Interacting particle systems and their large scale behavior

Tadahisa Funaki (舟木 直久)

BIMSA

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Self-Introduction:

Tadahisa Funaki, Professor Emeritus, University of Tokyo

Research Interests:
Probability Theory
Mathematical Physics (related to Statistical Physics)

Education:
1970.4–1974.3: University of Tokyo, Bachelor
1974.4–1976.3: University of Tokyo, Master Course
1976.4–1977.3: University of Tokyo, PhD Course, drop out

Academic Career:
1977.4–1979.1: Hiroshima University, Research Associate
1979.2–1995.9: Nagoya University, RA, ..., Professor
1995.10–2017.3: University of Tokyo, Professor
2017.4–2022.3: Waseda University, Professor

Mini-Courses at Yau Mathematical Sciences Center:
2020.11.17-12.17: KPZ limit for interacting particle systems
2022.3.7–6.10: Stochastic analysis and its applications
Abstract (Goal of the course):

- I will explain in some details our recent results on the derivation of interface motion such as motion by mean curvature or free boundary problem from particle systems via hydrodynamic space-time scaling limit.

Prerequisite:

- It is desirable that the audience is familiar with Modern Probability Theory and some tools in Stochastic Analysis such as martingales and stochastic differential equations.
- But I will try to quickly explain these in the course.
- For example, Parts I and II of my course given at YMSC, 2022 fit to this purpose; see slides of Lect-1 to Lect-20 posted on the web page of YMSC.
Plan of the course (three parts):

Part I: Interacting particle systems

▶ Quick introduction to probability theory and stochastic analysis: *Basic concepts, facts, tools, e.g. martingale theory in continuous time having jumps, ...*

▶ Exclusion process (Kawasaki dynamics), Zero-range process, Glauber dynamics, *Construction and facts*

References


Part II: Hydrodynamic scaling limit and fluctuation limit

- Entropy method, One block estimate, Two blocks estimate
- Relative entropy method
- Equilibrium fluctuation, Boltzmann-Gibbs principle


Part III: Applications and extensions of the methods explained in Part II

- Derivation of motion by mean curvature in phase separation phenomena
- Derivation of free boundary problem describing segregation of species
- Boltzmann-Gibbs principle revisited, Discrete Schauder estimate

**References**


In Parts II and III, we will discuss **Hydrodynamic Limit and Fluctuation Limit for Stochastic Models**

*(Micro)* Large scale interacting systems with stochastic mechanism  
(molecules in statistical physics, individuals in biology, .......)  
\[\Rightarrow\]  
scaling limit

*(Macro)* Nonlinear PDEs, Stochastic PDEs (PDEs with stoch terms)

**Space-time scaling limit:**  
Averaging effect under **Local ergodicity and Local equilibrium**

- The structure behind is similar to the derivation of the hydrodynamic equations from the Boltzmann equation via Hilbert expansion or Chapman-Enskog expansion passing-by Maxwellian distributions (equilibria in velocity fields) which change in macroscopic space-time variables.

- Boltzmann equation was derived from microscopic particle system (Newtonian classical mechanics) under molecular chaos or ergodic hypothesis.
Cloud (Image)

Local equilibrium:

- Microscopic states (random) are parametrized by mean particles density, mean velocity of particles, mean energy density, ....... (related to conserved quantities) .......
- They change for each macroscopic space-time point.
We will consider interacting random walks on a square lattice as the microscopic model.

Simulation of Glauber-Kawasaki dynamics

\[
N = 10 \quad N = 200 \quad N = 1000
\]

(by Yoshiki Otobe, $N$ is the size of the lattice)

“Kawasaki” means interacting random walks with hard core exclusive interactions (Total number of particles is conserved).

“Glauber” means creation and annihilation of particles (No conserved quantity).

Before discussing models, we recall some basic concepts and facts in probability theory and stochastic analysis.

In particular, we discuss jump processes (to cover random walks on lattice).
Part I:

Part I-A. Basic Concepts in Probability Theory

§1 Probability space, Random variables, Probability distributions, Expectation and variance

1.1 Probability space

- $\Omega$: a certain set
- $\mathcal{F}$: $\sigma$-field of $\Omega$ i.e., $\mathcal{F} \subset \mathcal{P}(\Omega)$ (i.e., $\mathcal{F}$ is a family of subsets of $\Omega$) and satisfies
  
  1. $\Omega \in \mathcal{F}$
  2. $A \in \mathcal{F} \implies A^c := \Omega \setminus A \in \mathcal{F}$
  3. $A_n \in \mathcal{F}, n = 1, 2, \ldots \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$

- $\Omega$ is called a sample space, $\omega \in \Omega$ a sample and $A \in \mathcal{F}$ an event.
A measure $P$ on a measurable space $(\Omega, \mathcal{F})$ satisfying $P(\Omega) = 1$ is called a probability measure i.e.

1. $P : \mathcal{F} \to [0, 1]$ and $P(\Omega) = 1$
2. ($\sigma$-additivity) If $A_n \in \mathcal{F}$, $n = 1, 2, \ldots$ are disjoint (i.e., $A_i \cap A_j = \emptyset$ for $\forall i \neq j$), then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

$(\Omega, \mathcal{F}, P)$ is called a probability space.

The reason that probability theory is built on measure theory lies, for example, in closeness of the classes of events or random variables under taking limits, strong law of large numbers, ...

For $\mathcal{A} \subset \mathcal{P}(\Omega)$, the smallest $\sigma$-field which containsof $\mathcal{A}$ is denoted by $\sigma(\mathcal{A})$ and called the $\sigma$-field generated by $\mathcal{A}$. Indeed, it is given by $\sigma(\mathcal{A}) = \bigcap_{\mathcal{G} : \sigma\text{-field}, \mathcal{A} \subset \mathcal{G}} \mathcal{G}$. 
[Borel-Cantelli’s lemma] If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then

$$P\left( \lim_{n \to \infty} A_n \right) = 0, \text{ i.e., } P\left( \lim_{n \to \infty} A_n^c \right) = 1,$$

where

$$\omega \in \lim_{n \to \infty} A_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \iff \omega \text{ belongs infinitely many } A_n$$

$$\omega \in \lim_{n \to \infty} A_n^c := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n^c \iff \exists k \text{ s.t. } \omega \in A_n^c \text{ for } \forall n \geq k$$

We write the former as “$A_n$ i.o. (infinitely often)”, the latter as “$A_n^c$ e.v. (eventually)”. \hfill \Box

\[\vdash P\left( \lim_{n \to \infty} A_n \right) \leq P\left( \bigcup_{n=k}^{\infty} A_n \right) \leq \sum_{n=k}^{\infty} P(A_n) \to 0 \text{ (}k \to \infty) \] \hfill \Box
1.2 Random variables (denoted by r.v.’s)

Let \((S, \mathcal{S})\) be a measurable space and let 
\(X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})\) be an \(S\)-valued measurable function, 
i.e., for \(\forall A \in \mathcal{S}, X^{-1}(A) \equiv \{\omega \in \Omega; X(\omega) \in A\} \in \mathcal{F}\), 
then \(X\) is called an \(S\)-valued random variable (r.v.).

In particular, when \((S, \mathcal{S}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))\), \(X\) is called a 
real-valued random variable, where \(\mathcal{B}(\mathbb{R}) := \sigma\{\text{open sets of } \mathbb{R}\}\) is a Borel field of \(\mathbb{R}\).
Limits of r.v.'s: $X_n$, $n = 1, 2, \ldots$: real-valued r.v.'s, then

$$\inf_{n \geq 1} X_n, \quad \sup_{n \geq 1} X_n, \quad \liminf_{n \to \infty} X_n, \quad \limsup_{n \to \infty} X_n$$

are all r.v.'s (if they take finite values). In particular, if

$X = \lim_{n \to \infty} X_n$ exists, $X$ is a r.v.

[Note] To show this, it is essential that $\mathcal{F}$ is a $\sigma$-field.

$\sigma$-field generated by a r.v.: For $S$-valued r.v. $X$, set

$$\mathcal{F}_X \equiv \sigma(X) := \{X^{-1}(A); A \in S\} (\subset \mathcal{F}).$$

We call $\mathcal{F}_X$ or $\sigma(X)$ a $\sigma$-field generated by $X$. 
1.3 Probability distribution

For $S$-valued r.v. $X$,

$$P_X(A) := P(X^{-1}(A)), \quad A \in S$$

determines a probability measure (image measure) on $(S, S)$. It is called a distribution of $X$.

1.4 Expectation and variance

For a real-valued r.v. $X$, its integral $\int_{\Omega} X(\omega) \, P(d\omega)$ with respect to $P$ in Lebesgue’s sense is called an expectation (or mean) of $X$ and denoted by $E[X]$ (if exists).
For real-valued r.v.’s $X, X_1, X_2$, we define

$$\text{Var}(X) := E[(X - E[X])^2]$$ variance of $X$

$$\text{Cov}(X_1, X_2) := E [(X_1 - E[X_1])(X_2 - E[X_2])]$$
covariance of $X_1$ and $X_2$

Jensen’s inequality: Assume $\psi : \mathbb{R} \to \mathbb{R}$ is convex, i.e., for $\forall \lambda \in (0, 1), \forall x, y \in \mathbb{R}$

$$\psi(\lambda x + (1 - \lambda)y) \leq \lambda \psi(x) + (1 - \lambda)\psi(y).$$

Then, we have

$$\psi(E[X]) \leq E[\psi(X)].$$
Interchange of the expectation and limits

Let \((X_n)_{n=1,2,...}, X\) be real-valued r.v.’s defined on \((\Omega, \mathcal{F}, P)\).

1. **Lebesgue’s convergence theorem:** If \(X_n \to X\) \((a.s.)\) (a.s.-convergence, i.e. \(P(\exists \lim_{n \to \infty} X_n = X) = 1\)) and \(\exists\) non-negative integrable r.v. \(Y\) s.t. \(|X_n| \leq Y\) for \(\forall n\), then
   \[
   \lim_{n \to \infty} E[X_n] = E[X]
   \]

2. **Monotone convergence theorem:** If \(0 \leq X_1 \leq X_2 \leq \cdots\) and \(X_n \to X\) \((a.s.)\), then
   \[
   \lim_{n \to \infty} E[X_n] = E[X]
   \]

3. **Fatou’s lemma:** If \(X_n \geq 0\),
   \[
   E \left[ \liminf_{n \to \infty} X_n \right] \leq \liminf_{n \to \infty} E[X_n]
   \]
[Gaussian (normal) distribution] For \( m = (m_i) \in \mathbb{R}^d \) and \( d \times d \) positive definite real symmetric matrix \( V = (V_{ij}) \), the measure on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\)\]

\[
\mu_{m,V}(dx) = \frac{1}{(2\pi)^{d/2}(\det V)^{1/2}} \exp \left\{ -\frac{1}{2}(x - m) \cdot V^{-1}(x - m) \right\} \, dx
\]

is called a Gaussian distribution with mean \( m \) and covariance matrix \( V \). Here, \( dx \) is the Lebesgue measure on \( \mathbb{R}^d \), \( V^{-1} \) denotes the inverse matrix of \( V \) and \( \cdot \) denotes the inner product in \( \mathbb{R}^d \). Moreover, an \( \mathbb{R}^d \)-valued r.v. \( X = (X_1, X_2, \ldots, X_d) \) having distribution \( \mu_{m,V} \) is called a Gaussian random variable. It holds

\[
E[X] \equiv (E[X_i])_{i=1,2,\ldots,d} = m,
\]

\[
\text{Cov}(X) \equiv (\text{Cov}(X_i, X_j))_{i,j=1,2,\ldots,d} = V.
\]
§2 Dynkin’s $\pi$–$\lambda$ theorem

- This is useful when we want to extend “some fact” (or “some property”) to all sets in a $\sigma$-field.

- For example, this is used to show the uniqueness of measure: coincidence on $\pi$-system $\mathcal{P}$ implies that on $\sigma(\mathcal{P})$. 
Several concepts of convergence of random variables

Let real-valued r.v.’s \((X_n)_{n=1,2,...}\) and \(X\) be given on \((\Omega, \mathcal{F}, P)\).

1. (a.s.-convergence) \(X_n \to X\) a.s.
   \[\iff P\left(\lim_{n \to \infty} X_n = X\right) = 1\] (i.e. \(P\left(\{\omega \in \Omega; \lim_{n \to \infty} X_n(\omega) = X(\omega)\}\right) = 1\))

2. (Convergence in probability) \(X_n \to X\) in prob.
   \[\iff \text{For } \forall \varepsilon > 0, \lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0\]

3. \((L^p\text{-convergence, Convergence in mean of } p\text{th order})\)
   \(X_n \to X\) in \(L^p\) \((p \geq 1)\) \[\iff \lim_{n \to \infty} E[|X_n - X|^p] = 0\]

4. (Convergence in law) \(X_n \to X\) in law
   \[\iff \text{For } \forall f \in C_b(\mathbb{R}), \lim_{n \to \infty} E[f(X_n)] = E[f(X)]\]

where \(C_b(\mathbb{R}) = \{\text{all bounded continuous functions: } \mathbb{R} \to \mathbb{R}\}\).

\[\text{a.s.-conv. } \iff \text{conv. in prob. } \implies \text{conv. in law}\]

\[L^p\text{-conv. } \iff \]
§4 Independence

- $\Lambda$ is an arbitrary parameter set.
- Independence of events $\{A_\lambda \in \mathcal{F}\}_{\lambda \in \Lambda}$, sub $\sigma$–fields $\{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{F}$, $(S_\lambda, S_\lambda)$-valued r.v.'s $\{X_\lambda\}_{\lambda \in \Lambda}$ was defined.
- We denote Independence as $A \perp B$, $\{A_\lambda\} \perp$, $\{\mathcal{F}_\lambda\} \perp$, $\{X_\lambda\} \perp$, etc.
§5 Product of infinitely many probability spaces

- $\{(S_n, \mathcal{S}_n, \mu_n)\}_{n=1,2,\ldots}$: countably many probability spaces
- $\Omega := \prod_{n=1}^{\infty} S_n$: infinite product space
- $\mathcal{C} := \{C : \text{cylinder sets of } \Omega\}$, where
  
  \[ C \equiv C_A^{(n)} = \{\omega = (\omega_1, \omega_2, \ldots) \in \Omega ; (\omega_1, \ldots, \omega_n) \in A\}, \]

  with $n \in \mathbb{N}$ and
  
  \[ A \in \mathcal{S}_1 \times \cdots \times S_n := \sigma\{A_1 \times \cdots \times A_n ; A_k \in \mathcal{S}_k\}. \]

- $\mathcal{F} := \sigma(C)$: Kolmogorov’s $\sigma$–field
- $P(C_A^{(n)}) := (\mu_1 \times \cdots \times \mu_n)(A)$ on $\mathcal{C}$, where $\mu_1 \times \cdots \times \mu_n$ is a product measure of finitely many measures.
- Note that $P(C_A^{(n)})$ is well-defined on $\mathcal{C}$ independently of the choice of $n, A$.  

\[ \text{22 / 122} \]
Then,

- Probability measure $P$ on $(\Omega, \mathcal{F})$ can be uniquely constructed as an extension of $P$ on $(\Omega, \mathcal{C})$. For the proof, we use Carathéodory-Hopf extension theorem.

- We denote $P = \prod_{n=1}^{\infty} \mu_n$ and call infinite product measure.

- Based on infinite product of probability spaces, one can construct an infinite sequence of independent r.v.’s.