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Asymptotic stability of planar rarefaction waves under periodic perturbations for 3-d Navier-Stokes equations

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ABSTRACT

In this paper, we study a Cauchy problem for the 3-d compressible isentropic Navier-Stokes equations, in which the initial data is a 3-d periodic perturbation around a planar rarefaction wave. We prove that the solution of the Cauchy problem exists globally in time and tends to the background rarefaction wave in the $L^\infty(\mathbb{R}^3)$ space as $t \to +\infty$. The result reveals that even though the initial perturbation has infinite oscillations at the far field and is not integrable along any direction of space, the planar rarefaction wave is nonlinearly stable for the 3-d N-S equations. The key point is to construct a suitable ansatz $(\tilde{\rho}, \tilde{\mathbf{u}})$ to carry the same oscillations as those of the solution $(\rho, \mathbf{u})$ at the far field in the normal direction of the rarefaction wave, so that the difference $(\rho - \tilde{\rho}, \mathbf{u} - \tilde{\mathbf{u}})$

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belongs to some Sobolev space and the energy method is available.

1. Introduction

The three-dimensional (3-d) compressible isentropic Navier-Stokes (N-S) equations read

$$\begin{cases} \partial_t \rho + \operatorname{div}\big(\rho \mathbf{u}\big) = 0, \\ \partial_t\big(\rho \mathbf{u}\big) + \operatorname{div}\big(\rho \mathbf{u} \otimes \mathbf{u}\big) + \nabla p(\rho) = \mu \triangle \mathbf{u} + \big(\mu + \lambda\big) \nabla \operatorname{div} \mathbf{u}, \end{cases} \qquad t > 0, x \in \mathbb{R}^3, \tag{1.1}$$

where $\rho(x,t) > 0$ is the density, $\mathbf{u}(x,t) = \big(u_1, u_2, u_3\big)(x,t) \in \mathbb{R}^3$ is the velocity, the pressure $p(\rho)$ is gamma law, i.e. $p(\rho) = \rho^\gamma$ with $\gamma > 1$, and the viscous coefficients μ and λ satisfy

$$\mu > 0 \quad \text{and} \quad \lambda + \mu \geq 0. \tag{1.2}$$

Note that the condition (1.2) is of mathematical interests, which is a relaxed version of the physical one that $\mu > 0$ and $\lambda + \frac{2}{3}\mu \geq 0$.

When $\mu = \lambda = 0$, (1.1) turns to the 3-d Euler equations. A planar centered rarefaction wave $\big(\rho^r, \mathbf{u}^r\big)(x,t) = \big(\rho^r, u_1^r, 0, 0\big)(x_1, t)$ is a weak entropy solution to this hyperbolic system, where $\big(\rho^r, u_1^r\big)$ solves the following 1-d Riemann problem,

$$\begin{cases} \partial_t \rho + \partial_1\big(\rho u_1\big) = 0, \\ \partial_t\big(\rho u_1\big) + \partial_1\big(\rho u_1^2\big) + \partial_1 p(\rho) = 0, \\ \big(\rho, u_1\big)(x_1, 0) = \begin{cases} \big(\bar{\rho}^-, \bar{u}_1^-\big), & x_1 < 0, \\ \big(\bar{\rho}^+, \bar{u}_1^+\big), & x_1 > 0. \end{cases} \end{cases} \tag{1.3}$$

For $\rho > 0$, the system (1.3) is strictly hyperbolic with two distinct eigenvalues

$$\lambda_1(\rho, u_1) = u_1 - \sqrt{p'(\rho)}, \quad \lambda_2(\rho, u_1) = u_1 + \sqrt{p'(\rho)}.$$

The i-Riemann invariant $(i = 1, 2)$ is given by

$$Z_i(\rho, u_1) = u_1 + (-1)^{i+1} \int_1^\rho \frac{\sqrt{p'(s)}}{s} ds, \tag{1.4}$$

which takes constant values along i-right eigenvectors. Denote $\lambda_2^\pm := \lambda_2\big(\bar{\rho}^\pm, \bar{u}_1^\pm\big)$ and $Z_2^\pm := Z_2\big(\bar{\rho}^\pm, \bar{u}_1^\pm\big)$.

In this paper, we consider only a 2-rarefaction wave (a 1-rarefaction wave and a combination of full families of rarefaction waves are similar), i.e. the constant states $(\bar{\rho}^{\pm}, \bar{u}_1^{\pm})$ in (1.3) satisfy the relation

$$\bar{u}_1^+ = \bar{u}_1^- + \int_{\bar{\rho}^-}^{\bar{\rho}^+} \frac{\sqrt{p'(s)}}{s} ds \quad \text{with} \quad \bar{\rho}^- < \bar{\rho}^+, \tag{1.5}$$

which yields that $Z_2^- = Z_2^+$. The 2-rarefaction wave $(\rho^r, u_1^r)(x_1, t) = (\rho^r, u_1^r)(\frac{x_1}{t})$ can be solved exactly through

$$\begin{cases} \lambda_2\big(\rho^r, u_1^r\big) = u_1^r + \sqrt{p'(\rho^r)} = \omega\big(\frac{x_1}{t}\big), \\ Z_2\big(\rho^r, u_1^r\big) = u_1^r - \int_1^{\rho^r} \frac{\sqrt{p'(s)}}{s} ds = Z_2^- \big(= Z_2^+\big), \end{cases} \tag{1.6}$$

where

$$\omega\left(\frac{x_1}{t}\right) := \begin{cases} \lambda_2^-, & x_1 < \lambda_2^- t, \\ \frac{x_1}{t}, & \lambda_2^- t \le x_1 < \lambda_2^+ t, \\ \lambda_2^+, & x_1 \ge \lambda_2^+ t. \end{cases}$$

It is well-known that the hyperbolic conservation laws, such as the compressible Euler equations, have rich wave phenomena including shock, rarefaction wave and contact discontinuity. The linear superpositions of these waves are called Riemann solutions. It is known that the Riemann solutions characterize the large time behaviors of the solutions of the 1-d hyperbolic conservation laws, if the initial data tend to different constant states as $x_1 \to \pm\infty$. In other words, the Riemann solutions are nonlinearly stable under localized (e.g. compactly supported) perturbations; see [13,20]. For the viscous conservation laws such as the compressible N-S equations, the large time behaviors of the solutions are governed by the viscous versions of the corresponding Riemann solutions, which have been widely studied by many literatures. For instance, we refer to [8,32,21,22,37,29,30,25], [23,34,36] and [24,10,11] for the stability of viscous shocks, rarefaction waves and contact discontinuities, respectively.

A 1-d basic wave pattern becomes a planar wave in multiple dimensions. The stability of the planar wave patterns for the multi-dimensional (multi-d) conservation laws is a challenging problem. For the inviscid systems, only local existence of some piecewise smooth solutions were obtained by [26,27,1,5,6]. And [2] proved the uniqueness and L^2_{loc}-stability of planar rarefaction waves in a broad class of general entropy solutions for the multi-d compressible Euler equations. On the other hand, if the initial data of the 2-d compressible Euler equations is either periodic or a Riemann data containing at least one planar shock, then the uniqueness of admissible solutions fails in the L^∞ space; see [3,4,16]. Hence, the theory of the multi-d conservation laws is essentially different from the 1-d case. If conservation laws have viscosity, [38] showed the asymptotic stability of the planar rarefaction wave through an L^2-energy method. This work was then further improved by [14,35,15], where both the wave-strength and initial perturbation can be arbitrarily large and an optimal decay rate was obtained. For the 2-d isentropic Euler equations with a specially selected viscosity, [9] used the elementary energy method to obtain the asymptotic stability of the planar rarefaction wave. Recently, [18,19] proved the stability of planar rarefaction waves for the multi-d N-S equations, in which the initial perturbations are L^2-integrable in the normal direction of the wave propagation and periodic in the transverse directions.

The stability of wave patterns with space-periodic perturbations is also full of interest and importance in the theories of conservation laws, in which some resonant phenomena may happen for the non-isentropic Euler equations; see [28]. For the 1-d hyperbolic systems with at most two equations, Lax and Glimm [17,7] proved that the periodic solutions exist globally in time and tend asymptotically to the constant averages at rates t^{-1}. Recently, [39–41] proved the asymptotic stability of shocks and rarefaction waves with periodic perturbations for the 1-d scalar conservation laws in both inviscid and viscous cases. In [12], we began to study the stability of planar rarefaction waves under multi-d periodic perturbations for the multi-d scalar viscous conservation laws and also obtained the time-decay rates. In particular, we introduced a Gagliardo-Nirenberg (G-N) type inequality, which plays an important role in obtaining a priori estimates on an unbounded domain $\mathbb{R} \times \mathbb{T}^{n-1}$ without zero boundary conditions, here $\mathbb{T}$ denotes a torus.

In this paper, we continue to study the nonlinear stability of planar rarefaction waves with periodic perturbations for the 3-d isentropic N-S equations. Different from the perturbations in the previous works [9,18,19], the ones which are periodic on the whole space $\mathbb{R}^3$ have infinite oscillations at the far field and are not integrable along any direction of space. Note that the L^2_{loc}-stability of planar rarefaction waves for the M-D Euler equations in [2] also applies to the case for the periodic perturbations which belong to the L^2_{loc} space. However, a global stability on $\mathbb{R}^3$ like the $L^\infty(\mathbb{R}^3)$-stability for the hyperbolic system is truly difficult to prove due to the oscillations at the far field. Thanks to the dissipative mechanism of (1.1), we are able to prove that if the wave-strength and periodic perturbation are both small, the Cauchy problem (1.1), (1.8) admits a unique global solution, which tends asymptotically to the background rarefaction wave in the $L^\infty(\mathbb{R}^3)$ space as $t \to +\infty$. The key point is to construct a suitable ansatz $(\tilde{\rho}, \tilde{\mathbf{u}})$ to carry the same oscillations as those of the solution $(\rho, \mathbf{u})$ at the far field in the normal direction of the wave propagation, i.e. the difference $(\rho-\tilde{\rho}, \mathbf{u}-\tilde{\mathbf{u}})(x,t) \to 0$ as $|x_1| \to +\infty$. Although $(\rho - \tilde{\rho}, \mathbf{u} - \tilde{\mathbf{u}})$ still oscillates in the transverse directions x_2 and x_3, it is still feasible to use the energy method with the aid of the G-N type inequality on $\mathbb{R} \times \mathbb{T}^2$, which was given in [12], to achieve the main result, Theorem 1.3.

Now we start to formulate the main result. Since the centered rarefaction wave $(\rho^r, u_1^r)(x_1, t)$ is only Lipschitz continuous, we need to construct a smooth approximation as in [33]. Inspired by (1.6), let $(\tilde{\rho}^r, \tilde{u}_1^r)(x_1, t)$ be the unique smooth solution solved through

$$\begin{cases}\tilde{u}_1^r(x_1,t)+\sqrt{p'(\tilde{\rho}^r)}(x_1,t)=\tilde{\omega}(x_1,t),\\ \tilde{u}_1^r(x_1,t)-\int_1^{\tilde{\rho}^r(x_1,t)}\frac{\sqrt{p'(s)}}{s}ds=Z_2^-\big(=Z_2^+\big),\end{cases} \tag{1.7}$$

where $\tilde{\omega}(x_1,t)$ is the unique smooth solution to the problem,

$$\begin{cases}\partial_t\tilde{\omega}+\partial_1\big(\frac{\tilde{\omega}^2}{2}\big)=0,\\ \tilde{\omega}(x_1,0)=\frac{\lambda_2^-+\lambda_2^+}{2}+\frac{\lambda_2^+-\lambda_2^-}{2}\tanh(x_1).\end{cases}$$

Note that this smooth rarefaction wave is time-asymptotically equivalent to the centered rarefaction wave (ρ^r,u_1^r) in the $L^\infty(\mathbb{R})$ space; see Lemma 2.1 below.

To study the stability of the planar rarefaction wave under a 3-d periodic perturbation, we prescribe the initial data for (1.1) as

$$(\rho,\rho\mathbf{u})(x,0)=\big(\tilde{\rho}^r,\tilde{\rho}^r\tilde{u}_1^r,0,0\big)(x_1,0)+(v_0,\mathbf{w}_0)(x),\quad x\in\mathbb{R}^3, \tag{1.8}$$

where $v_0(x)\in\mathbb{R}$ and $\mathbf{w}_0(x)=(w_{1,0},w_{2,0},w_{3,0})(x)\in\mathbb{R}^3$ are periodic functions on the 3-d torus $\mathbb{T}^3=[0,1]^3$, satisfying

$$\int_{\mathbb{T}^3}(v_0,\mathbf{w}_0)(x)dx=0. \tag{1.9}$$

Remark 1.1. The periodic perturbations with zero averages should be imposed on the conservative quantities, i.e. the density and the momentum. If (1.9) does not hold, by adding the constant averages onto $(\bar{\rho}^\pm,\bar{\rho}^\pm\bar{u}_1^\pm,0,0)$, the problem (1.1), (1.8) turns to be connected with other kinds of Riemann solutions, which may contain a shock and is not the topic of this paper.

Remark 1.2. The solution $(\rho,\mathbf{u})$ to the Cauchy problem (1.1), (1.8) is periodic with respect to x_2 and x_3, but not to x_1. Thus one cannot study the problem on the bounded torus $\mathbb{T}^3$, but on the unbounded domain $\mathbb{R}\times\mathbb{T}^2$ instead.

Before presenting the main result, we first introduce some notations and construct the ansatz.

Let $\big(\rho^\pm,\mathbf{u}^\pm\big)(x,t)=\big(\rho^\pm,u_1^\pm,u_2^\pm,u_3^\pm\big)(x,t)$ be the unique periodic solutions to (1.1) with the periodic initial data

$$\big(\rho^\pm,\rho^\pm\mathbf{u}^\pm\big)(x,0)=\big(\bar{\rho}^\pm,\bar{\rho}^\pm\bar{u}_1^\pm,0,0\big)+(v_0,\mathbf{w}_0)(x),\quad x\in\mathbb{R}^3, \tag{1.10}$$

respectively, here the global existence can be found in Lemma 2.3. For later use, we define

$$\big(v^\pm,\mathbf{w}^\pm\big)(x,t):=\big(\rho^\pm,\rho^\pm\mathbf{u}^\pm\big)(x,t)-\big(\bar{\rho}^\pm,\bar{\rho}^\pm\bar{u}_1^\pm,0,0\big), \tag{1.11}$$

which are periodic functions with zero averages over $\mathbb{T}^3$ for all $t \geq 0$; see Lemma 2.3. Comparing (1.8) with (1.10), one has that

$$(\rho, \rho\mathbf{u})(x,0) - (\rho^{\pm}, \rho^{\pm}\mathbf{u}^{\pm})(x,0) = (\tilde{\rho}^r, \tilde{\rho}^r\tilde{u}_1^r, 0, 0)(x_1, 0) - (\bar{\rho}^{\pm}, \bar{\rho}^{\pm}\bar{u}_1^{\pm}, 0, 0) \quad \forall (x_2, x_3) \in \mathbb{R}^2,$$

which tends to zero as $x_1 \to \pm\infty$, respectively. If $\|v_0\|_{L^\infty(\mathbb{R}^3)}$ is suitably small, the difference of the velocity satisfies that

$$\begin{aligned}\mathbf{u}(x,0) - \mathbf{u}^{\pm}(x,0) &= \frac{(\tilde{\rho}^r\tilde{\mathbf{u}}^r)(x_1,0) + \mathbf{w}_0(x)}{\tilde{\rho}^r(x_1,0) + v_0(x)} - \frac{\bar{\rho}^{\pm}\bar{\mathbf{u}}^{\pm} + \mathbf{w}_0(x)}{\bar{\rho}^{\pm} + v_0(x)} \\ &= \frac{\big(\tilde{\rho}^r(x_1,0) - \bar{\rho}^{\pm}\big)\big(v_0(x)\bar{\mathbf{u}}^{\pm} - \mathbf{w}_0(x)\big)}{(\tilde{\rho}^r(x_1,0) + v_0(x))(\bar{\rho}^{\pm} + v_0(x))} + \frac{\tilde{\rho}^r(x_1,0)(\tilde{\mathbf{u}}^r(x_1,0) - \bar{\mathbf{u}}^{\pm})}{\tilde{\rho}^r(x_1,0) + v_0(x)},\end{aligned}$$

which yields that

$$\sup_{x_2,x_3\in\mathbb{R}} \big|\mathbf{u}(x,0) - \mathbf{u}^{\pm}(x,0)\big| \leq C\big(\big|\tilde{\rho}^r(x_1,0) - \bar{\rho}^{\pm}\big| + \big|u_1^r(x_1,0) - \bar{u}_1^{\pm}\big|\big) \to 0 \quad \text{as } x_1 \to \pm\infty.$$

Observing from this, it is plausible to expect that, for all $t > 0$, the solution $(\rho, \mathbf{u})$ of (1.1), (1.8) tends to the periodic solutions $(\rho^{\pm}, \mathbf{u}^{\pm})$ as $x_1 \to \pm\infty$, respectively. Inspired by the expression of the background smooth planar rarefaction wave,

$$\begin{aligned}\tilde{\rho}^r(x_1,t) &= \bar{\rho}^-\big(1 - \sigma(x_1,t)\big) + \bar{\rho}^+\sigma(x_1,t), \\ \tilde{u}_1^r(x_1,t) &= \bar{u}_1^-\big(1 - \eta(x_1,t)\big) + \bar{u}_1^+\eta(x_1,t),\end{aligned} \tag{1.12}$$

where

$$\sigma(x_1,t) := \frac{\tilde{\rho}^r(x_1,t) - \bar{\rho}^-}{\bar{\rho}^+ - \bar{\rho}^-}, \quad \eta(x_1,t) := \frac{\tilde{u}_1^r(x_1,t) - \bar{u}_1^-}{\bar{u}_1^+ - \bar{u}_1^-}, \tag{1.13}$$

we construct the ansatz as follows.

Ansatz. Set the ansatz $(\tilde{\rho}, \tilde{\mathbf{u}}) = (\tilde{\rho}, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ as

$$\begin{aligned}\tilde{\rho}(x,t) &:= \rho^-(x,t)\big(1 - \sigma(x_1,t)\big) + \rho^+(x,t)\sigma(x_1,t) \\ &= \tilde{\rho}^r(x_1,t) + v^-(x,t)(1 - \sigma(x_1,t)) + v^+(x,t)\sigma(x_1,t), \\ \tilde{\mathbf{u}}(x,t) &:= \mathbf{u}^-(x,t)\big(1 - \eta(x_1,t)\big) + \mathbf{u}^+(x,t)\eta(x_1,t) \\ &= \tilde{u}_1^r(x_1,t)\mathbf{e}_1 + \mathbf{z}^-(x,t)(1 - \eta(x_1,t)) + \mathbf{z}^+(x,t)\eta(x_1,t),\end{aligned} \tag{1.14}$$

where $\mathbf{e}_1 := (1,0,0)$ is a unit vector, $v^{\pm}$ is given in (1.11) and $\mathbf{z}^{\pm}(x,t) := \mathbf{u}^{\pm}(x,t) - \bar{\mathbf{u}}^{\pm}$ with $\bar{\mathbf{u}}^{\pm} := \bar{u}_1^{\pm}\mathbf{e}_1$. If $\|v_0\|_{L^\infty(\mathbb{R}^3)}$ is suitably small, it follows from (1.11) that

$$\mathbf{z}^{\pm}(x,t) = \frac{\mathbf{w}^{\pm}(x,t) - v^{\pm}(x,t)\bar{\mathbf{u}}^{\pm}}{\rho^{\pm}(x,t)}. \tag{1.15}$$

Same as the solution of (1.1), (1.8), the ansatz (1.14) is also periodic with respect to x_2 and x_3, but not to x_1.

Define the domain $\Omega = \mathbb{R} \times \mathbb{T}^2$, then we are ready to state the main result.

Theorem 1.3. *Assume that (1.2) and (1.5) holds and the periodic perturbation* $(v_0, \mathbf{w}_0) \in H^5(\mathbb{T}^3)$ *satisfies (1.9). Then there exist* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$ *such that, if*

$$\left|\bar{\rho}^+ - \bar{\rho}^-\right| \le \delta_0 \quad \text{and} \quad \|v_0, \mathbf{w}_0\|_{H^5(\mathbb{T}^3)} \le \varepsilon_0, \tag{1.16}$$

the Cauchy problem (1.1), (1.8) admits a unique global solution $(\rho, \mathbf{u})(x,t)$*, which is periodic with respect to* x_2 *and* x_3*, satisfying*

$$\begin{aligned} (\rho - \tilde{\rho}, \mathbf{u} - \tilde{\mathbf{u}}) &\in C\big(0,+\infty; H^2(\Omega)\big), \\ \nabla(\rho - \tilde{\rho}) &\in L^2\big(0,+\infty; H^1(\Omega)\big), \\ \nabla(\mathbf{u} - \tilde{\mathbf{u}}) &\in L^2\big(0,+\infty; H^2(\Omega)\big), \end{aligned} \tag{1.17}$$

and

$$\sup_{x\in\mathbb{R}^3} \left|(\rho, \mathbf{u})(x,t) - \big(\rho^r, u_1^r, 0, 0\big)(x_1,t)\right| \to 0 \quad \text{as } t \to +\infty. \tag{1.18}$$

Remark 1.4. If the periodic perturbation in (1.8) is independent of x_2 or x_3, then the 3-d problem is reduced to a 2-d one. Hence, Theorem 1.3 is also valid for the 2-d compressible N-S equations.

The rest of the paper is organized as follows. In the next section, some useful lemmas are given. Then the a priori estimates and the main result are obtained in Section 3. In the last two appendices, exponential decay rates of both the periodic solutions and error terms of the ansatz are proved.

2. Preliminaries

Notations. Let $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}, \|\cdot\|_l := \|\cdot\|_{H^l(\Omega)}$ for $l \ge 1$, and $\delta := |\bar{\rho}^+ - \bar{\rho}^-|$, $\varepsilon := \|v_0, \mathbf{w}_0\|_{H^5(\mathbb{T}^3)}$. Throughout the paper $C > 0$ denotes a generic constant, independent of ε, δ and t.

Lemma 2.1 *([33], Lemma 2.1). The smooth rarefaction wave* $\big(\tilde{\rho}^r, \tilde{u}_1^r\big)$ *given by (1.7) satisfies the following properties.*

i) $\big(\tilde{\rho}^r, \tilde{u}_1^r\big)$ *solves the 1-d isentropic Euler equations;*
ii) $\partial_1\tilde{\rho}^r > 0, \partial_1\tilde{u}_1^r > 0$ *and* $\left|\partial_1^2\tilde{u}_1^r\right| \le C\partial_1\tilde{u}_1^r$ *for all* $t \ge 0$ *and* $x_1 \in \mathbb{R}$*;*

iii) $\left\|\left(\tilde{\rho}^r, \tilde{u}_1^r\right) - \left(\rho^r, u_1^r\right)\right\|_{L^\infty(\mathbb{R})} \to 0$ *as* $t \to +\infty$;

iv) *For any* $p \in [1, +\infty]$ *and* $t \geq 0$, *it holds that*

$$
\begin{aligned}
&\left\|\nabla_{t,x_1}\left(\tilde{\rho}^r, \tilde{u}_1^r\right)\right\|_{L^p(\mathbb{R})} \leq C \min\left\{\delta, \delta^{1/p} t^{-1+1/p}\right\}, \\
&\left\|\nabla_{t,x_1}^m\left(\tilde{\rho}^r, \tilde{u}_1^r\right)\right\|_{L^p(\mathbb{R})} \leq C \min\left\{\delta, t^{-1}\right\} \qquad \text{for } m = 2, 3.
\end{aligned}
$$

Lemma 2.2. *The functions* σ *and* η *given in (1.13) are smooth and satisfy the following properties.*

i) $\partial_1\sigma(x_1, t) > 0, \partial_1\eta(x_1, t) > 0$ *for any* $(x_1, t) \in \mathbb{R} \times [0, +\infty)$.

ii) *For any* $p \in [1, +\infty]$, *it holds that*

$$
\begin{aligned}
\|\sigma(1-\sigma), \sigma(1-\eta), \eta(1-\sigma), \eta(1-\eta)\|_{L^p(\mathbb{R})} &\leq C(1+t)^{1/p}, \\
\|\sigma - \eta\|_{L^p(\mathbb{R})} &\leq C\delta(1+t)^{1/p}, \\
\left\|\nabla_{t,x_1}^m(\sigma, \eta)\right\|_{L^p(\mathbb{R})} &\leq C \quad \text{for } m = 1, 2, 3.
\end{aligned}
$$

Proof. From Lemma 2.1, it is direct to get i). Then it remains to show ii). We prove only $\sigma(1-\eta)$ and $\sigma - \eta$, since the proofs of the others are similar.

It follows from (1.5) that $C^{-1}\delta \leq \left|\bar{u}_1^+ - \bar{u}_1^-\right| \leq C\delta$. By (1.7), one has that

$$
\begin{aligned}
\partial_1\tilde{u}_1^r &= \frac{\sqrt{p'(\tilde{\rho}^r)}}{\tilde{\rho}^r}\partial_1\tilde{\rho}^r \;\Rightarrow\; C^{-1}\partial_1\tilde{\rho}^r \leq \partial_1\tilde{u}_1^r \leq C\partial_1\tilde{\rho}^r, \\
\partial_1\tilde{\rho}_1^r &= \left(\frac{p''(\tilde{\rho}^r)}{2\sqrt{p'(\tilde{\rho}^r)}} + \frac{\sqrt{p'(\tilde{\rho}^r)}}{\tilde{\rho}^r}\right)^{-1}\partial_1\tilde{\omega} \;\Rightarrow\; C^{-1}\partial_1\tilde{\omega} \leq \partial_1\tilde{\rho}_1^r \leq C\partial_1\tilde{\omega},
\end{aligned}
$$

which yields that

$$
0 < \tilde{\rho}^r - \bar{\rho}^- = \int_{-\infty}^{x_1} \partial_y\tilde{\rho}^r(y,t)dy \leq C\int_{-\infty}^{x_1} \partial_y\tilde{\omega}(y,t)dy \leq C\left(\tilde{\omega} - \lambda_2^-\right),
$$

and similarly,

$$
0 < \bar{\rho}^+ - \tilde{\rho}^r \leq C\left(\lambda_2^+ - \tilde{\omega}\right).
$$

Thus, one has that

$$
\begin{aligned}
\|\sigma(1-\eta)\|_{L^p(\mathbb{R})} &\leq C\delta^{-2}\left\|(\tilde{\rho}^r - \bar{\rho}^-)(\tilde{u}_1^r - \bar{u}_1^+)\right\|_{L^p(\mathbb{R})} \leq C\delta^{-2}\left\|(\tilde{\rho}^r - \bar{\rho}^-)(\tilde{\rho}^r - \bar{\rho}^+)\right\|_{L^p(\mathbb{R})} \\
&\leq C\delta^{-2}\left\|\left(\tilde{\omega} - \lambda_2^-\right)\left(\tilde{\omega} - \lambda_2^+\right)\right\|_{L^p(\mathbb{R})} \leq C(1+t)^{1/p},
\end{aligned}
$$

where the last inequality can be derived from the characteristic curve method and the fact that $\left|\lambda_2^+ - \lambda_2^-\right| \le C\delta$.

Next we prove $|\sigma - \eta|$. Denote $A(\rho) := \int_1^\rho \frac{\sqrt{p'(s)}}{s} ds$. It follows from (1.4) and (1.7) that

$$\tilde{u}_1^r = \bar{u}_1^- + A(\tilde{\rho}^r) - A(\bar{\rho}^-),$$

which, together with (1.5), yields that

$$\begin{aligned}\sigma - \eta &= \sigma \left[1 - \frac{\int_0^1 A'\big(\bar{\rho}^- + s(\tilde{\rho}^r - \bar{\rho}^-)\big) ds}{\int_0^1 A'\big(\bar{\rho}^- + s(\bar{\rho}^+ - \bar{\rho}^-)\big) ds} \right] \\ &= \sigma(1-\sigma)\big(\bar{\rho}^+ - \bar{\rho}^-\big) \frac{\int_0^1 \int_0^1 A''\big(\bar{\rho}^- + s(\tilde{\rho}^r - \bar{\rho}^-) + \tau s(\bar{\rho}^+ - \tilde{\rho}^r)\big) s d\tau ds}{\int_0^1 A'\big(\bar{\rho}^- + s(\bar{\rho}^+ - \bar{\rho}^-)\big) ds}.\end{aligned}$$

Since $\bar{\rho}^- \le \bar{\rho}^- + s(\tilde{\rho}^r - \bar{\rho}^-) + \tau s(\bar{\rho}^+ - \tilde{\rho}^r) \le \bar{\rho}^- + s(\bar{\rho}^+ - \bar{\rho}^-) \le \bar{\rho}^+$ for any $\tau, s \in [0,1]$, one has that

$$\|\sigma - \eta\|_{L^p(\mathbb{R})} \le C\delta \, \|\sigma(1-\sigma)\|_{L^p(\mathbb{R})} \le C\delta(1+t)^{1/p},$$

which finishes the proof. □

Lemma 2.3. *Consider the Cauchy problem (1.1) with the periodic initial data*

$$(\rho, \rho\mathbf{u})(x,0) = (\bar{\rho}, \bar{\rho}\bar{\mathbf{u}}) + (v_0, \mathbf{w}_0)(x), \quad x \in \mathbb{R}^3, \tag{2.1}$$

where $\bar{\rho} > 0, \bar{\mathbf{u}} \in \mathbb{R}^3$ are constants, and $(v_0, \mathbf{w}_0) \in H^{k+2}(\mathbb{T}^3)$ $(k \ge 1)$ is periodic with period $\mathbb{T}^3$, satisfying

$$\int_{\mathbb{T}^3} (v_0, \mathbf{w}_0) dx = 0.$$

Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon = \|v_0, \mathbf{w}_0\|_{H^{k+2}(\mathbb{T}^3)} < \varepsilon_0$, the unique global periodic solution $(\rho, \mathbf{u}) \in C\big((0,+\infty); H^{k+2}(\mathbb{T}^3)\big)$ exists and satisfies that

$$\begin{aligned} &\int_{\mathbb{T}^3} (\rho - \bar{\rho}, \rho\mathbf{u} - \bar{\rho}\bar{\mathbf{u}})(x,t) dx = 0, \\ &\|(\rho, \mathbf{u}) - (\bar{\rho}, \bar{\mathbf{u}})\|_{W^{k,\infty}(\mathbb{R}^3)} \le C\varepsilon e^{-ct}, \end{aligned} \qquad t \ge 0,$$

where the constant $c > 0$ is independent of ε and t.

The proof of Lemma 2.3 is given in the appendix, which is based on the energy method with the aid of the Poincaré inequality on $\mathbb{T}^3$. Thus, under the assumptions of Theorem 1.3, there exists a constant $\alpha > 0$, independent of ε and t, such that

$$\big\|\big(v^{\pm}, \mathbf{z}^{\pm}\big)\big\|_{W^{3,\infty}(\mathbb{R}^3)} = \big\|\big(\rho^{\pm}, \mathbf{u}^{\pm}\big) - \big(\bar{\rho}^{\pm}, \bar{u}_1^{\pm}\mathbf{e}_1\big)\big\|_{W^{3,\infty}(\mathbb{R}^3)} \leq C\varepsilon e^{-2\alpha t}, \quad t \geq 0. \tag{2.2}$$

Although the ansatz (1.14) is not a solution to (1.1), their errors

$$h_0 := \partial_t\tilde{\rho} + \operatorname{div}(\tilde{\rho}\tilde{\mathbf{u}}), \tag{2.3}$$

$$\begin{aligned} \mathbf{h} = \big(h_1, h_2, h_3\big) &:= \partial_t\big(\tilde{\rho}\tilde{\mathbf{u}}\big) + \operatorname{div}\big(\tilde{\rho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}\big) + \nabla p(\tilde{\rho}) - \mu\triangle\tilde{\mathbf{u}} - (\mu+\lambda)\nabla\operatorname{div}\tilde{\mathbf{u}} \\ &= h_0\tilde{\mathbf{u}} + \tilde{\rho}\partial_t\tilde{\mathbf{u}} + \tilde{\rho}\tilde{\mathbf{u}}\cdot\nabla\tilde{\mathbf{u}} + \nabla p(\tilde{\rho}) - \mu\triangle\tilde{\mathbf{u}} - (\mu+\lambda)\nabla\operatorname{div}\tilde{\mathbf{u}}, \end{aligned} \tag{2.4}$$

decay exponentially fast with respect to t. More precisely, it holds that

Lemma 2.4. *Under the assumptions of Theorem 1.3, the errors (2.3) and (2.4) satisfy that*

$$\|h_0\|_{W^{2,p}(\Omega)} + \big\|\mathbf{h} + \big(2\mu+\lambda\big)\partial_1^2\tilde{u}_1^r\mathbf{e}_1\big\|_{W^{1,p}(\Omega)} \leq C\varepsilon e^{-\alpha t}, \quad p \in [1, +\infty], \tag{2.5}$$

where $\Omega = \mathbb{R}\times\mathbb{T}^2$ and $\alpha > 0$ is the constant given in (2.2).

The proof of Lemma 2.4 is similar to [12, Lemma 2.3], which depends on Lemmas 2.2 and 2.3. We still place it in the appendix for easy reading.

As indicated in [12], the functions that are integrable on $\Omega = \mathbb{R}\times\mathbb{T}^2$ and periodic with respect to x_2 and x_3 might not satisfy the 3-d G-N inequalities in general (counterexample: the 1-d functions in the $C_c^\infty(\mathbb{R})$ space are periodic with respect to x_2 and x_3 and satisfy the 1-d G-N inequalities, but not the 3-d ones). Thus we list the following G-N type inequality on Ω.

Lemma 2.5 *([12], Theorem 1.4). Assume that $u(x)$ is in the $L^q(\Omega)$ space with $\nabla^m u \in L^r(\Omega)$, where $1 \leq q, r \leq +\infty$ and $m \geq 1$, and u is periodic with respect to x_2 and x_3. Then there exists a decomposition $u(x) = \sum_{k=1}^{3} u^{(k)}(x)$ such that each $u^{(k)}$ satisfies the k-dimensional G-N inequality,*

$$\Big\|\nabla^j u^{(k)}\Big\|_{L^p(\Omega)} \leq C\,\|\nabla^m u\|_{L^r(\Omega)}^{\theta_k}\,\|u\|_{L^q(\Omega)}^{1-\theta_k}, \tag{2.6}$$

where $0 \leq j < m$ is any integer and $1 \leq p \leq +\infty$ is any number, satisfying

$$\frac{1}{p} = \frac{j}{k} + \Big(\frac{1}{r} - \frac{m}{k}\Big)\theta_k + \frac{1}{q}\big(1-\theta_k\big) \quad \text{with} \quad \frac{j}{m} \leq \theta_k \leq 1.$$

Moreover, it holds that

$$\left\|\nabla^j u\right\|_{L^p(\Omega)} \leq C \sum_{k=1}^{3} \|\nabla^m u\|_{L^r(\Omega)}^{\theta_k} \|u\|_{L^q(\Omega)}^{1-\theta_k}. \tag{2.7}$$

3. Proof of Theorem 1.3

Define the perturbation terms,

$$\phi := \rho - \tilde{\rho} \quad \text{and} \quad \psi = \big(\psi_1, \psi_2, \psi_3\big) := \mathbf{u} - \tilde{\mathbf{u}}. \tag{3.1}$$

It follows from (1.1), (2.3) and (2.4) that

$$\begin{cases} \partial_t \phi + \rho \mathrm{div}\psi + \mathbf{u} \cdot \nabla \phi + \phi \mathrm{div}\tilde{\mathbf{u}} + \nabla \tilde{\rho} \cdot \psi = -h_0, \\ \rho \partial_t \psi + \rho \mathbf{u} \cdot \nabla \psi + \rho \psi \cdot \nabla \tilde{\mathbf{u}} + p'(\rho) \nabla \phi + \Big(p'(\rho) - \frac{\rho}{\tilde{\rho}} p'(\tilde{\rho})\Big) \nabla \tilde{\rho} \\ \quad = \mu \triangle \psi + (\mu + \lambda) \nabla \mathrm{div}\psi + \mathbf{f} - \phi \mathbf{g} + \big(2\mu + \lambda\big) \partial_1^2 \tilde{u}_1^r \mathbf{e}_1, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \mathbf{f} &= (f_1, f_2, f_3)^T = h_0 \tilde{\mathbf{u}} - \mathbf{h} - \big(2\mu + \lambda\big) \partial_1^2 \tilde{u}_1^r \mathbf{e}_1, \\ \mathbf{g} &= (g_1, g_2, g_3)^T = \frac{1}{\tilde{\rho}} \left[\mu \triangle \tilde{\mathbf{u}} + (\mu + \lambda) \nabla \mathrm{div}\tilde{\mathbf{u}} + \mathbf{h} - h_0 \tilde{\mathbf{u}}\right] \\ &\qquad = \frac{1}{\tilde{\rho}} \big[\mu \triangle (\tilde{\mathbf{u}} - \tilde{u}_1^r \mathbf{e}_1) + (\mu + \lambda) \nabla \mathrm{div}(\tilde{\mathbf{u}} - \tilde{u}_1^r \mathbf{e}_1) - \mathbf{f}\big] ? \end{aligned}$$

By Lemma 2.4, one has that

$$\|\mathbf{f}\|_{W^{1,p}(\Omega)} \leq C\varepsilon e^{-\alpha t} \quad \forall p \in [1, +\infty] \quad \text{and} \quad \|\mathbf{g}\|_{W^{1,\infty}(\Omega)} \leq C\varepsilon e^{-\alpha t}. \tag{3.3}$$

It follows from (1.8), (1.14) and (1.15) that the perturbations (3.1) have the initial data

$$\begin{aligned} \phi(x,0) &= \tilde{\rho}^r(x_1,0) + v_0(x) - \tilde{\rho}^r(x_1,0) - v_0(x)(1 - \sigma(x_1,0)) - v_0(x)\sigma(x_1,0) = 0, \\ \psi(x,0) &= \frac{(\tilde{\rho}^r \tilde{u}_1^r)(x_1,0)\mathbf{e}_1 + \mathbf{w}_0(x)}{\tilde{\rho}^r(x_1,0) + v_0(x)} - \tilde{u}_1^r(x_1,0)\mathbf{e}_1 - \mathbf{z}^-(x,0)(1 - \eta(x_1,0)) \\ &\quad - \mathbf{z}^+(x,0)\eta(x_1,0) \\ &= \underbrace{\frac{\bar{\rho}^+ - \bar{\rho}^-}{\tilde{\rho}^r(x_1,0) + v_0(x)} \left[\frac{(\eta(1-\sigma))(x_1,0)}{\bar{\rho}^+ + v_0(x)} - \frac{(\sigma(1-\eta))(x_1,0)}{\bar{\rho}^- + v_0(x)}\right]}_{:=q_1} \mathbf{w}_0(x) \\ &\quad - \underbrace{\frac{\bar{\rho}^+ - \bar{\rho}^-}{\tilde{\rho}^r(x_1,0) + v_0(x)} \left[\frac{\bar{u}_1^-(\eta(1-\sigma))(x_1,0)}{\bar{\rho}^+ + v_0(x)} - \frac{\bar{u}_1^+(\sigma(1-\eta))(x_1,0)}{\bar{\rho}^- + v_0(x)}\right]}_{:=q_2} v_0(x)\mathbf{e}_1. \end{aligned}$$

Then Lemma 2.2 implies that

$$\|\psi(\cdot,0)\|_2 \le \|\mathbf{w}_0, v_0\|_{W^{2,\infty}(\mathbb{R}^3)} \|q_1, q_2\|_2 \le C_0\delta\varepsilon, \tag{3.4}$$

where $C_0 > 0$ is independent of ε and δ. In the following, we aim to solve the Cauchy problem (3.2) with the initial data

$$\begin{cases} \phi(x,0) = 0, \\ \psi(x,0) = q_1(x)\mathbf{w}_0(x) - q_2(x)v_0(x)\mathbf{e}_1, \end{cases} \tag{3.5}$$

which satisfies (3.4). For any $T > 0$, define the solution space

$$\begin{aligned} X(0,T) := \big\{&(\phi,\psi) \text{ is periodic with respect to } x_2 \text{ and } x_3 : \\ &(\phi,\psi) \in C(0,T;H^2(\Omega)),\ \nabla\phi \in L^2\big(0,T;H^1(\Omega)\big),\ \nabla\psi \in L^2\big(0,T;H^2(\Omega)\big)\big\}. \end{aligned}$$

Theorem 3.1. *Under the assumptions of Theorem 1.3, there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that if*

$$\delta = \left|\bar\rho^+ - \bar\rho^-\right| \le \delta_0 \quad \textit{and} \quad \varepsilon = \|v_0, \mathbf{w}_0\|_{H^5(\mathbb{T}^3)} \le \varepsilon_0,$$

then the Cauchy problem (3.2) with (3.5) admits a unique solution $(\phi,\psi) \in X(0,+\infty)$, satisfying

$$\|\phi,\psi\|_{L^\infty(\mathbb{R}^3)} \to 0 \quad \textit{as } t \to +\infty. \tag{3.6}$$

Once Theorem 3.1 is proved, it is straightforward to imply Theorem 1.3. Now we focus on proving Theorem 3.1. Since the local existence is standard, we only need to give the a priori estimates.

3.1. A priori estimates

Proposition 3.2. *Under the assumptions of Theorem 3.1, for any $T > 0$, assume that $(\phi,\psi) \in X(0,T)$ solves the problem (3.2) with the initial data $(\phi,\psi)(x,0) = (\phi_0,\psi_0)(x) \in H^2(\Omega)$. Then there exist $\varepsilon_0 > 0, \delta_0 > 0$ and $\nu_0 > 0$ such that if $\varepsilon < \varepsilon_0, \delta < \delta_0$ and*

$$\nu := \sup_{t\in(0,T)} \|\phi,\psi\|_2 < \nu_0, \tag{3.7}$$

it holds that

$$\sup_{t\in(0,T)} \|\phi,\psi\|_2^2 + \int_0^T \big(\|\nabla\phi\|_1^2 + \|\nabla\psi\|_2^2\big)dt \le C_1\big(\|\phi_0,\psi_0\|_2^2 + \varepsilon + \delta^{\frac{1}{3}}\big), \tag{3.8}$$

where the constant $C_1 > 0$ is independent of ε, δ, ν and T.

To prove Proposition 3.2, it first follows from (3.7) and Lemma 2.5 that

$$\begin{aligned}
&\sup_{t\in(0,T)} \|\phi,\psi\|_{L^\infty(\Omega)} \\
&\leq C \sup_{t\in(0,T)} \left\{ \|\nabla(\phi,\psi)\|^{\frac{1}{2}} \|\phi,\psi\|^{\frac{1}{2}} + \|\nabla(\phi,\psi)\| + \left\|\nabla^2(\phi,\psi)\right\|^{\frac{3}{4}} \|\phi,\psi\|^{\frac{1}{4}} \right\} \\
&\leq C \sup_{t\in(0,T)} \|\phi,\psi\|_2 \leq C\nu.
\end{aligned}$$

Thus, one can let $\varepsilon_0 > 0$ and $\nu_0 > 0$ small such that

$$\frac{1}{2}\bar{\rho}^- \leq \inf_{\substack{x\in\Omega\\ t\in(0,T)}} \rho(x,t) \leq \sup_{\substack{x\in\Omega\\ t\in(0,T)}} \rho(x,t) \leq 2\bar{\rho}^+. \tag{3.9}$$

Lemma 3.3. *Under the assumptions of Proposition 3.2, there exist $\varepsilon_0 > 0, \delta_0 > 0$ and $\nu_0 > 0$ such that if $\varepsilon < \varepsilon_0, \delta < \delta_0$ and $\nu < \nu_0$, then*

$$\sup_{t\in(0,T)} \|\phi,\psi\|^2 + \int_0^T \left\| \left(\partial_1 \tilde{u}_1^r\right)^{\frac{1}{2}} (\phi,\psi_1) \right\|^2 dt + \int_0^T \|\nabla\psi\|^2 \, dt \leq C\left(\|\phi_0,\psi_0\|^2 + \varepsilon + \delta^{\frac{1}{3}} \right). \tag{3.10}$$

Proof. Define

$$\Phi(\rho,\tilde{\rho}) = \int_{\tilde{\rho}}^{\rho} \frac{p(s)-p(\tilde{\rho})}{s^2} ds = \frac{1}{(\gamma-1)\rho} \left[p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho})(\rho-\tilde{\rho}) \right],$$

which satisfies $C^{-1}|\phi|^2 \leq \Phi(\rho,\tilde{\rho}) \leq C|\phi|^2$. By the fact that

$$\partial_\rho \Phi = \frac{p(\rho)-p(\tilde{\rho})}{\rho^2}, \qquad \partial_{\tilde{\rho}} \Phi = p'(\tilde{\rho}) \Big(\frac{1}{\rho} - \frac{1}{\tilde{\rho}} \Big),$$

one can multiply Φ on $(1.1)_1$ to get that

$$\begin{aligned}
\partial_t(\rho\Phi) + \operatorname{div}(\rho\Phi\mathbf{u}) &= \rho\big(\partial_t\Phi + \mathbf{u}\cdot\nabla\Phi\big) \\
&= -\left[p(\rho)-p(\tilde{\rho})\right] \operatorname{div}\mathbf{u} - p'(\tilde{\rho})\phi \Big[\frac{1}{\tilde{\rho}} \big(h_0 + \psi\cdot\nabla\tilde{\rho} \big) - \operatorname{div}\tilde{\mathbf{u}} \Big] \\
&= -\left[p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho})\phi \right] \operatorname{div}\tilde{\mathbf{u}} - \operatorname{div}\left[\big(p(\rho)-p(\tilde{\rho}) \big) \psi \right] \\
&\quad + \psi\cdot\nabla\big(p(\rho) - p(\tilde{\rho}) \big) - \frac{p'(\tilde{\rho})}{\tilde{\rho}} \phi \big(h_0 + \psi\cdot\nabla\tilde{\rho} \big),
\end{aligned}$$

which gives that

$$
\begin{aligned}
&\partial_t(\rho\Phi) + [p(\rho) - p(\tilde\rho) - p'(\tilde\rho)\phi]\,\partial_1\tilde u_1^r - p'(\rho)\psi\cdot\nabla\phi - \big(p'(\rho) - \frac{\rho}{\tilde\rho}p'(\tilde\rho)\big)\psi\cdot\nabla\tilde\rho \\
&\quad = \mathrm{div}\{\cdots\} - [p(\rho) - p(\tilde\rho) - p'(\tilde\rho)\phi]\,\big(\mathrm{div}\tilde{\mathbf u} - \partial_1\tilde u_1^r\big) - \frac{p'(\tilde\rho)}{\tilde\rho}h_0\phi,
\end{aligned}
\tag{3.11}
$$

where $\{\cdots\} = -\rho\Phi\mathbf u - \big(p(\rho) - p(\tilde\rho)\big)\psi$. On the other hand, multiplying $\cdot\psi$ on $(3.2)_2$ yields that

$$
\begin{aligned}
&\partial_t\Big(\frac12\rho|\psi|^2\Big)\underbrace{-\frac12\partial_t\rho|\psi|^2 + \rho(\mathbf u\cdot\nabla\psi)\cdot\psi}_{I_{1,1}} + \rho(\psi\cdot\nabla\tilde{\mathbf u})\cdot\psi + p'(\rho)\nabla\phi\cdot\psi \\
&\quad + \big(p'(\rho) - \frac{\rho}{\tilde\rho}p'(\tilde\rho)\big)\nabla\tilde\rho\cdot\psi + \mu|\nabla\psi|^2 - \mathrm{div}\Big(\frac{\mu}{2}\nabla|\psi|^2\Big) + (\mu+\lambda)|\mathrm{div}\psi|^2 \\
&\quad - (\mu+\lambda)\mathrm{div}\big(\psi\mathrm{div}\psi\big) = \mathbf f\cdot\psi - \phi\mathbf g\cdot\psi + \big(2\mu+\lambda\big)\partial_1^2\tilde u_1^r\psi_1.
\end{aligned}
\tag{3.12}
$$

Note that

$$
I_{1,1} = -\frac12\left[\partial_t\rho + \mathrm{div}(\rho\mathbf u)\right]|\psi|^2 + \mathrm{div}\Big(\frac12\rho|\psi|^2\,\mathbf u\Big) = \mathrm{div}\Big(\frac12\rho|\psi|^2\,\mathbf u\Big),
$$

and

$$
\rho(\psi\cdot\nabla\tilde{\mathbf u})\cdot\psi = \rho\partial_1\tilde u_1^r\psi_1^2 + \underbrace{\rho(\psi\cdot\nabla(\tilde{\mathbf u} - \tilde u_1^r\mathbf e_1))\cdot\psi}_{I_{1,2}},
$$

then adding up (3.11) and (3.12) yields that

$$
\begin{aligned}
&\partial_t\big(\rho\Phi + \frac12\rho|\psi|^2\big) + [p(\rho) - p(\tilde\rho) - p'(\tilde\rho)\phi]\,\partial_1\tilde u_1^r + \rho\partial_1\tilde u_1^r\psi_1^2 + \mu|\nabla\psi|^2 + (\mu+\lambda)|\mathrm{div}\psi|^2 \\
&\quad = \mathrm{div}\{\cdots\} - [p(\rho) - p(\tilde\rho) - p'(\tilde\rho)\phi]\,\big(\mathrm{div}\tilde{\mathbf u} - \partial_1\tilde u_1^r\big) - I_{1,2} \\
&\quad\quad - \frac{p'(\tilde\rho)}{\tilde\rho}h_0\phi + \mathbf f\cdot\psi - \phi\mathbf g\cdot\psi + \big(2\mu+\lambda\big)\partial_1^2\tilde u_1^r\psi_1,
\end{aligned}
\tag{3.13}
$$

where $\mathrm{div}\{\cdots\}$ vanishes after integration on Ω. Note that from Lemma 2.3, one has that

$$
\begin{aligned}
\|\mathrm{div}\tilde{\mathbf u} - \partial_1\tilde u_1^r\|_{L^\infty(\Omega)} &\le C\left\|\mathbf z^\pm\right\|_{W^{1,\infty}(\Omega)} \le C\varepsilon e^{-\alpha t}, \\
|I_{1,2}| &\le C\left\|\mathbf z^\pm\right\|_{W^{1,\infty}(\Omega)}|\psi|^2 \le C\varepsilon e^{-\alpha t}|\psi|^2.
\end{aligned}
$$

Then integrating (3.13) on Ω yields that

$$
\sup_{t\in(0,T)}\|\phi,\psi\|^2 + \int_0^T\|\nabla\psi\|^2\,dt + \int_0^T\left\|\big(\partial_1\tilde u_1^r\big)^{\frac12}(\phi,\psi_1)\right\|^2 dt
$$

$$\leq C\|\phi_0,\psi_0\|^2 + C\int_0^T \Big\{\varepsilon e^{-\alpha t}\big(\|\phi\|^2+\|\psi\|^2\big) + \|h_0\|\,\|\phi\|$$
$$+ \big(\|\mathbf{f}\| + \|\mathbf{g}\|_{L^\infty(\Omega)}\|\phi\|\big)\|\psi\| + \int_\Omega \left|\partial_1^2\tilde{u}_1^r\right| |\psi_1|\,dx\Big\}dt$$
$$\leq C\|\phi_0,\psi_0\|^2 + C\varepsilon \sup_{t\in(0,T)}\|\phi,\psi\|^2 + C\varepsilon + C\int_0^T\int_\Omega \left|\partial_1^2\tilde{u}_1^r\right| |\psi|\,dxdt, \tag{3.14}$$

where we have used (3.3) and the fact that $\partial_1\tilde{u}_1^r > 0$. Decompose $\psi = \sum_{k=1}^{3}\psi^{(k)}$ as in Lemma 2.5 such that $\psi^{(k)}$ satisfies the k-dimensional G-N inequalities. Then by Lemma 2.1, the last term in (3.14) satisfies that

$$C\int_0^T\int_\Omega \left|\partial_1^2\tilde{u}_1^r\right| |\psi|\,dxdt \leq C\int_0^T\int_\Omega \sum_{k=1}^{3}\left|\partial_1^2\tilde{u}_1^r\right| \left|\psi^{(k)}\right| dxdt$$
$$\leq C\int_0^T \Big[\left\|\partial_1^2\tilde{u}_1^r\right\|_{L^1(\Omega)}\Big(\left\|\psi^{(1)}\right\|_{L^\infty(\Omega)} + \left\|\psi^{(2)}\right\|_{L^\infty(\Omega)}\Big) + \left\|\partial_1^2\tilde{u}_1^r\right\|_{L^{\frac{6}{5}}(\Omega)}\left\|\psi^{(3)}\right\|_{L^6(\Omega)}\Big]dt$$
$$\leq C\int_0^T \min\{\delta,t^{-1}\}\big(\|\nabla\psi\|^{\frac12}\|\psi\|^{\frac12} + \|\nabla\psi\|\big)dt$$
$$\leq \frac12\int_0^T\|\nabla\psi\|^2\,dt + C\delta^{\frac13}\sup_{t\in(0,T)}\|\psi\|^2 + C\delta^{\frac13}. \tag{3.15}$$

Collecting (3.14) and (3.15), one can obtain (3.10) if $\varepsilon>0$ and $\delta>0$ are small. □

Lemma 3.4. *Under the assumptions of Proposition 3.2, there exist $\varepsilon_0>0, \delta_0>0$ and $\nu_0>0$ such that if $\varepsilon<\varepsilon_0, \delta<\delta_0$ and $\nu<\nu_0$, then*

$$\sup_{t\in(0,T)}\|\nabla\phi\|^2 + \int_0^T\|\nabla\phi\|^2\,dt \leq C\Big(\|\phi_0\|_1^2 + \|\psi_0\|^2 + \varepsilon + \delta^{\frac13} + \nu\int_0^T\left\|\nabla^3\psi\right\|^2 dt\Big). \tag{3.16}$$

Proof. By taking the gradient ∇ on $(3.2)_1$ and multiplying the result by $\cdot\frac{\nabla\phi}{\rho^2}$, one has that

$$\partial_t\Big(\frac{|\nabla\phi|^2}{2\rho^2}\Big) - \frac{\mathrm{div}\tilde{\mathbf{u}}}{2\rho^2}|\nabla\phi|^2 - \frac{\mathrm{div}\psi}{2\rho^2}|\nabla\phi|^2 + \frac{\mathrm{div}\psi}{\rho^2}\nabla\tilde{\rho}\cdot\nabla\phi + \frac1\rho\nabla\mathrm{div}\psi\cdot\nabla\phi + \mathrm{div}\Big(\frac{|\nabla\phi|^2}{2\rho^2}\mathbf{u}\Big)$$
$$+\frac{1}{\rho^2}\nabla\phi\cdot\big[\nabla\big(\tilde{\mathbf{u}}+\psi\big)\nabla\phi + \phi\nabla\mathrm{div}\tilde{\mathbf{u}} + \nabla^2\tilde{\rho}\psi + \nabla\psi\nabla\tilde{\rho}\big] = -\frac{1}{\rho^2}\nabla h_0\cdot\nabla\phi, \tag{3.17}$$

where we have used the fact that

$$\begin{aligned}\frac{|\nabla\phi|^2}{\rho^3}\partial_t\rho+\frac{1}{2\rho^2}(\mathbf{u}\cdot\nabla)|\nabla\phi|^2&=\frac{|\nabla\phi|^2}{\rho^3}\partial_t\rho+\mathrm{div}\Big(\frac{|\nabla\phi|^2}{2\rho^2}\mathbf{u}\Big)-\mathrm{div}\Big(\frac{\mathbf{u}}{2\rho^2}\Big)|\nabla\phi|^2\\&=\mathrm{div}\Big(\frac{|\nabla\phi|^2}{2\rho^2}\mathbf{u}\Big)-\frac{3}{2\rho^2}\mathrm{div}\mathbf{u}\,|\nabla\phi|^2.\end{aligned}\tag{3.18}$$

Multiplying $\cdot\frac{\nabla\phi}{\rho}$ on $(3.2)_2$ yields that

$$\begin{aligned}&\partial_t\psi\cdot\nabla\phi+(\mathbf{u}\cdot\nabla\psi)\cdot\nabla\phi+(\psi\cdot\nabla\tilde{\mathbf{u}})\cdot\nabla\phi+\frac{p'(\rho)}{\rho}|\nabla\phi|^2+\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\tilde\rho)}{\tilde\rho}\Big)\nabla\tilde\rho\cdot\nabla\phi\\&\quad\underbrace{-\frac{1}{\rho}[\mu\triangle\psi+(\mu+\lambda)\nabla\mathrm{div}\psi]\cdot\nabla\phi}_{I_{2,1}}=\frac{1}{\rho}\mathbf{f}\cdot\nabla\phi-\frac{\phi}{\rho}\mathbf{g}\cdot\nabla\phi+\frac{2\mu+\lambda}{\rho}\partial_1^2\tilde u_1^r\partial_1\phi.\end{aligned}\tag{3.19}$$

Note that

$$\begin{aligned}\partial_t\psi\cdot\nabla\phi&=\partial_t\big(\psi\cdot\nabla\phi\big)-\psi\cdot\nabla\partial_t\phi\\&=\partial_t\big(\psi\cdot\nabla\phi\big)-\mathrm{div}(\psi\partial_t\phi)-\mathrm{div}\psi\big(h_0+\rho\mathrm{div}\psi+\mathbf{u}\cdot\nabla\phi+\phi\mathrm{div}\tilde{\mathbf{u}}+\nabla\tilde\rho\cdot\psi\big),\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}I_{2,1}&=\frac{2\mu+\lambda}{\rho}\nabla\mathrm{div}\psi\cdot\nabla\phi+\frac{\mu}{\rho}\big(\triangle\psi-\nabla\mathrm{div}\psi\big)\cdot\nabla\phi\\&=\frac{2\mu+\lambda}{\rho}\nabla\mathrm{div}\psi\cdot\nabla\phi+\mu\mathrm{div}\Big(\frac{\nabla\phi\times\mathrm{curl}\psi}{\rho}\Big)+\frac{\mu\nabla\rho\cdot\big(\nabla\phi\times\mathrm{curl}\psi\big)}{\rho^2},\end{aligned}\tag{3.21}$$

here and hereafter "×" denotes the cross product of two vectors. By (3.20) and (3.21), adding up (3.19) and $(2\mu+\lambda)\cdot$(3.17) gives that

$$\partial_t\Big(\frac{2\mu+\lambda}{2\rho^2}|\nabla\phi|^2+\psi\cdot\nabla\phi\Big)+\frac{p'(\rho)}{\rho}|\nabla\phi|^2=\mathrm{div}\{\cdots\}+\sum_{j=2}^{5}I_{2,j},\tag{3.22}$$

where $\mathrm{div}\{\cdots\}$ vanishes after integration on Ω and

$$\begin{aligned}I_{2,2}&=\frac{2\mu+\lambda}{\rho^2}\Big(\frac{1}{2}\mathrm{div}\tilde{\mathbf{u}}|\nabla\phi|^2-\mathrm{div}\psi\nabla\tilde\rho\cdot\nabla\phi-\nabla\phi\cdot\nabla\tilde{\mathbf{u}}\nabla\phi-\nabla\phi\cdot\nabla\psi\nabla\tilde\rho\Big)\\&\quad+\rho(\mathrm{div}\psi)^2+\mathrm{div}\psi\mathbf{u}\cdot\nabla\phi-(\mathbf{u}\cdot\nabla\psi)\cdot\nabla\phi+\frac{\mu}{\rho^2}\nabla\tilde\rho\cdot\big(\nabla\phi\times\mathrm{curl}\psi\big),\\I_{2,3}&=-\frac{2\mu+\lambda}{\rho^2}\big(\phi\nabla\mathrm{div}\tilde{\mathbf{u}}\cdot\nabla\phi+\nabla\phi\cdot\nabla^2\tilde\rho\psi\big)+\mathrm{div}\psi\big(\phi\mathrm{div}\tilde{\mathbf{u}}+\nabla\tilde\rho\cdot\psi\big)-(\psi\cdot\nabla\tilde{\mathbf{u}})\cdot\nabla\phi\end{aligned}$$

$$
\begin{aligned}
&\quad -\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\tilde\rho)}{\tilde\rho}\Big)\nabla\tilde\rho\cdot\nabla\phi,\\
I_{2,4}&=\frac{2\mu+\lambda}{\rho^2}\Big(\frac{1}{2}\mathrm{div}\psi\,|\nabla\phi|^2-\nabla\phi\cdot\nabla\psi\nabla\phi\Big),\\
I_{2,5}&=-\frac{2\mu+\lambda}{\rho^2}\nabla h_0\cdot\nabla\phi+\mathrm{div}\psi h_0+\frac{1}{\rho}\big(\mathbf{f}-\mathbf{g}\phi\big)\cdot\nabla\phi+\frac{2\mu+\lambda}{\rho}\partial_1^2\tilde u_1^r\partial_1\phi.
\end{aligned}
$$

Now we estimate $I_{2,j}$ one by one. Since

$$
\begin{aligned}
\|\nabla\tilde{\mathbf{u}}\|_{L^\infty(\Omega)}&\le\|\partial_1\tilde u_1^r\|_{L^\infty(\mathbb{R})}+C\left\|\mathbf{z}^\pm\right\|_{W^{1,+\infty}(\Omega)}\le C(\delta+\varepsilon),\\
\|\nabla\tilde\rho\|_{L^\infty(\Omega)}&\le\|\partial_1\tilde\rho^r\|_{L^\infty(\mathbb{R})}+C\left\|v^\pm\right\|_{W^{1,+\infty}(\Omega)}\le C(\delta+\varepsilon),
\end{aligned}\tag{3.23}
$$

it holds that

$$
\int_0^T\|I_{2,2}\|_{L^1(\Omega)}\,dt\le\Big(\frac14+C(\delta+\varepsilon)\Big)\int_0^T\int_\Omega\frac{p'(\rho)}{\rho}\,|\nabla\phi|^2\,dxdt+C\int_0^T\|\nabla\psi\|^2\,dt.
$$

Note that $|\nabla\mathrm{div}\tilde{\mathbf{u}}|\le\left|\partial_1^2\tilde u_1^r\right|+C\varepsilon e^{-\alpha t}$, $\left|\nabla^2\tilde\rho\right|\le\left|\partial_1^2\tilde\rho^r\right|+C\varepsilon e^{-\alpha t}$ and

$$
\begin{aligned}
|\nabla\tilde\rho\cdot\psi-\partial_1\tilde\rho^r\psi_1|&\le C\varepsilon e^{-\alpha t}\,|\psi|\,,\\
\big|\big(\psi\cdot\nabla\tilde{\mathbf{u}}\big)\cdot\nabla\phi-\partial_1\tilde u_1^r\psi_1\partial_1\phi\big|&\le C\varepsilon e^{-\alpha t}\,|\psi|\,|\nabla\phi|\,,\\
|\nabla\tilde\rho\cdot\nabla\phi-\partial_1\tilde\rho^r\partial_1\phi|&\le C\varepsilon e^{-\alpha t}\,|\nabla\phi|\,,
\end{aligned}
$$

then it follows from Lemma 2.1 and (3.7) that

$$
\begin{aligned}
\int_0^T\|I_{2,3}\|_{L^1(\Omega)}\,dt&\le C\nu\int_0^T\big(\left\|\partial_1^2\tilde u_1^r\right\|+\left\|\partial_1^2\tilde\rho^r\right\|\big)\,\|\nabla\phi\|\,dt+C\varepsilon\int_0^T e^{-\alpha t}\,\|\phi,\psi\|\,\|\nabla(\phi,\psi)\|\,dt\\
&\quad+C\int_0^T\int_\Omega\partial_1\tilde u_1^r\big(\,|\phi|+|\psi_1|\,\big)\big(\,|\partial_1\phi|+|\nabla\psi|\,\big)dxdt\\
&\le C\delta^{\frac12}+C(\delta^{\frac12}+\varepsilon)\int_0^T\left\|\nabla\big(\phi,\psi\big)\right\|^2dt+C\varepsilon\sup_{t\in(0,T)}\|\phi,\psi\|^2+C\int_0^T\left\|\big(\partial_1\tilde u_1^r\big)^{\frac12}(\phi,\psi_1)\right\|^2dt.
\end{aligned}
$$

By Lemma 2.5, one has that

$$
\|\nabla\psi\|_{L^\infty(\Omega)}\le C\big(\left\|\nabla^2\psi\right\|^{\frac12}\|\nabla\psi\|^{\frac12}+\left\|\nabla^2\psi\right\|+\left\|\nabla^3\psi\right\|^{\frac34}\|\nabla\psi\|^{\frac14}\,\big)\le C\,\|\nabla\psi\|_2\,.\tag{3.24}
$$

Thus, one can estimate $I_{2,4}$ by

$$\int_0^T \|I_{2,4}\|_{L^1(\Omega)}\,dt \le C\int_0^T \|\nabla\psi\|_{L^\infty(\Omega)}\|\nabla\phi\|^2\,dt \le C\int_0^T \|\nabla\psi\|_2\|\nabla\phi\|^2\,dt$$
$$\le C\nu\int_0^T \|\nabla\phi\|^2\,dt + C\nu\int_0^T \left\|\nabla^3\psi\right\|^2 dt.$$

By (3.3) and Lemmas 2.1 and 2.4, the last one $I_{2,5}$ satisfies that

$$\int_0^T \|I_{2,5}\|_{L^1(\Omega)}\,dt \le C(\varepsilon+\delta^{\frac{1}{2}})\int_0^T \|\nabla\phi\|^2\,dt + C\varepsilon\int_0^T \|\nabla\psi\|^2\,dt + C\varepsilon\sup_{t\in(0,T)}\|\phi\|^2$$
$$+ C(\varepsilon+\delta^{\frac{1}{2}}).$$

Thus, collecting the estimates of $I_{2,2}$ to $I_{2,5}$ and using Lemma 3.3, one can integrate (3.22) over $\Omega\times(0,T)$ to obtain (3.16). $\square$

Lemma 3.5. *Under the assumptions of Proposition 3.2, there exist $\varepsilon_0>0,\delta_0>0$ and $\nu_0>0$ such that if $\varepsilon<\varepsilon_0,\delta<\delta_0$ and $\nu<\nu_0$, then*

$$\sup_{t\in(0,T)}\|\nabla\psi\|^2 + \int_0^T \left\|\nabla^2\psi\right\|^2 dt \le C\big(\|\phi_0,\psi_0\|_1^2 + \varepsilon + \delta^{\frac{1}{3}} + \nu\int_0^T \left\|\nabla^3\psi\right\|^2 dt\big). \tag{3.25}$$

Proof. By multiplying $-\frac{\triangle\psi}{\rho}\cdot$ on $(3.2)_2$ and using the fact that

$$\begin{aligned}\frac{1}{\rho}\nabla\mathrm{div}\psi\cdot\triangle\psi &= \frac{1}{\rho}|\nabla\mathrm{div}\psi|^2 + \frac{1}{\rho}\nabla\mathrm{div}\psi\cdot(\triangle\psi-\nabla\mathrm{div}\psi)\\ &= \frac{1}{\rho}|\nabla\mathrm{div}\psi|^2 + \mathrm{div}\Big[\frac{\mathrm{div}\psi}{\rho}\big(\triangle\psi-\nabla\mathrm{div}\psi\big)\Big] + \frac{\mathrm{div}\psi}{\rho^2}(\nabla\tilde\rho+\nabla\phi)\cdot\big(\triangle\psi-\nabla\mathrm{div}\psi\big),\end{aligned} \tag{3.26}$$

one can get that

$$\frac{1}{2}\partial_t|\nabla\psi|^2 + \frac{\mu}{\rho}|\triangle\psi|^2 + \frac{\mu+\lambda}{\rho}|\nabla\mathrm{div}\psi|^2 = \mathrm{div}\{\cdots\} + \sum_{j=1}^4 I_{3,j}, \tag{3.27}$$

where $\{\cdots\} = \nabla\psi\partial_t\psi - \frac{\mu+\lambda}{\rho}\mathrm{div}\psi\big(\triangle\psi-\nabla\mathrm{div}\psi\big)$,

$$I_{3,1} = (\mathbf{u}\cdot\nabla\psi)\cdot\triangle\psi + \frac{p'(\rho)}{\rho}\nabla\phi\cdot\triangle\psi - \frac{\mu+\lambda}{\rho^2}\mathrm{div}\psi\nabla\tilde\rho\cdot\big(\triangle\psi-\nabla\mathrm{div}\psi\big),$$
$$I_{3,2} = (\psi\cdot\nabla\tilde{\mathbf{u}})\cdot\triangle\psi + \Big(\frac{p'(\rho)}{\rho}-\frac{p'(\tilde\rho)}{\tilde\rho}\Big)\nabla\tilde\rho\cdot\triangle\psi,$$

$$I_{3,3} = -\frac{\mu+\lambda}{\rho^2}\mathrm{div}\psi\nabla\phi\cdot(\triangle\psi - \nabla\mathrm{div}\psi),$$
$$I_{3,4} = -\frac{1}{\rho}(\mathbf{f}-\phi\mathbf{g})\cdot\triangle\psi - \frac{2\mu+\lambda}{\rho}\partial_1^2\tilde{u}_1^r\triangle\psi_1.$$

Note that $\int_\Omega \frac{\mu}{\rho}|\triangle\psi|^2\,dx \ge C\|\triangle\psi\|^2 \ge a_0\|\nabla^2\psi\|^2$ for some constant $a_0 > 0$. Then similar to the estimates of $I_{2,2}$ to $I_{2,5}$ in Lemma 3.4, one can prove that

$$\int_0^T \|I_{3,1}\|_{L^1(\Omega)}\,dt \le \frac{a_0}{4}\int_0^T \|\nabla^2\psi\|^2\,dt + C\int_0^T \|\nabla(\phi,\psi)\|^2\,dt,$$
$$\int_0^T \|I_{3,2}\|_{L^1(\Omega)}\,dt \le C\varepsilon \sup_{t\in(0,T)}\|\phi,\psi\|^2 + C(\varepsilon+\delta)\int_0^T \|\nabla^2\psi\|^2\,dt$$
$$+ C\int_0^T \left\|(\partial_1\tilde{u}_1^r)^{\frac{1}{2}}(\phi,\psi_1)\right\|^2 dt,$$
$$\int_0^T \|I_{3,3}\|_{L^1(\Omega)}\,dt \le C\int_0^T \|\mathrm{div}\psi\|_{L^\infty(\Omega)}\|\nabla\phi\|\,\|\nabla^2\psi\|\,dt \le C\nu\int_0^T \|\nabla\psi\|_2\,\|\nabla^2\psi\|\,dt$$
$$\le C\nu\int_0^T \|\nabla\psi\|_1^2\,dt + C\nu\int_0^T \|\nabla^3\psi\|^2\,dt,$$
$$\int_0^T \|I_{3,4}\|_{L^1(\Omega)}\,dt \le C(\varepsilon+\delta^{\frac{1}{2}})\int_0^T \|\nabla^2\psi\|^2\,dt + C\varepsilon\sup_{t\in(0,T)}\|\phi\|^2 + C(\varepsilon+\delta^{\frac{1}{2}}).$$

Then integrating (3.27) over $\Omega\times(0,T)$, together with Lemmas 3.3 and 3.4, yields (3.25). □

Lemma 3.6. *Under the assumptions of Proposition 3.2, there exist $\varepsilon_0 > 0, \delta_0 > 0$ and $\nu_0 > 0$ such that if $\varepsilon < \varepsilon_0, \delta < \delta_0$ and $\nu < \nu_0$, then*

$$\sup_{t\in(0,T)}\|\nabla^2\phi\|^2 + \int_0^T \|\nabla^2\phi\|^2\,dt \le C\big(\|\phi_0\|_2^2 + \|\psi_0\|_1^2 + \varepsilon + \delta^{\frac{1}{3}} + \nu\int_0^T \|\nabla^3\psi\|^2\,dt\big). \tag{3.28}$$

Proof. Let $i\in\{1,2,3\}$ be fixed. By taking the second derivative $\nabla\partial_i$ on $(3.2)_1$ and multiplying the result by $\cdot\frac{\nabla\partial_i\phi}{\rho^2}$, one can get that

$$\partial_t\Big(\frac{|\nabla\partial_i\phi|^2}{2\rho^2}\Big) + \frac{1}{\rho}\nabla\partial_i\mathrm{div}\psi\cdot\nabla\partial_i\phi = -\mathrm{div}\Big(\frac{|\nabla\partial_i\phi|^2}{2\rho^2}\mathbf{u}\Big) + I_{4,1} + I_{4,2}, \tag{3.29}$$

where

$$\begin{aligned} I_{4,1} &= \frac{3}{2\rho^2}\mathrm{div}\mathbf{u}\,|\nabla\partial_i\phi|^2 - \frac{1}{\rho^2}\left[\nabla\partial_i(\rho\mathrm{div}\psi) - \rho\nabla\partial_i\mathrm{div}\psi\right]\cdot\nabla\partial_i\phi \\ &\quad - \frac{1}{\rho^2}\left[\nabla\partial_i(\mathbf{u}\cdot\nabla\phi) - (\mathbf{u}\cdot\nabla)\nabla\partial_i\phi\right]\cdot\nabla\partial_i\phi - \frac{1}{\rho^2}\left[\nabla\partial_i(\phi\mathrm{div}\tilde{\mathbf{u}}) - \phi\nabla\partial_i\mathrm{div}\tilde{\mathbf{u}}\right]\cdot\nabla\partial_i\phi \\ &\quad - \frac{1}{\rho^2}\left[\nabla\partial_i(\nabla\tilde{\rho}\cdot\psi) - \nabla^2\partial_i\tilde{\rho}\psi\right]\cdot\nabla\partial_i\phi, \\ I_{4,2} &= -\frac{\phi}{\rho^2}\nabla\partial_i\mathrm{div}\tilde{\mathbf{u}}\cdot\nabla\partial_i\phi - \frac{1}{\rho^2}\nabla^2\partial_i\tilde{\rho}\ \psi\cdot\nabla\partial_i\phi - \frac{1}{\rho^2}\nabla\partial_i h_0\cdot\nabla\partial_i\phi. \end{aligned}$$

Here we have used (3.18) with $\nabla\phi$ being replaced by $\nabla\partial_i\phi$, and the bad term $\frac{1}{\rho}\nabla\partial_i\mathrm{div}\psi\cdot\nabla\partial_i\phi$ in (3.29) can be canceled by the equation $(3.2)_2$. Using Lemma 2.1 and (3.23), one can prove that

$$\begin{aligned} \int_0^T \|I_{4,1}\|_{L^1(\Omega)}\,dt &\le C(\varepsilon+\delta)\int_0^T \left\|\nabla^2\phi\right\|\left(\|\nabla\phi\|_1 + \|\nabla\psi\|_1\right)dt + C\int_0^T \|\nabla\psi\|_{L^\infty(\Omega)}\left\|\nabla^2\phi\right\|^2 dt \\ &\quad + C\int_0^T \left\|\nabla^2\phi\right\|\|\nabla\phi\|_{L^4(\Omega)}\left\|\nabla^2\psi\right\|_{L^4(\Omega)}dt \\ &\le C(\varepsilon+\delta+\nu)\int_0^T \left\|\nabla(\phi,\psi)\right\|_1^2 dt + C\nu\int_0^T \left\|\nabla^3\psi\right\|^2 dt, \end{aligned} \tag{3.30}$$

where we have used (3.7), (3.24) and the fact

$$\begin{aligned} \|\nabla\phi\|_{L^4(\Omega)} &\le C\sum_{k=1}^{3}\left\|\nabla^2\phi\right\|^{k/4}\|\nabla\phi\|^{1-k/4} \le C\nu, \\ \left\|\nabla^2\psi\right\|_{L^4(\Omega)} &\le C\sum_{k=1}^{3}\left\|\nabla^3\psi\right\|^{k/4}\left\|\nabla^2\psi\right\|^{1-k/4} \le C\left\|\nabla^2\psi\right\|_1. \end{aligned} \tag{3.31}$$

In addition, similar to the estimates of $I_{2,3}$ and $I_{2,5}$, one can prove that

$$\int_0^T \|I_{4,2}\|_{L^1(\Omega)}\,dt \le C(\delta^{\frac12}+\varepsilon)\int_0^T \left\|\nabla^2\phi\right\|^2 dt + C\varepsilon\sup_{t\in(0,T)}\|\phi,\psi\|^2 + C(\delta^{\frac12}+\varepsilon). \tag{3.32}$$

On the other hand, taking the derivative ∂_i on $\frac{1}{\rho}\cdot(3.2)_2$ and multiplying the result by $\cdot\nabla\partial_i\phi$ yield that

$$\partial_t\left(\partial_i\psi\cdot\nabla\partial_i\phi\right) + \frac{p'(\rho)}{\rho}|\nabla\partial_i\phi|^2 - \frac{2\mu+\lambda}{\rho}\nabla\mathrm{div}\partial_i\psi\cdot\nabla\partial_i\phi = \mathrm{div}\{\cdots\} + \sum_{j=3}^{5} I_{4,j}, \tag{3.33}$$

where $\{\cdots\} = \partial_i\partial_t\phi\partial_i\psi + \frac{\mu}{\rho}\nabla\partial_i\phi \times \mathrm{curl}\partial_i\psi$ and

$$
\begin{aligned}
I_{4,3} &= \partial_i h_0 \mathrm{div}\partial_i\psi + \partial_i\Big(\frac{\mathbf{f}-\phi\mathbf{g}}{\rho}\Big)\cdot\nabla\partial_i\phi + \partial_i\Big(\frac{2\mu+\lambda}{\rho}\partial_1^2\tilde{u}_1^r\Big)\partial_{1i}\phi,\\
I_{4,4} &= \partial_i\big(\mathrm{div}\tilde{\mathbf{u}}\phi + \nabla\tilde{\rho}\cdot\psi\big)\mathrm{div}\partial_i\psi - \partial_i\big(\psi\cdot\nabla\tilde{\mathbf{u}}\big)\cdot\nabla\partial_i\phi - \partial_i\Big[\Big(\frac{p'(\rho)}{\rho} - \frac{p'(\tilde{\rho})}{\tilde{\rho}}\Big)\nabla\tilde{\rho}\Big]\cdot\nabla\partial_i\phi,\\
I_{4,5} &= \partial_i\big(\rho\mathrm{div}\psi + \mathbf{u}\cdot\nabla\phi\big)\mathrm{div}\partial_i\psi - \partial_i\big(\mathbf{u}\cdot\nabla\psi\big)\cdot\nabla\partial_i\phi - \partial_i\Big(\frac{p'(\rho)}{\rho}\Big)\nabla\phi\cdot\nabla\partial_i\phi\\
&\quad + \partial_i\Big(\frac{\mu}{\rho}\Big)\triangle\psi\cdot\nabla\partial_i\phi + \partial_i\Big(\frac{\mu+\lambda}{\rho}\Big)\nabla\mathrm{div}\psi\cdot\nabla\partial_i\phi + \frac{\mu\nabla\rho\cdot\big(\nabla\partial_i\phi\times\mathrm{curl}\partial_i\psi\big)}{\rho^2}.
\end{aligned}
$$

Here we have used (3.20) and (3.21) with ψ and ϕ being replaced by $\partial_i\psi$ and $\partial_i\phi$, respectively.

Similar to the estimate of $I_{2,5}$, one can prove that

$$
\begin{aligned}
\int_0^T \|I_{4,3}\|_{L^1(\Omega)}\,dt &\le C(\varepsilon+\delta^{\frac12})\int_0^T \big\|\nabla^2\phi\big\|^2 dt + C\int_0^T \big\|\nabla\phi,\nabla^2\psi\big\|^2 dt\\
&\quad + C\varepsilon\sup_{t\in(0,T)}\|\phi\|^2 + C(\varepsilon+\delta^{\frac12}).
\end{aligned}
$$

In addition, it holds that

$$
\begin{aligned}
\int_0^T \|I_{4,4}\|_{L^1(\Omega)}\,dt &\le C\int_0^T\int_\Omega \Big\|\partial_1^2\tilde{u}_1^r, \partial_1^2\tilde{\rho}^r, |\partial_1\tilde{\rho}^r|^2\Big\| \|\phi,\psi\|_{L^\infty(\Omega)} \big\|\nabla^2\psi,\nabla^2\phi\big\|\,dxdt\\
&\quad + C\int_0^T \varepsilon e^{-\alpha t}\|\phi,\psi\| \big\|\nabla^2\psi,\nabla^2\phi\big\|\,dt + C(\varepsilon+\delta)\int_0^T \|\nabla\phi,\nabla\psi\| \big\|\nabla^2\psi,\nabla^2\phi\big\|\,dt\\
&\le C(\varepsilon+\delta+\nu)\int_0^T \big\|\nabla^2\phi\big\|^2 dt + C\delta + C\varepsilon\sup_{t\in(0,T)}\|\phi,\psi\|^2 + C\int_0^T (\|\nabla\phi\|^2 + \|\nabla\psi\|_1^2)dt,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T \|I_{4,5}\|_{L^1(\Omega)}\,dt &\le C\int_0^T \big[\,\big\|\nabla^2\psi\big\|\big(\|\nabla\psi\|_1 + \|\nabla\phi\|_1\big) + \big\|\nabla^2\phi\big\|\big(\|\nabla\psi\| + \|\nabla\phi\|\big)\big]dxdt\\
&\quad + C\int_0^T \|\nabla\psi\|_{L^\infty}\big(\|\nabla\phi\|\big\|\nabla^2\psi\big\| + \|\nabla\psi\|\big\|\nabla^2\phi\big\|\big)dt
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_0^T \|\nabla^2\phi\|\,\|\nabla^2\psi\|_{L^4(\Omega)}\,\|\nabla\phi\|_{L^4(\Omega)}\,dt + C\int_0^T \|\nabla\phi\|_{L^4(\Omega)}^2\,\|\nabla^2\phi\|\,dt\\
&\le \big(\frac{1}{4}+C\nu\big)\int_0^T\int_\Omega \frac{p'(\rho)}{\rho}\left|\nabla^2\phi\right|^2 dxdt + C\int_0^T \big(\,\|\nabla\psi\|_1^2 + \|\nabla\phi\|^2\,\big)dt + C\nu\int_0^T \left\|\nabla^3\psi\right\|^2 dt,
\end{aligned}
$$

where the last inequality follows from similar estimates of (3.30) and the fact that

$$
\|\nabla\phi\|_{L^4(\Omega)}^2\,\|\nabla^2\phi\| \le C\sum_{k=1}^{3}\|\nabla\phi\|^{2-k/2}\left\|\nabla^2\phi\right\|^{1+k/2}.
$$

Collecting the estimates of $I_{4,1}$ to $I_{4,5}$ above and applying Lemmas 3.3–3.5, one can add up $(2\mu+\lambda)\cdot$(3.29) and (3.33) and sum the results with respect to i from 1 to 3 to obtain (3.28). □

Lemma 3.7. *Under the assumptions of Proposition 3.2, there exist $\varepsilon_0>0,\delta_0>0$ and $\nu_0>0$ such that if $\varepsilon<\varepsilon_0,\delta<\delta_0$ and $\nu<\nu_0$, then*

$$
\sup_{t\in(0,T)}\left\|\nabla^2\psi\right\|^2 + \int_0^T \left\|\nabla^3\psi\right\|^2 dt \le C\big(\,\|\phi_0,\psi_0\|_2^2+\varepsilon+\delta^{\frac{1}{3}}\big). \tag{3.34}
$$

Proof. For fixed $i\in\{1,2,3\}$, taking the derivative ∂_i on $\frac{1}{\rho}\cdot(3.2)_2$ and multiplying the result by $\cdot(-\triangle\partial_i\psi)$, one has that

$$
\frac{1}{2}\partial_t\big(\,|\nabla\partial_i\psi|^2\,\big)+\frac{\mu}{\rho}\,|\triangle\partial_i\psi|^2+\frac{\mu+\lambda}{\rho}\,|\nabla\mathrm{div}\partial_i\psi|^2=\mathrm{div}\{\cdots\}+I_{5,1}+I_{5,2}, \tag{3.35}
$$

where $\{\cdots\}=\nabla\partial_i\psi\partial_t\partial_i\psi-\frac{\mu+\lambda}{\rho}\mathrm{div}\partial_i\psi\big(\triangle\partial_i\psi-\nabla\mathrm{div}\partial_i\psi\big)$ and

$$
\begin{aligned}
I_{5,1}&=\triangle\partial_i\psi\cdot\partial_i\Big[\mathbf{u}\cdot\nabla\psi+\psi\cdot\nabla\tilde{\mathbf{u}}+\frac{p'(\rho)}{\rho}\nabla\phi+\Big(\frac{p'(\rho)}{\rho}-\frac{p'(\tilde\rho)}{\tilde\rho}\Big)\nabla\tilde\rho-\frac{\mathbf{f}-\phi\mathbf{g}}{\rho}\\
&\qquad-\frac{2\mu+\lambda}{\rho}\partial_1^2\tilde u_1^r\mathbf{e}_1\Big],\\
I_{5,2}&=\frac{\partial_i\rho}{\rho^2}\left[\mu\triangle\psi+(\mu+\lambda)\nabla\mathrm{div}\psi\right]\cdot\triangle\partial_i\psi-\frac{\mu+\lambda}{\rho^2}\mathrm{div}\partial_i\psi\nabla\rho\cdot\big(\triangle\partial_i\psi-\nabla\mathrm{div}\partial_i\psi\big).
\end{aligned}
$$

Since $\int_\Omega\frac{\mu}{\rho}\,|\partial_i\triangle\psi|^2\,dx\ge a_0\left\|\nabla^2\partial_i\psi\right\|^2$ for some constant $a_0>0$, then

$$
\int_0^T\|I_{5,1}\|_{L^1(\Omega)}\,dt\le C\int_0^T\|\nabla^3\psi\|\,\|\nabla(\psi,\phi)\|_1\,dxdt+C\int_0^T\left\|\nabla^3\psi\right\|\|\nabla(\psi,\phi)\|_{L^4(\Omega)}^2\,dt
$$

$$+ C\int_0^T \|\nabla^3\psi\| \left(\|\mathbf{f}\|_1 + \|\partial_1^2 \tilde{u}_1^r\|_1 + \|\mathbf{g}\|_{W^{1,\infty}} \|\phi\| \right) dt + C\nu \int_0^T \|\nabla^3\psi\|^2 dt$$

$$\le \left(\frac{a_0}{4} + C(\varepsilon+\delta+\nu)\right)\int_0^T \|\nabla^3\psi\|^2 dt + C\int_0^T \|\nabla(\psi,\phi)\|_1^2 dt$$

$$+ C\varepsilon \sup_{t\in(0,T)} \|\phi\|^2 + C(\delta+\varepsilon),$$

where we have used (3.31) and the Holder inequality for $1 = \frac{1}{2} + \frac{k}{8} + \frac{4-k}{8}$. In addition, by using (3.24), one can prove that

$$\int_0^T \|I_{5,2}\|_{L^1(\Omega)} dt \le C\int_0^T \|\nabla^2\psi\| \|\nabla^3\psi\| dt + C\int_0^T \|\nabla\phi\|_{L^\infty} \|\nabla^2\psi\| \|\nabla^3\psi\| dt$$

$$\le \left(\frac{a_0}{4} + C\nu\right)\int_0^T \|\nabla^3\psi\|^2 dt + C\int_0^T \|\nabla^2\psi\|^2 dt.$$

Then integrating (3.35) over $\Omega \times (0,T)$ and summing the results with respect to i from 1 to 3 can lead to the completion of the proof of Lemma 3.7. □

Collecting Lemmas 3.3–3.7, Proposition 3.2 follows directly.

3.2. Proof of Theorem 3.1

The local existence of the solution $(\phi,\psi) \in X(0,T_0)$ to (3.2) and (3.5) can be proved by the standard contraction mapping theorem. Hence, with Proposition 3.2 and (3.4), one can let ε_0 and δ_0 be small such that $C_1(C_0^2\delta_0^2\varepsilon_0^2 + \varepsilon_0 + \delta_0^{\frac{1}{3}}) < \nu_0^2$, then the a priori assumptions (3.7) can be closed. Through a standard continuation argument, one can obtain a global in time solution $(\phi,\psi) \in X(0,+\infty)$. Once the estimate (3.8) with $T = +\infty$ is obtained, (3.6) can be proved by the similar argument as in [33]. Thus, the proof of Theorem 3.1 is completed.

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Appendix A. Proof of Lemma 2.3

Proof. In this proof, we denote $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{T}^3)}$ and $\|\cdot\|_l = \|\cdot\|_{H^l(\mathbb{T}^3)}$ for $l \geq 1$, and let $C > 0$ and $C_j > 0, j = 0, 1, 2, \cdots$ be generic constants, independent of ε and t. By the Galilean transformation, one can assume without loss of generality that $\bar{\mathbf{u}} = 0$.

It is known (see [31]) that if $\varepsilon = \|v_0, \mathbf{w}_0\|_{k+2} > 0$ is small, the problem (1.1) with (2.1) admits a unique global periodic solution $(\rho, \mathbf{u}) \in C\big(0, +\infty; H^{k+2}(\mathbb{T}^3)\big)$, satisfying

$$\sup_{t>0} \|(\rho - \bar{\rho}, \mathbf{u})\|_{W^{k,+\infty}(\mathbb{T}^3)} \leq C \sup_{t>0} \|(\rho - \bar{\rho}, \mathbf{u})\|_{k+2} \leq C\varepsilon.$$

Then it remains to prove the exponential decay rate of the solution. Define $v := \rho - \bar{\rho}, \mathbf{z} := \mathbf{u} - \bar{\mathbf{u}} = \mathbf{u}$ and $\mathbf{w} := \rho\mathbf{u} - \bar{\rho}\bar{\mathbf{u}} = \rho\mathbf{z}$. Due to the conservative form of (1.1), it holds that

$$\int_{\mathbb{T}^3} v(x,t) = 0, \quad \int_{\mathbb{T}^3} \mathbf{w}(x,t)dx = 0, \quad t \geq 0.$$

Thus, it follows from the Poincaré inequalities on $\mathbb{T}^3$ that

$$\begin{aligned} &\|v\|^2 \leq C_0 \|\nabla v\|^2, \\ &\|\mathbf{z}\|^2 \leq C\|\mathbf{w}\|^2 \leq C\|\nabla \mathbf{w}\|^2 \leq C\big(\|\mathbf{z}\|_{L^\infty}^2 \|\nabla v\|^2 + \|\nabla \mathbf{z}\|^2\big) \leq C_1\big(\varepsilon^2 \|\nabla v\|^2 + \|\nabla \mathbf{z}\|^2\big). \end{aligned} \tag{A.1}$$

Note that the perturbations $v = \rho - \bar{\rho}$ and $\mathbf{z} = \mathbf{u} - \bar{\mathbf{u}}$ satisfy the same equations as (3.2), if one replaces $(\phi, \psi), (\tilde{\rho}, \tilde{u})$ by $(v, \mathbf{z}), (\bar{\rho}, \bar{\mathbf{u}})$, respectively and set $h_0 = \mathbf{f} = \mathbf{g} = 0$. Thus, similar to the estimates of Lemmas 3.3–3.7, one can get that

$$E'(t) + C_2\big(\|\nabla v\|_1^2 + \|\nabla \mathbf{z}\|_2^2\big) \leq 0, \tag{A.2}$$

where $E(t)$ is an energy functional satisfying $C^{-1}\|v, \mathbf{z}\|_2^2 \leq E(t) \leq C\|v, \mathbf{z}\|_2^2$. Collecting (A.1) and (A.2), one has that

$$E'(t) + \frac{C_2}{2}\big(\|\nabla v\|_1^2 + \|\nabla \mathbf{z}\|_2^2\big) + \frac{C_2}{2C_0}\|v\|^2 + \frac{C_2}{2C_1}\|\mathbf{z}\|^2 - \frac{C_2\varepsilon^2}{2}\|\nabla v\|^2 \leq 0,$$

which yields that $E'(t) + 2C_3E(t) \leq 0$, if $\varepsilon > 0$ is small. Thus, it holds that $\|v, \mathbf{z}\|_2(t) \leq Ce^{-C_3 t}$. The higher order estimates $\|v, \mathbf{z}\|_{k+2}(t) \leq Ce^{-C_4 t}$ can be proved similarly, which is omitted for convenience. At last, one can finish the proof by using the Sobolev inequality,

$$\|v, \mathbf{z}\|_{W^{k,\infty}(\mathbb{R}^3)} = \|v, \mathbf{z}\|_{W^{k,\infty}(\mathbb{T}^3)} \leq C\|v, \mathbf{z}\|_{k+2}. \quad \square$$

Appendix B. Proof of Lemma 2.4

Proof. The idea is to extract the "well-decay" terms R_i $(i = 1, 2, \cdots)$ from the equations (2.3) and (2.4), where all R_i are the products of periodic functions decaying exponentially fast with respect to t (e.g. $v^\pm, \mathbf{z}^\pm, \tilde{\rho} - \tilde{\rho}^r, \partial_t \mathbf{u}^\pm, \nabla\rho^\pm, \cdots$) and integrable functions with respect to $x_1 \in \mathbb{R}$ (e.g. $\eta(1-\sigma), \sigma - \eta, \partial_t\sigma, \partial_1\eta, \cdots$).

Denote $\varepsilon = \|v_0, \mathbf{z}_0\|_{H^5(\mathbb{T}^3)}$. It follows from (2.2) that

$$\left\|v^\pm, \mathbf{z}^\pm\right\|_{W^{3,+\infty}(\mathbb{R}^3)} \leq C\varepsilon e^{-2\alpha t}.$$

i) For h_0 given by (2.3), first note that

$$\partial_t\tilde{\rho} = \partial_t\rho^-(1-\sigma) + \partial_t\rho^+\sigma + \partial_t\tilde{\rho}^r + R_1, \tag{B.1}$$

where the remainder $R_1 = \left(v^+ - v^-\right)\partial_t\sigma$, satisfying that

$$\|R_1(\cdot,t)\|_{W^{2,p}(\Omega)} \leq C\varepsilon e^{-2\alpha t}\, \|\partial_t\sigma(\cdot,t)\|_{W^{2,p}(\mathbb{R})} \leq C\varepsilon e^{-2\alpha t}.$$

Similarly,

$$\operatorname{div}\left(\tilde{\rho}\tilde{\mathbf{u}}\right) = \operatorname{div}\left(\rho^-\mathbf{u}^-\right)(1-\sigma) + \operatorname{div}\left(\rho^+\mathbf{u}^+\right)\sigma + \partial_1\left(\tilde{\rho}^r\tilde{u}_1^r\right) + R_2, \tag{B.2}$$

where

$$\begin{aligned} R_2 = {} & \operatorname{div}\left(\rho^-\mathbf{u}^+\right)(1-\sigma)\eta + \operatorname{div}\left(\rho^+\mathbf{u}^-\right)\sigma(1-\eta) + (\tilde{\rho} - \tilde{\rho}^r)\partial_1\tilde{u}_1^r + (\tilde{u}_1 - \tilde{u}_1^r)\partial_1\tilde{\rho}^r \\ & + \tilde{\rho}\left(z_1^+ - z_1^-\right)\partial_1\eta + \tilde{u}_1\left(v^+ - v^-\right)\partial_1\sigma - \operatorname{div}\left(\rho^-\mathbf{u}^-\right)(1-\sigma)\eta - \operatorname{div}\left(\rho^+\mathbf{u}^+\right)\sigma(1-\eta), \end{aligned}$$

satisfies that $\|R_2\|_{W^{2,p}(\Omega)} \leq C\varepsilon e^{-\alpha t}$. Collecting (B.1) and (B.2) gives that $h_0 = R_1 + R_2$, which yields (2.5) for h_0.

ii) The estimate of $\mathbf{h}$ can be obtained by decomposing the terms in (2.4) one by one. For brevity, we give only the decomposition of the most difficult term $\nabla p(\tilde{\rho})$. In fact,

$$\begin{aligned} \nabla p(\tilde{\rho}) &= \left(p'(\tilde{\rho}) - p'(\rho^-)\right)\nabla\rho^-(1-\sigma) + \nabla p(\rho^-)(1-\sigma) + \left(p'(\tilde{\rho}) - p'(\rho^+)\right)\nabla\rho^+\sigma \\ &\quad + \nabla p(\rho^+)\sigma + \left(p'(\tilde{\rho}) - p'(\tilde{\rho}^r)\right)\partial_1\tilde{\rho}^r\mathbf{e}_1 + \nabla p(\tilde{\rho}^r) + p'(\tilde{\rho})\left(v^+ - v^-\right)\partial_1\sigma\mathbf{e}_1 \\ &= \nabla p(\rho^-)(1-\sigma) + \nabla p(\rho^+)\sigma + \partial_1 p(\tilde{\rho}^r)\mathbf{e}_1 + R_3, \end{aligned} \tag{B.3}$$

where

$$\begin{aligned} R_3 = {} & \left(\rho^+ - \rho^-\right)\left[a(\rho^-,\tilde{\rho})\nabla\rho^- - a(\rho^+,\tilde{\rho})\nabla\rho^+\right]\sigma(1-\sigma) + a(\tilde{\rho}^r,\tilde{\rho})(\tilde{\rho} - \tilde{\rho}^r)\partial_1\tilde{\rho}^r\mathbf{e}_1 \\ & + p'(\tilde{\rho})\left(v^+ - v^-\right)\partial_1\sigma\mathbf{e}_1 \qquad \text{with } a(u,v) := \int_0^1 p''(u + \theta(v-u))d\theta, \end{aligned}$$

which satisfies that $\|R_3\|_{W^{1,p}(\Omega)} \leq C\varepsilon e^{-\alpha t}$. $\square$

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