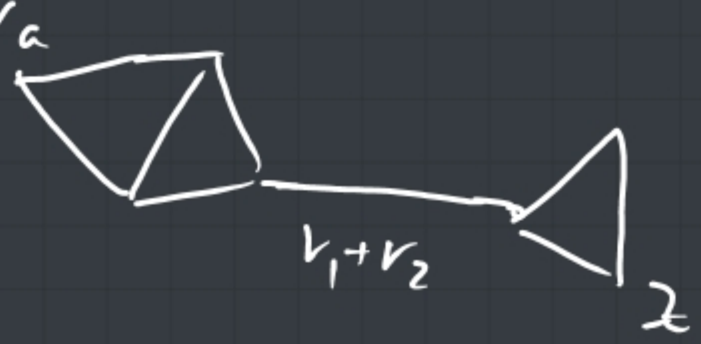
$$R_{eff}(a \leftrightarrow z) = R(a \leftrightarrow z) = \frac{W(a)-W(z)}{\|I\|}$$

I flow corresp. to W (current flow)

Same for all volt. functions.

$\exists$ unique volt. func. with $W(a)=1$, $W(z)=0$.

$R\Big($ 

$\Big) = R\Big($ 

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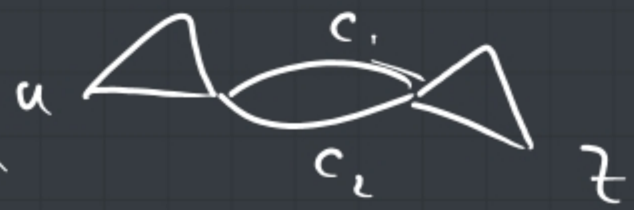
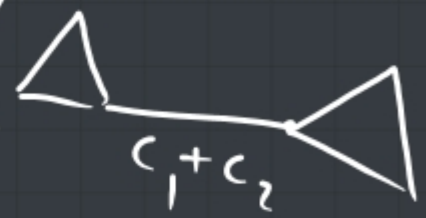
$R\Big($ 

$\Big) = R\Big($ 

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We took $W(z)=0$
$W(a) = R(a\leftrightarrow z)\|I\|$

Escape prob: $\mathbb{P}_a(\tau_z < \tau_a^+) = \sum_{x\sim a} p(a,x)\, \mathbb{P}_x(\tau_z < \tau_a) =$

$$\sum_{x\sim a} \frac{c_{ax}}{c_a} \frac{W(a)-W(x)}{W(a)} = \frac{1}{c_a R(a\leftrightarrow z)\|I\|} \sum_{x\sim a} I(ax) = \frac{1}{c_a R(a\leftrightarrow z)}$$

$$= \frac{C(a\leftrightarrow z)}{c_a}$$