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On singular homology theories of digraphs and quivers*

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Singular cubical homology theory can be constructed for different categories of digraphs and quivers based on two types of digraph cubes. Furthermore, there exist various categories of digraphs and quivers that share the same set of objects but have different sets of morphisms. Additionally, there are different definitions of homotopy, leading to the existence of several homotopy categories in graph theory. Similarly, singular simplicial homology theories can be defined on various categories (homotopy categories) of digraphs and quivers.

This paper systematically explores the relationships between singular homology theories for different categories of digraphs and quivers. We also introduce new notions of homology for these categories and describe the functorial and homotopy properties of them.

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1. Introduction

At the present time, the development of new methods from algebraic topology in graph theory is an active topic of research (see, for example, [3, 5, 6, 18, 19, 20, 21, 22, 23, 25, 30, 32, 37, 31, 40] connection with discrete approaches

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to mathematical physics [4, 11, 12, 14, 16] and non-commutative differential geometry [7, 8, 9, 13, 17, 35, 44].

There are two fundamental approaches to construct homology theory in a category of digraph or quivers. The first is the path homology theory created and developed by Grigor'yan, Lin, Muranov, and Yau in the series of seminal works [19, 21, 22]. This theory, known as GLMY-theory in honor of the authors involved [1, 10, 26, 30, 42], possesses functoriality, homotopy invariance, and a variety of desirable properties similar to the classical homology theory of topological spaces.

The singular cubical homology theory in the category of digraphs and quivers was constructed in [18] based on singular cubes defined using digraph maps $f\colon I^n \to G$, where G is a digraph (or quiver) and I^n is the n-fold box product of the digraph $I = (0 \to 1)$. A *transitive* singular cubical homology theory arises naturally in the category of transitive digraphs and discrete topological spaces [2, 37]. This homology theory is based on singular cubes given by maps from the *transitive digraph cube* $\widehat{I^n}$ to a digraph, where the transitive cube $\widehat{I^n}$ is obtained by adding directed edges to the digraph cube I^n. The singular cubic homology theory coincides with the path homology theory in the category of cubical digraphs [23] but is different in general [18, Pr. 13, Pr. 14]. In the category of transitive digraphs the path homology theory, the singular cubical homology theory, and the transitive singular homology theory coincide [37].

In contrast to classical algebraic topology, there are two natural definitions of a digraph cube and many possible choices of graph theoretic morphism and homotopy. Therefore, we obtain many categories (homotopy categories) of digraphs or quivers in which we can define different singular cubical homology theories. Similarly, the same choices are available when defining the singular simplicial homology theories of a digraph or quiver.

In this work, we prove that the homotopy invariance of a singular cubical homology depends on the choice of cube and homotopy under consideration. We define several different categories and homotopy categories of digraphs and quivers and describe the relations between them. With the appropriate digraph cube constructions, we are then able to describe singular cubical homology theories corresponding to the underlying categories. Likewise, by defining the digraph n-simplex, we can describe the singular simplicial homology theories in our chosen categories of digraphs or quivers in the usual way [41].

We note that the homology theories considered in papers [18, 33, 41], and [37] are defined in the categories of digraphs or quivers that admit "maps sending arrows to verities" (see Definitions 2.2 and 2.7 below). In the present paper, we construct also homology theories that are defined in the categories of digraphs or quivers that only admit “maps sending arrows to arrows.”

The paper is organized as follows: The basic categorical construction used throughout the rest of the paper are introduced in Section 2. In Section 3, we introduce several notions of homotopy and describe the relationships between their homotopy categories of digraphs and quivers. In particular, we introduce a new notion of *strong homotopy* and its corresponding homotopy category. In Section 4, we define the digraph simplexes and digraph cubes used in the definitions of the singular homology theories set out in the next section. We describe the relationships between these fundamental objects through the construction of natural digraph maps arising among the digraph simplexes and digraph cubes. This leads in Section 5 to the definition of a collection of singular homology theories in categories described in Section 2 based on the singular cubes and simplexes defined in Section 4. We describe functorial and homotopy properties of these theories and the relationships between them. We also give examples to demonstrate the non-equivalence of the distinct homology theories introduced in this Section. We explain briefly in Section 6 some further constructions of homotopy categories of digraph and quivers and their associated singular homology theories. In particular, we define non-trivial homology theories based on singular cubes (simplexes) given by inclusions.

2. Categories of digraphs and quivers

In this section we give basic definitions and introduce the categories of digraphs and quivers which we consider in the paper (see [24, 21, 28, 33, 34, 41]).

Definition 2.1. *A digraph* $G = (V_G, E_G)$ consists of a set V_G of *vertices* and a subset $E_G \subset \{V_G \times V_G\}$ of ordered pairs (v, w) of vertices called *arrows.* The arrow (v, w) is denoted $v \to w$. An arrow $v \to v$ is called a *loop.* The vertex $v = \text{orig}\,(v \to w)$ is called the *origin of the arrow* and the vertex $w = \text{end}(v \to w)$ is called the *end of the arrow.* A digraph is *simple* if it has no loops.

A digraph G is finite if it contains a finite number of vertices. We say that a digraph H is a *subgraph* of a digraph G and write $H \subset G$ if $V_H \subset V_G$ and $E_H \subset E_G$.

Definition 2.2. Let G and H be two digraphs. A *digraph map* $f\colon G \to H$ is given by a pair of maps $f_V\colon V_G \to V_H$ and $f_E\colon E_G \to E_H \cup V_H$ such that for every $(v \to w) \in E_G$ one of the following two conditions is satisfied:

(1) $f_E(v \to w) = (f_V(v) \to f_V(w)) \in E_H$,

(2) $f_E(v \to w) = f_V(v) = f_V(w) \in V_H$.

The map f of digraphs is a *homomorphism* if for every arrow $(v \to w) \in E_G$ condition (1) is satisfied. In particular, in this case $f_E\colon E_G \to E_H$. Two digraphs G and G' are *isomorphic* if there exist homomorphisms $f\colon G \to G'$ and $g\colon G' \to G$ such that

$$f \circ g = \mathrm{Id}_{G'}, \qquad g \circ f = \mathrm{Id}_G\,.$$

Remark 2.3. *The difference between a digraph map and a digraph homomorphism is: a digraph homomorphism can not map an arrow to a vertex but digraph map can. In the following two examples, the first one is a digraph map but not a digraph homomorphism, the second one is a digraph homomorphism.*

Example 2.4. *The digraph map f in Figure 1 is given on the set of vertices by $f(0) = f(2) = 0$, $f(1) = f(3) = 1$.*

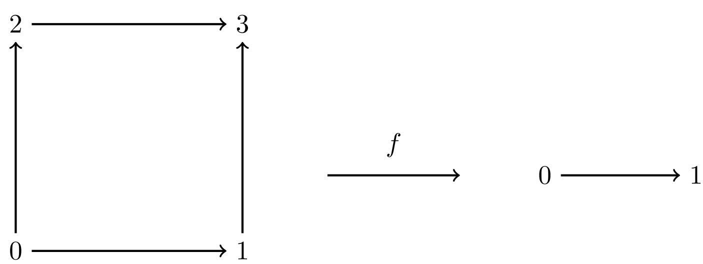

Figure 1: The digraph map f.

The digraph map f is not a digraph homomorphism since it maps the arrows $0 \to 2, 1 \to 3$ to the vertices 0 and 1 respectively.

Example 2.5. The digraph map g in Figure 2 is given on the set of vertices by $g(0) = 0$, $g(1) = g(2) = 1$, $g(3) = 2$. Here g is a digraph homomorphism.

In this article, we often write f instead of f_V or f_E if it clear from the context which map is under consideration. Let $\mathbf{D}$ be the category of finite digraphs and digraph maps and $\mathbb{D}$ be the category of finite digraphs and homomorphisms. The categories $\mathbf{D}$ and $\mathbb{D}$ have the same objects and there is the natural inclusion of categories $\mathbb{D} \subset \mathbf{D}$.

Definition 2.6. A *quiver* $Q = (V_Q, E_Q, s, t)$ consists of a set of *vertices* V_Q, a set of *arrows* E_Q, and two maps $s, t\colon E_Q \to V_Q$. For an arrow $a \in E_Q$, the vertex $s(a) \in V_Q$ is called the *origin vertex* of a, and the vertex $t(a)$ is called the *end vertex* of a. An arrow $a \in E_Q$ is called a *loop* if $s(a) = t(a)$.

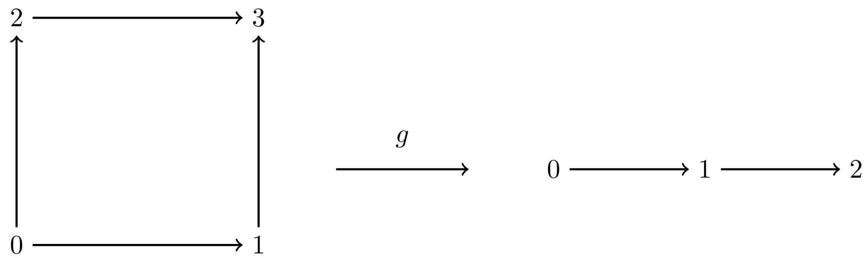

Figure 2: The digraph homomorphism g.

We say that a quiver $Q' = (V', E', s', t')$ is a *subquiver* of a quiver $Q = (V, E, s, t)$ and write $Q' \subset Q$ if $V' \subset V$, $E' \subset E$, and the maps s', t' are the restrictions of the maps s, t, respectively.

Definition 2.7. Let $Q = (V, E, s, t)$ and $Q' = (V', E', s', t')$ be two quivers. A *quiver map* $f : Q \to Q'$ is given by a pair of maps $f_V \colon V \to V'$ and $f_E \colon E \to E' \cup V'$ such that for every $a \in E$ one of the following two conditions is satisfied:
(1) $f_E(a) \in E'$ and $f_V(s(a)) = s'(f_E(a))$, $f_V(t(a)) = t'(f_E(a))$,
(2) $f_E(a) = v' \in V'$ and $f_V(s(a)) = f_V(t(a)) = v'$.

The map $f = (f_V, f_E)$ of quivers is a *homomorphism* if for every arrow $a \in E$ condition (1) is satisfied. In particular, in this case $f_E \colon E \to E'$.

Two quivers Q and Q' are *isomorphic* if there exist homomorphisms $f \colon Q \to Q'$ and $g \colon Q' \to Q$ such that

$$f \circ g = \mathrm{Id}_{Q'}, \qquad g \circ f = \mathrm{Id}_Q.$$

In what follows, we often write f instead of f_V or f_E if it clear from the context which map is under consideration. Let $\mathbf{Q}$ be the category of quivers and quiver maps and $\mathbb{Q}$ be the category of quivers and homomorphisms.

Every digraph $G = (V_G, E_G)$ defines a quiver $Q = \mathbf{I}(G) = (V_Q, E_Q, s, t)$ where $V_Q = V_G$, $E_Q = E_G$ and, for $(v \to w) \in E_Q$, we have $s(v \to w) = \mathrm{orig}(v \to w) = v$, $t(v \to w) = \mathrm{end}(v \to w) = w$. Similarly, every digraph map $f = (f_V, f_E) \colon G \to G'$, in a natural way, defines a quiver map $\mathbf{I}(f) = (f_V, f_E) \colon \mathbf{I}(G) \to \mathbf{I}(G')$.

Proposition 2.8. *The maps* $\mathbf{I}$ *define a functor* $\mathbf{D} \to \mathbf{Q}$ *which we continue to denote* $\mathbf{I}$*. The restriction of* $\mathbf{I}$ *to the subcategory* $\mathbb{D} \subset \mathbf{D}$ *defines a functor* $\mathbb{I} \colon \mathbb{D} \to \mathbb{Q}$.

Using Proposition 2.8 we can identify the corresponding categories of digraphs as the subcategories of quivers. Thus we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Q} & \subset & \mathbf{Q} \\ \cup & & \cup \\ \mathbb{D} & \subset & \mathbf{D}. \end{array} \tag{2.1}$$

3. Homotopy theories

In this section, we recall the basic notions of homotopy theory for quivers and digraphs in the categories $\mathbf{D} \subset \mathbf{Q}$ [21, 33, 41]. Following this, we introduce a new notion of homotopy, which we call *h-homotopy*, that arises naturally in the categories $\mathbb{D} \subset \mathbb{Q}$ in diagram (2.1). We then give an example which shows the difference between homotopy and h-homotopy.

Furthermore, we define the *strong product* of digraphs and the strong product of a line digraph with a quiver. Using these definitions we introduce a new notion *s-homotopy* which can be used for all categories in diagram (2.1). Similarly to the h-homotopy we define the *sh-homotopy* which also arises naturally from the strong product in the categories $\mathbb{D} \subset \mathbb{Q}$.

Let $I_0 = \{0\}$ be the one-vertex digraph. For $n \in \mathbb{N}$, consider a digraph $I_n = (V_{I_n}, E_{I_n})$ with $V_{I_n} = \{0, 1, \dots, n\}$ and with a set of arrows E_{I_n} consisting of exactly one arrow between any two consecutive vertices i and $i+1$ and no other arrows. A digraph I_n is called a *line* digraph. A line digraph I_n is called *direct* if all its arrows have the form of $i \to i+1$. Let $I = (0 \to 1)$ be the direct line digraph with one arrow.

Definition 3.1. Let G and H be two digraphs. We define the *Box product* digraph $G \Box H = \Pi = (V_\Pi, E_\Pi)$ by setting $V_\Pi = V_G \times V_H$ and $[(x, y) \to (x', y')] \in E_\Pi$, for $x, x' \in V_G$ and $y, y' \in V_H$, if $x = x'$ and $y \to y'$, or $x \to x'$ and $y = y'$.

Definition 3.2. Let $Q = (V, E, s, t)$ be a quiver and $I_n = (V_{I_n}, E_{I_n})$ be a line digraph. We define the *Box product* quiver $I_n \Box Q = \Pi = (V_\Pi, E_\Pi, s_\Pi, t_\Pi)$ by setting $V_\Pi = V_{I_n} \times V_Q$,

$$E_\Pi = \{(v, a) | v \in V_{I_n}, a \in E\} \cup \{[(v, w) \to (v', w)] \,|\, (v \to v') \in E_{I_n}, w \in V\},$$

and the maps s_Π, t_Π are given as follows: $s_\Pi(i, a) = (i, s(a))$, $t_\Pi(i, a) = (i, t(a))$ for $i \in V_{I_n}, a \in E$ and $s_\Pi[(i, w) \to (i', w)] = (i, w)$, $t_\Pi[(i, w) \to (i', w)] = (i', w)$ for $(i \to i') \in E_{I_n}, w \in V$.

Remark 3.3. For any quiver $Q = (V, E, s, t)$ and any line digraph I_n, we have two natural inclusions $j_0, j_n \colon Q \to I_n \Box Q$ where $[j_0]_V(v) = (0, v), [j_n]_V(v) =$

(n, v), $[j_0]_E(a) = (0, a), [j_n]_E(a) = (n, a)$ for $v \in V, a \in E$. Thus the sub-quiver $0\Box Q \subset I_n\Box Q$ is identified with Q and called the *bottom boundary* of $I_n\Box Q$. Similarly the sub-quiver $n\Box Q \subset I_n\Box Q$ is identified with Q and it is called the *top boundary* of $I_n\Box Q$. For a digraph G, the natural inclusions $j_0, j_n : G \to I_n\Box G$ are defined similarly.

Definition 3.4. i) Two quiver maps $f, g : Q \to Q'$ are called *homotopic* if there exists a line digraph I_n $(n \geq 0)$ and a quiver map

$$F : I_n\Box Q \to Q' \tag{3.1}$$

such that

$$F|_{0\Box Q} = f \quad \text{and} \quad F|_{n\Box Q} = g. \tag{3.2}$$

The map F is called a *homotopy* between f and g and we write $f \simeq g$. In the case $n = 1$ the map F is called a *one-step homotopy.*

ii) Two quiver homomorphisms $f, g : Q \to Q'$ are called *h-homotopic* if there exists a line digraph I_n $(n \geq 0)$ and a quiver homomorphism (3.1) which satisfies conditions (3.2). In this case, the map F is called an *h-homotopy* between homomorphisms f and g and we write $f \doteq g$. In the case $n = 1$ the map F is called a *one-step h-homotopy.*

Definition 3.5. i) Quivers Q and Q' are called *homotopy equivalent* if there exist quiver maps

$$f : Q \to Q', \quad g : Q' \to Q \tag{3.3}$$

such that

$$f \circ g \simeq \mathrm{Id}_{Q'}, \qquad g \circ f \simeq \mathrm{Id}_Q. \tag{3.4}$$

In this case we write $Q \simeq Q'$ and the maps f and g are called *homotopy inverses* of each other.

ii) Quivers Q and Q' are called *h-homotopy equivalent* if there exist homomorphisms (3.3) such that

$$f \circ g \doteq \mathrm{Id}_{Q'}, \qquad g \circ f \doteq \mathrm{Id}_Q. \tag{3.5}$$

In this case we write $Q \doteq Q'$ and the maps f and g are called *h-homotopy inverses* of each other.

Note that by Proposition 2.8 we can apply the notions from Definitions 3.4 and 3.5 to digraphs. Examples of homotopy equivalent digraphs are given in [21, §3.3].

Proposition 3.6. *i) The relation "$\simeq$" is an equivalence relation on the set of quiver (digraph) maps and this relation is an equivalence relation on the set of quivers (digraphs). The relation "$\doteq$" is an equivalence relation on the set of quiver (digraph) homomorphisms and it is an equivalence relation on the set of quivers (digraphs).*

ii) All quivers (digraphs) with homotopy classes of quiver (digraph) maps form the homotopy category of quivers (digraphs) which is denoted by **HQ** *(***HD***). All quivers (digraphs) with h-homotopy classes of homomorphisms form the h-homotopy category of quivers (digraphs) which is denoted by* $\mathbb{HQ}$ *(*$\mathbb{HD}$*).*

A *directed walk in a quiver* $Q = (V, E, s, t)$ is an alternating sequence of vertices and arrows $v_0, a_0, v_1, a_1, \dots, v_n$ where $v_i \in V$, $a_i \in E$ such that for any pair (v_i, v_{i+1}) we have $s(a_i) = v_i, t(a_i) = v_{i+1}$. The integer n is called the *length* of the walk. A directed walk is *closed* if $v_0 = v_n$. A directed walk and a closed directed walk in a digraph are defined similarly. Let $G = (V, E)$ be a simple digraph. We note that any directed walk in G defines an *allowed path* $v_0 \to v_1 \to \cdots \to v_n$ (see [18, 20, 21, 23]) with $v_i \in V, (v_i \to v_{i+1}) \in E$. Let $f\colon Q \to Q'$ be a homomorphism of quivers. Then the image $f_V(v_0), f_E(a_0), f_V(v_1), f_E(a_1), \dots, f_V(v_n)$ of the directed walk $v_0, a_0, v_1, a_1, \dots, v_n$ is a directed walk and the image of a closed walk is a closed walk. Any homomorphism preserves the length of the walk.

Lemma 3.7. *Let Q be a quiver that does not a closed walk and $F\colon I_n\Box Q \to Q$ be an h-homotopy which is the identity homomorphism on the bottom boundary $0\Box Q$. Then $I_n = I_0$.*

Proof. It is sufficient to consider the case of a one-step h-homotopy F. Since Q is a finite quiver without closed walks there exists a directed walk of a maximal length n in Q. Consider this walk $v_0, a_0, v_1, a_1, \dots, a_{n-1}, v_n$ and the directed walk

$$(0, v_0), (0, a_0), (0, v_1), (0, a_1), \dots, (0, a_{n-1}), (0, v_n), (0 \to 1, v_n), (1, v_n)$$

in $I_1\Box Q$ of length $n + 1$. The image of this walk in Q under homomorphism F has length $n+1$ and we obtain a contradiction with the condition that the maximal length of a walk in Q is n. $\square$

Theorem 3.8. *Let Q and Q' be two quivers which do not have closed walks. Then Q and Q' are h-homotopy equivalent iff Q is isomorphic to Q'.*

Proof. It follows from Definitions 2.7, 3.4, and 3.5 that isomorphic quivers are h-homotopy equivalent. Now let $f\colon Q \to Q'$, $g\colon Q' \to Q$ be homomorphisms which are h-homotopy inverses of each other. Then $g \circ f \cong \mathrm{Id}_Q$ and, there is an h-homotopy $F\colon I_n \Box Q \to Q$ which is the identity map on the bottom boundary and which equals $g \circ f$ on the top boundary. By Lemma 3.7, $I_n = I_0$ and, hence, $g \circ f = \mathrm{Id}_Q$. A similar argument shows $f \circ g = \mathrm{Id}_{Q'}$, which completes the proof. □

Example 3.9. i) For $n, k \geq 0$, any two line digraphs I_n and I_k are homotopy equivalent to the one vertex digraph [21, Corollary 3.8]. However the directed line digraphs I_n and I_k are h-homotopy equivalent if and only if $n = k$, since for $n > k$ there are no homomorphisms from I_n to I_k.

ii) We now give a non-trivial example in which an h-homotopy is realized. Consider the digraph $G = (V, E)$ below

$$0 \ \underset{b}{\overset{a}{\rightleftarrows}} \ 1 \ \underset{d}{\overset{c}{\rightleftarrows}} \ 2$$

with $V = \{0, 1, 2\}$, $E = \{a, b, c, d\}$ and let $G' = (V', E')$ be the digraph

$$0' \ \underset{b'}{\overset{a'}{\rightleftarrows}} \ 1',$$

with $V' = \{0', 1'\}$, $E' = \{a', b'\}$.

Let $f\colon G \to G'$ be the homomorphism defined on the set of vertices by $f_V(0) = f_V(2) = 0', f_V(1) = 1'$, and let $g\colon G' \to G$ be the homomorphism defined on the set of vertices by $g_V(0') = 1, g_V(1') = 0$. Then $h = f \circ g\colon G' \to G'$ is an isomorphism which is given on the set of vertices by $h_V(0') = 1', h_V(1') = 0'$. Consider the digraph $I \Box G'$

$$\begin{array}{ccc} (1,0') & \underset{b'_1}{\overset{a'_1}{\rightleftarrows}} & (1,1') \\ \uparrow & & \uparrow \\ (0,0') & \underset{b'_0}{\overset{a'_0}{\rightleftarrows}} & (0,1'). \end{array}$$

Recall that we can identify bottom and top boundaries of $I \Box G'$ with G'. We define the h-homotopy $F\colon I \Box G' \to G'$ between $\mathrm{Id}_{G'}$ and h on the set of vertices by $F_V(1,0') = F_V(0,1') = 1', F_V(1,1') = F_V(0,0') = 0'$. We have $F|_{0\Box G'} = \mathrm{Id}_{0\Box G'}$ and $F|_{1\Box G'}$ coincides with h. Hence $f \circ g \doteq \mathrm{Id}_{G'}$.

The composition $k = g \circ f\colon G \to G$ is given on the set of vertices by the map $k_V(0) = k_V(2) = 1, k_V(1) = 0$. The digraph $I\Box G$ has the following form

$$\begin{array}{ccccc} (1,0) & \overset{a_1}{\underset{b_1}{\rightleftarrows}} & (1,1) & \overset{c_1}{\underset{d_1}{\rightleftarrows}} & (1,2) \\ \uparrow & & \uparrow & & \uparrow \\ (0,0) & \overset{a_0}{\underset{b_0}{\rightleftarrows}} & (0,1) & \overset{c_0}{\underset{d_0}{\rightleftarrows}} & (0,2). \end{array}$$

Define the h-homotopy $F\colon I\Box G \to G$ on the set of vertices by

$$F_V(1,2) = F_V(1,0) = F_V(0,1) = 1, F_V(1,1) = F_V(0,0) = 0, F_V(0,2) = 2.$$

The identification $0\Box G$ with G implies that $F|_{0\Box G} = \mathrm{Id}_{0\Box G}$, and the the identifications $1\Box G$ with G implies that $F|_{1\Box G}$ coincides with $g \circ f$. Hence $g \circ f \doteq \mathrm{Id}_G$. That is digraphs G and G' are h-homotopy equivalent.

Now we define the notion of *strong product* of digraphs, and the strong product of a line digraph and a quiver. Using strong products, we then introduce another notion of homotopy which we call *s-homotopy*. This homotopy is well defined for all categories in diagram (2.1). Finally, we define the notion *sh-homotopy* which arises naturally in the categories $\mathbb{D} \subset \mathbb{Q}$.

Definition 3.10. Let G and H be two digraphs. We define the *strong product digraph* $S = G \boxtimes H = (V_S, E_S)$ by setting $V_S = V_G \times V_H$ and, for $x, x' \in V_G$ and $y, y' \in V_H$, there is an arrow $[(x,y) \to (x',y')] \in E_S$ if and only if either $x = x'$ and $(y \to y') \in E_H$, or $(x \to x') \in E_G$ and $y = y'$, or $(x \to x') \in E_G$ and $(y \to y') \in E_H$.

Definition 3.11. Let $Q = (V, E, s, t)$ be a quiver and $I_n = (V_{I_n}, E_{I_n})$ be a line digraph. We define the *strong product* $\Psi = I_n \boxtimes Q = (V_\Psi, E_\Psi, s_\Psi, t_\Psi)$ *of the quiver* Q *and the line digraph* I_n by setting $V_\Psi = V_{I_n} \times V$ and E_Ψ as the union

$$\begin{aligned} \{(i,a) | i \in V_{I_n}, a \in E\} \cup \{(a,v) | a \in E_{I_n}, v \in V\} \\ \cup \{(i,a) \to (j,a) | (i \to j) \in E_{i_n}, a \in E\}. \end{aligned}$$

The maps s_Ψ, t_Ψ are given as follows: $s_\Psi(i,a) = (i, s(a))$ and $t_\Psi(i,a) = (i, t(a))$ for $a \in E$ and $i \in V_{I_n}$, $s_\Psi(a,v) = (i,v)$ and $t_\Psi(a,v) = (j,v)$ for $a = (i \to j) \in E_{I_n}$ and $v \in V$, $s_\Psi[(i,a) \to (j,a)] = (i, s(a))$ and $t_\Psi[(i,a) \to (j,a)] = (j, t(a))$ for $(i \to j) \in E_{I_n}$ and $a \in E$.

Remark 3.12. For a quiver Q, similarly to Remark 3.3 we have two natural inclusions of the bottom and top boundaries $j_0, j_n \colon Q \to I_n \boxtimes Q$ where $[j_0]_V(v) = (0,v), [j_n]_V(v) = (n,v), [j_0]_E(a) = (0,a), [j_n]_E(a) = (n,a)$ for $v \in V, a \in E$. Inclusions $j_0, j_n \colon G \to I_n \boxtimes G$ can also be defined for a digraph G.

Definition 3.13. i) Two quiver (digraph) maps $f, g \colon Q \to Q'$ are called *one-step s-homotopic* if there exists a quiver (digraph) map

$$F \colon I \boxtimes Q \to Q' \tag{3.6}$$

such that

$$F|_{0 \boxtimes Q} = f,\ F|_{1\boxtimes Q} = g \text{ or } F|_{0\boxtimes Q} = g,\ F|_{1\boxtimes Q} = f. \tag{3.7}$$

In this case we write $f \overset{1}{\cong} g$. Two quiver (digraph) maps $f, g \colon Q \to Q'$ are called *s-homotopic* if there exists a sequence of quiver (digraph) maps f_i $(0 \leq i \leq n)$ such that

$$f = f_0 \overset{1}{\cong} f_1 \overset{1}{\cong} \dots \overset{1}{\cong} f_n = g. \tag{3.8}$$

In this case we write $f \cong g$.

ii) Two quiver (digraph) homomorphisms $f, g \colon Q \to Q'$ are called *sh-homotopic* if they are s-homotopic by means a sequence of s-homotopies in (3.8) such that all homotopies $F \colon I \boxtimes Q \to Q'$ are given by homomorphisms. In this case we write $f \approx g$.

Definition 3.14. i) Quivers (digraphs) Q and Q' are called *s-homotopy equivalent* if there exist quiver (digraph) maps

$$f \colon Q \to Q', \quad g \colon Q' \to Q \tag{3.9}$$

such that

$$f \circ g \cong \mathrm{Id}_{Q'}, \qquad g \circ f \cong \mathrm{Id}_Q. \tag{3.10}$$

In this case we write $Q \cong Q'$ and the maps f and g are called *s-homotopy inverses*of each other.

ii) Quivers (digraphs) Q and Q' are called *sh-homotopy equivalent* if there exist homomorphisms (3.9) such that

$$f \circ g \approx \mathrm{Id}_{Q'}, \qquad g \circ f \approx \mathrm{Id}_Q. \tag{3.11}$$

In this case we write $Q \approx Q'$ and the maps f and g are called *sh-homotopy inverses* of each other.

Proposition 3.15. *i) The relation " $\cong$ " is an equivalence relation on the set of quiver (digraph) maps and it is an equivalence relation on the set of quivers (digraphs). The relation " $\approx$ " is an equivalence relation on the set of quiver (digraph) homomorphisms and it is an equivalence relation on the set of quivers (digraphs).*

ii) All quivers (digraphs) with s-homotopy classes of quiver (digraph) maps form the s-homotopy category of quivers (digraphs) which is denoted by $\mathbf{H}^s\mathbf{Q}$ *(*$\mathbf{H}^s\mathbf{D}$*). All quivers (digraphs) with sh-homotopy classes of homomorphisms form the sh-homotopy category of quivers (digraphs) which is denoted by* $\mathbb{H}^s\mathbb{Q}$ *(*$\mathbb{H}^s\mathbb{D}$*).*

Proposition 3.16. *The relations between homotopy categories are given by the following commutative diagram of categories and functors in which all arrows are given by the natural inclusions*

$$\begin{array}{ccccccc} \mathbb{H}^s\mathbb{D} & \longrightarrow & \mathbb{H}\mathbb{D} & \longrightarrow & \mathbf{HD} & \longleftarrow & \mathbf{H}^s\mathbf{D} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^s\mathbb{Q} & \longrightarrow & \mathbb{H}\mathbb{Q} & \longrightarrow & \mathbf{HQ} & \longleftarrow & \mathbf{H}^s\mathbf{Q}. \end{array} \tag{3.12}$$

Proof. It follows from the definitions that any s-homotopy between two quiver (digraph) maps gives a homotopy between these maps. Similarly, any sh-homotopy between two quiver (digraph) homomorphisms gives an h-homotopy between these homomorphisms. ☐

Now we give several examples which illustrate the difference between the categories of Proposition 3.16.

Example 3.17. i) Consider the digraph $G = (V_G, E_G)$

$$\begin{array}{ccccc} & & 2 & & \\ & {}^{c}\swarrow & & \nwarrow^{b} & \\ 0 & & \overset{a}{\to} & & 1. \end{array} \tag{3.13}$$

where $V_G = \{0, 1, 2\}$ and $E_G = \{a, b, c\}$. Let $f\colon G \to G$ be the homomorphism given on the set of vertices by $f_V(0) = 1$, $f_V(1) = 2$, $f_V(2) = 0$. Then f is one-step h-homotopic to the identity homomorphism, that is $f \doteq \mathrm{Id}_G$. The h-homotopy $F\colon I\Box G \to G$ is given on the set of vertices

$$V_{I\Box G} = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$

by the map

$$F_V(0,i) = i \, (0 \leq i \leq 3), \ F_V(1,0) = 1, F_V(1,1) = 2, F_V(1,2) = 0. \tag{3.14}$$

However, the homomorphism f is not sh-homotopic to the identity homomorphism Id_G. This fact can be deduced using the following trivial remark. In the digraph $I \boxtimes G$ there are two different directed walks from the vertex $(0,0)$ to $(1,1)$: the directed walk of length one $(0,0) \to (1,1)$ and the directed walk of length two $(0,0) \to (0,1) \to (1,1)$. Any sh-homotopy is a homomorphism and, hence, it preserves the length of any directed walk. However, in the digraph G there are no two vertices which satisfy this condition.

ii) Consider digraphs $I = (0 \to 1)$ and $I_2 = (0 \to 1 \to 2)$ and the map $f\colon I \to I_2$ given on the set of vertices by $f_V(i) = i$ for $i = 0, 1$, and the map $g\colon I \to I_2$ given on the set of vertices by $g_V(i) = i + 1$ for $i = 0, 1$. The maps f and g are homotopic. The one-step homotopy is given by the map

$$\begin{array}{ccc} (1,0) & \to & (1,1) \\ \uparrow & & \uparrow \\ (0,0) & \to & (0,1) \end{array} \xrightarrow{F} I_2$$

where $F_V(0,0) = 0, F_V(0,1) = F_V(1,0) = 1, F_V(1,1) = 2$. The same argument used in part i) of the example proves that the maps f and g are not s-homotopic.

iii) Now we give a non-trivial example of an sh-homotopy between two homomorphisms. Let G be the digraph in (3.13). Consider the complete digraph $H = (V_H, E_H)$ with three vertices $V_H = \{0, 1, 2\}$ given below

$$\begin{array}{ccccc} & & 2 & & \\ & \swarrow\nearrow & & \searrow\nwarrow & \\ 0 & & \leftrightarrows & & 1. \end{array}$$

Let $f\colon G \to H$ be the homomorphism given on the set of vertices by $f_V(0) = 0, f_V(1) = 1, f_V(2) = 2$ and $g\colon G \to H$ be the homomorphism given on the set of vertices by $g_V(0) = 1, g_V(1) = 2, g_V(2) = 0$. Then $f \approx g$. The sh-homotopy $F\colon I \boxtimes G \to H$ is given on the set of vertices in (3.14).

4. Digraph cubes and digraph simplexes

In this section we define digraph simplexes and two types of digraph cube which are necessary for defining singular homology on categories of quivers and digraphs previously introduced in [18, 21, 33, 41, 37]. We also describe natural maps between the objects introduced.

For $n \geq 0$, we define the *n-simplex digraph* Δ^n by setting $\Delta^0 = \{0\}$ and, for $n \geq 1$, $\Delta^n = (V_{\Delta^n}, E_{\Delta^n})$ where

$$V_{\Delta^n} = \{0, 1, \dots, n\},\ E_{\Delta^n} = \{i \to j | i, j \in V_{\Delta^n}, i < j\}.$$

Let $I = (0 \to 1)$ be the line digraph. For $n \geq 0$, we define *n-cube digraph* I^n by setting $I^0 = \{0\}$ and

$$I^n = \underbrace{I \Box I \Box I \Box \dots \Box I}_{n \text{ times}} \quad \text{for } n \geq 1.$$

For $n \geq 1$, the set of vertices of I^n consists of all 2^n binary sequences $(a_1, \dots, a_n)$, where $a_i = 0, 1$. There is an arrow $(a_1, \dots, a_n) \to (b_1, \dots, b_n)$ if and only if $(b_1, \dots, b_n)$ is obtained from $a = (a_1, \dots, a_n)$ by replacing a digit 0 by 1 at exactly one position. The set of vertices V_{I^n} is *partially ordered set* with order given by the relation: for any two vertices $(a_1, \dots, a_n), (b_1, \dots, b_n) \in V_{I^n}$ we have $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ iff $a_i \leq b_i$ for every $i = 1, 2, \dots, n$. We write $(a_1, \dots, a_n) < (b_1, \dots, b_n)$ if $(a_1, \dots, a_n) \leq (b_1, \dots, b_n)$ and $(a_1, \dots, a_n) \neq (b_1, \dots, b_n)$.

A digraph $G = (V_G, E_G)$ without loops is *transitive* if for any three different vertices $v, w, z \in V_G$ and arrows $(v \to w), (w \to z) \in E_G$ there is the arrow $(v \to z) \in E_G$.

Definition 4.1. For $n \geq 0$, define the *transitive cube digraph* $\widehat{I}^n = (V_{\widehat{I}^n}, E_{\widehat{I}^n})$ where $V_{\widehat{I}^n} = V_{I^n}$ and there is an arrow $[(a_1, \dots, a_n) \to (b_1, \dots, b_n)] \in E_{\widehat{I}^n}$ if and only if $(a_1, \dots, a_n) < (b_1, \dots, b_n)$.

We note that the strong product of digraphs is associative and hence for the digraph I and $n \geq 1$ the n-power $\underbrace{I \boxtimes I \boxtimes \dots \boxtimes I}_{n\text{-times}}$ is well defined [43].

Proposition 4.2. *For $n \geq 1$, we have*

$$\widehat{I}^n = \underbrace{I \boxtimes I \boxtimes \dots \boxtimes I}_{n\text{-times}}$$

Proof. By induction on n. □

Now we describe several natural maps between the digraphs defined above. For $n \geq 0$, we have the inclusion homomorphism

$$\mathcal{I} \colon I^n \subset \widehat{I}^n \tag{4.1}$$

given by the identity map on the set of vertices.

By the definitions, we have $\Delta^0 = I^0 = \widehat{I}^0$ and $\Delta^1 = I^1 = \widehat{I}^1 = (0 \to 1)$. For $n \geq 1$, consider the allowed path

$$p = [(0, \dots, 0) \to (0, \dots, 0, 1) \to (0, \dots, 0, 1, 1) \to \dots \to (1, \dots, 1)]$$

in the cube $\widehat{I}^n$. Consider also the subgraph $\widehat{\Delta} \subset \widehat{I}^n$ generated by the set of vertices in the path p. Digraph $\widehat{\Delta}$ is isomorphic to Δ^n. The isomorphism $\Delta^n \to \widehat{\Delta}$ is given on the set of vertices by the map sending each integer $0 \leq k \leq n$ to the vertex that is given by the binary sequence of length n consisting of exactly k units on the right side and $n-k$ zeros on the left side. The composition of this isomorphism with the inclusion $\widehat{\Delta} \subset \widehat{I}^n$ gives the inclusion homomorphism

$$\mathcal{T}\colon \Delta^n \to \widehat{I}^n \tag{4.2}$$

for $n \geq 1$. We denote also by $\mathcal{T}$ the isomorphisms in (4.2) for $n = 0, 1$.

For $n \geq 0$, we define the digraph map

$$\mathcal{P}\colon \widehat{I}^n \to \Delta^n \tag{4.3}$$

inductively as follows. For $n = 0, 1$, it is the identity map. For $n \geq 2$, we first observe that $\widehat{I}^n = \widehat{I}^{n-1} \boxtimes I$ by Proposition 4.2. For a vertex $(v, i) \in \widehat{I}^{n-1} \boxtimes I$, we define

$$\mathcal{P}\colon \widehat{I}^{n-1} \boxtimes I \to \Delta^n \quad \text{by} \quad \mathcal{P}(v, i) = \begin{cases} \mathcal{P}(v) & \text{for } i = 0, \\ n & \text{for } i = 1, \end{cases}$$

where $\mathcal{P}(v) \in V_{\Delta^{n-1}}$ is defined using the inductive assumption. We note, that the composition $\mathcal{P} \circ \mathcal{I}\colon I^n \to \Delta^n$ is a digraph map which can be written as the composition

$$I^{n-1} \square I \to \Delta^{n-1} \square I \to \Delta^n, \tag{4.4}$$

where the restriction to I^{n-1} is the map $\mathcal{P} \circ \mathcal{I}$ in dimension $n-1$.

Lemma 4.3. *Let $\mathcal{T}$ and $\mathcal{P}$ be the digraph maps in (4.2) and (4.3). Then the composition $\mathcal{P} \circ \mathcal{T}\colon \Delta^n \to \Delta^n$ is the identity map for $n \geq 0$.*

Given a cubical digraph I^n with $n \geq 1$, we define the *face inclusion homomorphisms* $\delta_i^c\colon I^{n-1} \to I^n$ for each $1 \leq i \leq n$ and $c = 0, 1$. For $n = 1$, set $\delta_1^c(0) = c \in V_I$. For $n \geq 2$, $(c_1, \dots, c_{n-1}) \in V_{I^{n-1}}$, and $1 \leq i \leq n$, set

$$\delta_i^c(c_1, \dots, c_{n-1}) = (c_1, \dots, c_{i-1}, c, c_i, \dots, c_{n-1}) \in V_{I^n}. \tag{4.5}$$

For $n \geq 1$ and $1 \leq i \leq n$, we define a *face projection* $\sigma_i \colon I^n \to I^{n-1}$ by setting $\sigma_1(c) = 0$ for $n = 1$ and $c \in V_I$. For $n \geq 2$, $(c_1, \ldots, c_n) \in V_{I^n}$, and $1 \leq i \leq n$, we set

$$\sigma_i(c_1, \ldots, c_n) = (c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) \in V_{I^{n-1}}. \tag{4.6}$$

Proposition 4.4. *Let $I = (0 \to 1)$ and $\delta_i^c \colon V_{I^{n-1}} \to V_{I^n}$, $\sigma_i \colon V_{I^n} \to V_{I^{n-1}}$ be the maps of vertices defined above for $c = 0, 1$ and $1 \leq i \leq n$. Then the map δ_i^c defines a face inclusion homomorphism*

$$\widehat{\delta}_i^c \colon \widehat{I}^{n-1} \to \widehat{I}^n \tag{4.7}$$

and the map σ_i defines a face projection digraph map

$$\widehat{\sigma}_i \colon \widehat{I}^n \to \widehat{I}^{n-1}. \tag{4.8}$$

For $n \geq 1$ and $0 \leq i \leq n$, define an inclusion homomorphism

$$\delta_i \colon \Delta^{n-1} \to \Delta^n \tag{4.9}$$

given on the set of vertices by non-decreasing injection that does not take the value $i \in V_{\Delta^n}$. For $n \geq 0$ and $0 \leq i \leq n$, define a digraph map

$$\sigma_i \colon \Delta^{n+1} \to \Delta^n \tag{4.10}$$

given on the set of vertices by non-increasing surjection that twice takes the value $i \in V_{\Delta^n}$.

5. Singular homology theories of quivers and digraphs

In this section, we review the definition and fundamental properties of well-known singular homology theories in graphs (refer to [18, 33, 41, 37] for further details), and introduce several new singular cubical and simplicial homology theories. Specifically, we establish homology theories that align with morphisms in categories $\mathbb{D} \subset \mathbb{Q}$. Additionally, we discuss the functorial and homotopy properties of the introduced homologies, as well as the relationships among various homology theories. It is worth noting that every digraph inherently possesses the natural structure of a quiver.

Definition 5.1. Let Q be a quiver. A *singular n-cube in Q* is a quiver map $\phi \colon I^n \to Q$, a *singular transitive n-cube in Q* is a quiver map $\phi \colon \widehat{I}^n \to Q$, and a *singular n-simplex in Q* is a quiver map $\phi \colon \Delta^n \to Q$.

Fix a ring R with a unit. For any quiver Q and $n \geq 0$, we define the following free R-modules:

i) $C_n = C_n(Q)$ generated by singular n-cubes $\phi\colon I^n \to Q$,
ii) $\widehat{C}_n = \widehat{C}_n(Q)$ generated by singular transitive n-cubes $\phi\colon \widehat{I}^n \to Q$ and
iii) $S_n = S_n(Q)$ generated by singular n-simplexes $\phi\colon \Delta^n \to Q$.

We set $C_{-1} = \widehat{C}_{-1} = S_{-1} = 0$. Let A_n denote one of the modules $C_n, \widehat{C}_n, S_n$. For $n \geq 0$, define differentials $\partial^n\colon A_n \to A_{n-1}$ on generators as follows. In all cases we have $\partial^0 = 0$. For $n \geq 1$, in case i) we set

$$\partial^n(\phi) = \sum_{j=1}^{n}(-1)^j(\phi \circ \delta_j^0 - \phi \circ \delta_j^1) \text{ for } \phi \in C_n, \tag{5.1}$$

where $\delta_j^c\colon I^{n-1} \to I^n\,(c = 0, 1)$ are defined in (4.5). Thus we obtain a chain complex C_*. For $n \geq 1$, a singular n-cube $\phi\colon I^n \to Q$ is *degenerate* if for some j it can be presented as a composition $\psi \circ \sigma_j$, where $\psi\colon I^{n-1} \to Q$ is a singular $(n-1)$-cube and σ_j is defined in (4.6). Let $B_n = B_n(Q)$ be the submodule of $C_n(Q)$ generated by all degenerate singular n-cubes and let $B_0 = B_{-1} = 0$. Then $\partial^n(B_n) \subset B_{n-1}$ and the *(normalized) quotient chain complex* $\Omega_*(Q) = C_*(Q)/B_*(Q)$ is well defined. The homology group $H_m(\Omega_*(Q))$ is called the *singular cubical homology group* of the quiver Q and is denoted by $H_m(Q) = H_m(Q, R)$.

Similarly, in case ii) we set

$$\widehat{\partial}^n(\phi) = \sum_{j=1}^{n}(-1)^j(\phi \circ \widehat{\delta}_j^0 - \phi \circ \widehat{\delta}_j^1) \text{ for } \phi \in \widehat{C}_n, \tag{5.2}$$

where $\widehat{\delta}_j^c\colon \widehat{I}^{n-1} \to \widehat{I}^n\,(c = 0, 1)$ are defined in Proposition 4.4. For $n \geq 1$, a transitive singular n-cube $\phi\colon \widehat{I}^n \to Q$ is *degenerate* if for some j it can be presented as a composition $\psi \circ \widehat{\sigma}_j$, where $\psi\colon \widehat{I}^{n-1} \to Q$ is a singular transitive $(n-1)$-cube in Q and $\widehat{\sigma}_j$ is defined in Proposition 4.4. We denote by $\widehat{B}_n = \widehat{B}_n(Q)$ the free submodule of $\widehat{C}_n$ generated by all degenerate singular transitive n-cubes for $n \geq 1$ and put $\widehat{B}_0 = \widehat{B}_{-1} = 0$. Then $\partial^n(\widehat{B}_n) \subset \widehat{B}_{n-1}$ and the (normalized) quotient chain complex $\widehat{\Omega}_*(Q) = \widehat{C}_*(Q)/\widehat{B}_*(Q)$ is well defined. The homology group $H_m(\widehat{\Omega}_*(Q))$ is called the *singular transitive cubical homology group* of the quiver Q and is denoted by $\widehat{H}_m(Q) = \widehat{H}_m(Q, R)$.

In case iii) we set

$$\partial^n(\phi) = \sum_{j=0}^{n}(-1)^j \phi \circ \delta_j \text{ for } \phi \in S_n \tag{5.3}$$

where $\delta_j\colon \Delta^{n-1} \to \Delta^n$ are defined in (4.9). Thus we obtain a chain complex $S_*(Q)$ [41]. The homology group $H_m(S_*(Q))$ is called the *singular simplicial homology group* of the quiver Q and is denoted by $H_m^{\Delta}(Q)$.

We observe that the homologies H_*, $\widehat{H}_*$ and H_*^{Δ} defined above in the category $\mathbf{Q}$ immediately give homologies in the subcategory $\mathbf{D}$ of digraphs.

Proposition 5.2 ([18, 33, 41, 37])**.** *Let $\mathcal{H}_*$ be one of the homology groups H_*, $\widehat{H}_*$, or H_*^{Δ}. Then any quiver map $f\colon Q \to Q'$ induces a homomorphism $\mathcal{H}_n(Q) \to \mathcal{H}_n(Q')$ for every $n \geq 0$.*

For any quiver Q, there are homomorphisms

$$\mathcal{I}_*\colon \widehat{H}_n(Q) \to H_n(Q),$$
$$\mathcal{T}_*\colon \widehat{H}_n(Q) \to H_n^{\Delta}(Q)$$

induced by the natural inclusions $\mathcal{I}$ in (4.1) and $\mathcal{T}$ in (4.2), respectively. These homomorphisms are functorial with respect to quiver maps.

Two homotopic maps $f \simeq g\colon Q \to Q'$ induce the same homomorphism $f_* = g_*\colon H_*(Q) \to H_*(Q')$.

We now show that there is an isomorphism between homology groups $\widehat{H}_*$ and H_*^{Δ}. Firstly, recall the Acyclic Carriers Theorem (see [29, §3.4] and [38, §1.2.1]) which we use in the proof.

A *geometric chain complex* C_* consists of finitely generated free abelian groups C_n for $n \geq 0$ and $C_n = 0$ for $n < 0$. Fix a basis in each group C_n. For two basis elements $b^{n-1} \in C_{n-1}, b^n \in C_n$, we write $b^{n-1} \prec b^n$ if b^{n-1} is non-trivial and lies in the subgroup generated by $\partial(b^n)$. Let $\widetilde{C}_*$ denote the complex C_* with the *augmentation*

$$\varepsilon\left(\sum_i k_i b_i^0\right) = \sum_i k_i, \quad k_i \in \mathbb{Z}.$$

A geometric chain complex C_* is *acyclic* if the homology groups of $\widetilde{C}_*$ are trivial. A chain map $\phi_*\colon C_* \to C'_*$ is *augmentation preserving* if $\varepsilon'\phi_0(c) = \varepsilon(c)$ for any $c \in C_0$.

Definition 5.3.

i) An algebraic carrier function E *from a geometric chain complex C_* to a chain complex D_* is defined on basis elements $b \in C_n$ for all $n \geq 0$ taking values $E(b) = E_*(b) \subset D_*$ that are subcomplexes of D_* satisfying $b \prec b'$ implies $E_*(b) \subset E_*(b')$.*

ii) The function E is called acyclic *if each subcomplex $E(b)$ is acyclic.*
iii) The chain map $f_*\colon C_* \to D_*$ is carried by *the algebraic carrier function E if $f_n(b) \in E_*(b)$ for any basis element $b \in C_n$.*

Theorem 5.4 (Acyclic Carriers Theorem)**.** *Let $f_*, g_*\colon C_* \to D_*$ be augmentation preserving chain maps of geometric chain complexes that are carried by the acyclic carrier function E, then the maps f_* and g_* are chain homotopic.*

Consider the composition $\mathcal{T} \circ \mathcal{P}\colon \widehat{I}^n \to \widehat{I}^n$ of digraph maps defined in Section 4. Thus, for any quiver Q we obtain an induced map of singular transitive cubic chain complexes

$$[\mathcal{T} \circ \mathcal{P}]_*\colon \widehat{\Omega}_*(Q) \to \widehat{\Omega}_*(Q)$$

defined on a singular transitive cube $\varphi\colon \widehat{I}^n \to Q$ by

$$[\mathcal{T} \circ \mathcal{P}]_*(\varphi) = \varphi \circ \mathcal{T} \circ \mathcal{P}\colon \widehat{I}^n \to Q.$$

Proposition 5.5. *Let $R = \mathbb{Z}$. There is a chain homotopy between $[\mathcal{T} \circ \mathcal{P}]_*\colon \widehat{\Omega}_*(Q) \to \widehat{\Omega}_*(Q)$ and the identity map $\mathrm{Id}_*\colon \widehat{\Omega}_*(Q) \to \widehat{\Omega}_*(Q)$.*

Proof. The chain complex $\widehat{\Omega}_*(Q)$ is geometric and the chain maps $[\mathcal{T} \circ \mathcal{P}]_*$ and Id_* are augmentation preserving. For a transitive singular cube $\varphi\colon \widehat{I}^n \to Q$ consider the image $\operatorname{Im} \varphi = Q_\varphi$ which is a sub-quiver of Q. Let $(0, \dots, 0) \in V_{\widehat{I}^n}$ be the initial vertex of $\widehat{I}^n$ and $(a_1, \dots, a_n) \in V_{\widehat{I}^n}$ be a vertex such that $\varphi(0, \dots, 0) \neq \varphi(a_1, \dots, a_n)$. Then there is an arrow $\varphi[(0, \dots, 0) \to (a_1, \dots, a_n)]$ in Q_φ and, hence, the quiver Q_φ is a contractible [21, Ex. 3.11]. To every basic element $\varphi \in \widehat{\Omega}_*(Q)$ we assign the subcomplex

$$E_*(\varphi) \overset{def}{=} \widehat{\Omega}_*(Q_\varphi) \subset \widehat{\Omega}_*(Q) \tag{5.4}$$

which is acyclic since Q_ϕ is contractible. We therefore obtain an algebraic carrier function E from the chain complex $\widehat{\Omega}_*(Q)$ into itself, with the chain maps $[\mathcal{T} \circ \mathcal{P}]_*$ and Id_* are carried by the function E. Now the proposition follows from the Acyclic Carriers Theorem 5.4. □

Theorem 5.6. *Let $R = \mathbb{Z}$. For any quiver Q, the chain maps*

$$\mathcal{P}_*\colon S_*(Q) \to \widehat{\Omega}_*(Q) \quad \textit{and} \quad \mathcal{T}_*\colon \widehat{\Omega}_*(Q) \to S_*(Q)$$

are chain homotopy inverse to each other.

The groups $\widehat{H}_n(Q)$ and $H_n^\Delta(Q)$ are isomorphic for every quiver Q and $n \geq 0$. These isomorphisms are natural with respect to quiver maps.

Proof. The proof follows from Lemma 4.3 and Proposition 5.5. □

Theorem 5.7. *The homology groups $H_*(Q)$ and $\widehat{H}_*(Q)$ are not isomorphic in general.*

The homology groups $H^{\Delta}_*(Q)$ and $\widehat{H}_*(Q)$ are not homotopy invariant in general.

Proof. The digraph square I^2 is contractible and hence

$$H_n(I^2,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n=0, \\ 0 & \text{for } n\geq 1. \end{cases}$$

By [37, Example 1] we have $\widehat{H}_1(I^2,\mathbb{Z})=\mathbb{Z}$ and the first statement follows. The digraph I^2 is homotopy equivalent to the one-vertex digraph $*$. It is easy to check, that $\widehat{H}_0(*,\mathbb{Z})=\mathbb{Z}$ and $\widehat{H}_n(*,\mathbb{Z})=0$ for $n\geq 1$. Hence homology groups $\widehat{H}_*(Q)$ are not homotopy invariant in general. Now the second statement follows from Theorem 5.6. □

Theorem 5.8. *Two s-homotopic quiver maps $f\simeq g\colon Q\to Q'$ induce the same homomorphisms*

$$\begin{aligned} f_*=g_*\colon H_*(Q)\to H_*(Q'),\\ f_*=g_*\colon \widehat{H}_*(Q)\to \widehat{H}_*(Q'),\\ f_*=g_*\colon H^{\Delta}_*(Q)\to H^{\Delta}_*(Q') \end{aligned}$$

of homology groups.

Proof. By Proposition 3.16 any s-homotopy is a homotopy and the first equality follows from Proposition 5.2. Now it is sufficient to consider only the groups $\widehat{H}_*$, as the case of groups H^{Δ}_* follows from Theorem 5.6. Consider a one-step homotopy $F\colon I\boxtimes Q\longrightarrow Q'$ with bottom map f and top map g. Every transitive singular n-cube $\varphi\colon \widehat{I}^n\to Q$ defines a transitive singular $(n+1)$-cube $\mathcal{D}(\phi)\colon \widehat{I}^{n+1}=I\boxtimes \widehat{I}^n\to Q'$ which is given by the composition

$$\widehat{I}^{n+1}=I\boxtimes \widehat{I}^n \xrightarrow{\operatorname{Id}\boxtimes\varphi} I\boxtimes Q \xrightarrow{F} Q'.$$

The remainder of the proof is similar to the case of homotopy invariance of normalized singular cubical homology in [39], [29, §8], [36, Chpt. II, §4], [15, Chpt. VII, §7], [27, p. 112]. □

Now we define homology theories which arise naturally in the categories $\mathbb{D}\subset\mathbb{Q}$. In particular, we describe functorial and homotopy properties of these theories and give several examples. We note that the concepts which we introduce below have no analogues in classical algebraic topology.

Definition 5.9. Let Q be a quiver or a digraph.

i) An *h-singular n-cube* in Q is a homomorphism $\phi\colon I^n \to Q$.
ii) An *transitive h-singular n-cube* in Q is a homomorphism $\phi\colon \widehat{I}^n \to Q$.
iii) An *h-singular n-simplex* in Q is a homomorphism $\phi\colon \Delta^n \to Q$.

Fix a ring R with a unit. For any quiver Q and $n \geq 0$, we define the following free R-modules:

i) $\mathbb{C}_n = \mathbb{C}_n(Q)$ generated by h-singular n-cubes $\phi\colon I^n \to Q$,
ii) $\widehat{\mathbb{C}}_n = \widehat{\mathbb{C}}_n(Q)$ generated by transitive h-singular n-cubes $\phi\colon \widehat{I}^n \to Q$ and
iii) $\mathbb{S}_n = \mathbb{S}_n(Q)$ generated by h-singular n-simplexes $\phi\colon \Delta^n \to Q$.

We set $\mathbb{C}_{-1} = \widehat{\mathbb{C}}_{-1} = \mathbb{S}_{-1} = 0$. Let $\mathbb{A}_n$ denote one of the modules $\mathbb{C}_n, \widehat{\mathbb{C}}_n, \mathbb{S}_n$. For $n \geq 0$, define differentials $\partial^n\colon \mathbb{A}_n \to \mathbb{A}_{n-1}$ on generators as follows. In all cases we have $\partial^0 = 0$. For $n \geq 1$, in case i) we set

$$\partial^n(\phi) = \sum_{j=1}^{n} (-1)^j (\phi \circ \delta_j^0 - \phi \circ \delta_j^1) \text{ for } \phi \in \mathbb{C}_n \tag{5.5}$$

where $\delta_j^c\colon I^{n-1} \to I^n$ $(c = 0, 1)$ are defined in (4.5). Thus we obtain a chain complex $\mathbb{C}_*$. We note that *degenerate h-singular cubes* do not exist. The homology group $H_m(\mathbb{C}_*(Q))$ is called the *h-singular cubical homology group* of the quiver Q and is denoted by $H_m^h(Q) = H_m^h(Q, R)$.

Similarly, in case ii) we set

$$\widehat{\partial}^n(\phi) = \sum_{j=1}^{n} (-1)^j (\phi \circ \widehat{\delta}_j^0 - \phi \circ \widehat{\delta}_j^1) \text{ for } \phi \in \widehat{\mathbb{C}}_n \tag{5.6}$$

where $\widehat{\delta}_j^c\colon \widehat{I}^{n-1} \to \widehat{I}^n$ $(c = 0, 1)$ are defined in Proposition 4.4. *Degenerate transitive h-singular cubes* do not exist either. The homology group $H_m(\widehat{\mathbb{C}}_*(Q))$ is called the *transitive h-singular cubic homology group* of the quiver Q and is denoted by $\widehat{H}_m^h(Q) = \widehat{H}_m^h(Q, R)$.

In case iii) we set

$$\partial^n(\phi) = \sum_{j=0}^{n} (-1)^j \phi \circ \delta_j \text{ for } \phi \in \mathbb{S}_n \tag{5.7}$$

where $\delta_j\colon \Delta^{n-1}\to\Delta^n$ are defined in (4.9). Thus we obtain a chain complex $\mathbb{S}_*(Q)$ [41]. The homology group $H_m(\mathbb{S}_*(Q))$ is called the *h-singular simplicial homology group* of the quiver Q and is denoted by $H_m^{\Delta h}(Q)$.

We note, that the homologies H_*^h, $\widehat{H}_*^h$, and $H_*^{\Delta h}$ defined above in the category $\mathbb{Q}$ immediately give the homologies in the subcategory $\mathbb{D}$ of digraphs.

Recall that a *tree* is an undirected graph in which any two vertices are connected by exactly one path. Let $G=(V,E)$ be a simple digraph without double edges, that is $(v\to w)\in E$ implies that $(w\to v)\notin E$. The *underlying graph* $G^u=(V^u,E^u)$ of the digraph G is defined to have the same set of vertices $V^u=V$ and for every arrow $(v\to w)\in E_G$ an edge $\{v,w\}\in E^u$.

Proposition 5.10. *Let $R=\mathbb{Z}$ and G be a simple digraph without double edges for which the underlying graph is a tree. Then*

$$\widehat{H}_n^h(G)=H_n^{\Delta h}(G)=\begin{cases}\mathbb{Z} & \text{for } n=0,\\ 0 & \text{for } n\geq 1.\end{cases}$$

Proof. It is easy to see that the chain complexes $\widehat{\mathbb{C}}_*(G)$ and $\mathbb{S}_*(G)$ coincide and are non-trivial only in dimensions $0,1$. We compute homology groups $\widehat{H}_*^h(G,\mathbb{Z})$ only for the case $G=(0\to 1)$. The computation in others cases is similar. The module $\widehat{\mathbb{C}}_0(G)$ is generated by the homomorphisms $\varphi_i\colon I^0=*\to i$ where $\varphi_i(*)=i$ for $i=0,1$. The module $\widehat{\mathbb{C}}_1(G)$ is generated by the isomorphism $\psi\colon I^1\to G$, and $\widehat{\mathbb{C}}_n(G)=0$ for $n\geq 2$. The non-trivial differential in $\widehat{\mathbb{C}}_*$ is given by $\partial^1(\psi)=\varphi_1-\varphi_0$. Hence $\widehat{H}_0^h(G,\mathbb{Z})=\mathbb{Z}$ and the homology is trivial in other dimensions. □

The differences between the newly introduced h-homology groups are illustrated in the following examples.

Example 5.11. Let $R=\mathbb{Z}$ and G be the digraph

$$\begin{array}{ccccc} & & 2 & & \\ & {}^{c}\nearrow & & \nwarrow^{b} & \\ 0 & & \overset{a}{\to} & & 1 \end{array} \tag{5.8}$$

which is isomorphic to Δ^2. Firstly, we compute homology groups $H_*^h(G)$. We have $\mathbb{C}_0(G)=\langle\varphi_0,\varphi_1,\varphi_2\rangle$, where $\varphi_i(*)=i$ for $i=0,1,2$, and $\mathbb{C}_1(G)=\langle\psi_a,\psi_b,\psi_c\rangle$ and ψ_α is the homomorphism of I onto the arrow $\alpha\in\{a,b,c\}$. It is easy to see that $\mathbb{C}_2(G)=\langle\gamma\rangle$, where the homomorphism γ is defined on the square

$$I^2 = \begin{array}{ccc} 2 & \to & 3 \\ \uparrow & & \uparrow \\ 0 & \to & 1 \end{array}$$

by $\gamma(0) = 0, \gamma(1) = \gamma(2) = 1, \gamma(3) = 2$. The modules $\mathbb{C}_n(G)$ are trivial for $n \geq 3$. Now we describe non-trivial differentials:

$$\partial^1(\psi_a) = \varphi_1 - \varphi_0, \quad \partial^1(\psi_b) = \varphi_2 - \varphi_1, \quad \partial^1(\psi_c) = \varphi_2 - \varphi_0,$$
$$\partial^2(\gamma) = \psi_a - \psi_b - (\psi_a - \psi_b) = 0.$$

Hence $\operatorname{Ker} \partial^2 = \langle\gamma\rangle$, $\operatorname{Im} \partial^2 = 0$, $\operatorname{Ker} \partial^1 = \langle\psi_a + \psi_b - \psi_c\rangle$, $\operatorname{Im} \partial^1 \cong \mathbb{Z}^2$ and so

$$H_n^h(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, 2, \\ 0 & \text{for } n \geq 3. \end{cases} \tag{5.9}$$

Now we compute homology groups $\widehat{H}_*^h(G)$. Similarly to above, we have $\widehat{\mathbb{C}}_0(G) = \mathbb{C}_0(G) = \langle\varphi_1, \varphi_2, \varphi_3\rangle$, $\widehat{\mathbb{C}}_1(G) = \mathbb{C}_1(G) = \langle\psi_a, \psi_b, \psi_c\rangle$ and $\mathbb{C}_2(G) = \langle\widehat{\gamma}\rangle$, where $\widehat{\gamma}|_{I^2} = \gamma$ with respect to the inclusion $I^2 \subset \widehat{I}^2$. The differentials are computed similarly to the case $\mathbb{C}_*$ and we obtain

$$\widehat{H}_n^h(G, \mathbb{Z}) = H_n^h(G, \mathbb{Z}) \ \text{ for } n \in \mathbb{Z}. \tag{5.10}$$

Now we compute homology groups $\widehat{H}_*^{\Delta h}(G)$. Similarly to above, we have $\mathbb{S}_0(G) = \mathbb{C}_0(G) = \langle\varphi_1, \varphi_2, \varphi_3\rangle$ and $\mathbb{S}_1(G) = \mathbb{C}_1(G) = \langle\psi_a, \psi_b, \psi_c\rangle$. The module $\mathbb{S}_2(G)$ is generated by the isomorphism $\gamma\colon \Delta^2 \to G$, and the modules $\mathbb{S}_n(G)$ are trivial for $n \geq 3$. In dimensions $0, 1$ the differentials of the chain complexes $\mathbb{S}_*(G)$ and $\mathbb{C}_*(G)$ coincide, and $\partial^2\colon \mathbb{C}_2(G) \to \mathbb{C}_1(G)$ is given by $\partial^2(\gamma) = \psi_b - \psi_c + \psi_a$. Hence, $\operatorname{Im} \partial^2 = \operatorname{Ker} \partial^1$ and we conclude that

$$\widehat{H}_n^{\Delta h}(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ 0 & \text{for } n \geq 1. \end{cases} \tag{5.11}$$

The equalities (5.9), (5.10), and (5.11) show that for the digraph G in (5.8) we have

$$H_*^h(G) = \widehat{H}_*^h(G) \neq H_*^{\Delta h}(G).$$

Example 5.12. Let $R = \mathbb{Z}$ and G be the digraph

$$\begin{array}{ccc} 2 & \overset{d}{\to} & 3 \\ b\uparrow & & \uparrow c \\ 0 & \overset{a}{\to} & 1 \end{array} \tag{5.12}$$

which is isomorphic to I^2. Firstly, we compute homology groups $H_*^h(G)$. We have $\mathbb{C}_0(G) = \langle\varphi_0, \varphi_1, \varphi_2, \varphi_3\rangle$, where $\varphi_i(*) = i$ for $i = 0, 1, 2, 3$, and $\mathbb{C}_1(G) = \langle\psi_a, \psi_b, \psi_c, \psi_d\rangle$, where ψ_α is the homomorphism of I onto the arrow $\alpha \in \{a, b, c, d\}$. The module $\mathbb{C}_2(G)$ is generated by the homomorphisms γ_0, γ_1, γ_2 and γ_3 of the square

$$I^2 = \begin{matrix} 2 & \to & 3 \\ \uparrow & & \uparrow \\ 0 & \to & 1 \end{matrix}$$

to the digraph G. The maps γ_0, γ_1, γ_2 and γ_3 are given on the set of vertices as follows:

$$\begin{aligned} &\gamma_0(i) = (i) \quad \text{for } i = 0, 1, 2, 3, \\ &\gamma_1(i) = (i) \quad \text{for } i = 0, 3, \qquad \gamma_1(2) = 1,\ \gamma_1(1) = 2, \\ &\gamma_2(i) = (i) \quad \text{for } i = 0, 1, 3, \qquad \gamma_2(2) = 1, \\ &\gamma_3(i) = (i) \quad \text{for } i = 0, 2, 3, \qquad \gamma_3(1) = 2. \end{aligned}$$

We have $\mathbb{C}_n(G) = 0$ for $n \geq 3$. The non-trivial differentials are:

$$\partial^1(\psi_a) = \varphi_1 - \varphi_0, \quad \partial^1(\psi_b) = \varphi_2 - \varphi_0, \quad \partial^1(\psi_c) = \varphi_3 - \varphi_1, \partial^1(\psi_d) = \varphi_3 - \varphi_2,$$
$$\begin{aligned} \partial^2(\gamma_0) &= \psi_a - \psi_d - (\psi_b - \psi_c), \\ \partial^2(\gamma_1) &= \psi_b - \psi_c - (\psi_a - \psi_d), \\ \partial^2(\gamma_2) &= \psi_a - \psi_c - (\psi_a - \psi_c) = 0, \\ \partial^2(\gamma_3) &= \psi_b - \psi_d - (\psi_b - \psi_d) = 0. \end{aligned}$$

Hence, $\operatorname{Ker}\partial^2 = \langle\gamma_0 - \gamma_1, \gamma_2, \gamma_3\rangle$, $\operatorname{Im}\partial^2 = \langle\psi_a + \psi_c - \psi_d - \psi_b\rangle = \operatorname{Ker}\partial^1$, $\operatorname{Im}\partial^1 \cong \mathbb{Z}^3$ and

$$H_n^h(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, \\ \mathbb{Z}^3 & \text{for } n = 2, \\ 0 & \text{for } n = 1 \text{ and } n \geq 3. \end{cases} \tag{5.13}$$

Now we compute homology groups $\widehat{H}_*^h(G)$. Similarly to above, we have $\widehat{\mathbb{C}}_0(G) = \mathbb{C}_0(G) = \langle\varphi_0, \varphi_1, \varphi_2, \varphi_3\rangle$, $\widehat{\mathbb{C}}_1(G) = \mathbb{C}_1(G) = \langle\psi_a, \psi_b, \psi_c, \psi_d\rangle$, and the modules $\widehat{\mathbb{C}}_n(G)$ are trivial for $n \geq 2$. The differentials in dimensions zero and one are the same as for $\mathbb{C}_*(G)$ and we conclude that

$$\widehat{H}_n^h(G, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2. \end{cases} \tag{5.14}$$

The equalities (5.13) and (5.14) show that

$$H^h(G) \neq \widehat{H}^h(G)$$

for the digraph G in (5.12).

Proposition 5.13. *The homology groups H_*^h, $\widehat{H}_*^h$, and $H_*^{\Delta h}$ are functorial on the categories $\mathbb{D}$ and $\mathbb{Q}$.*

Proposition 5.14. *For a quiver Q and $n \geq 0$, the homomorphisms*

$$\mathcal{I}_*\colon \widehat{H}_n^h(Q) \to H_n^h(Q),$$
$$\mathcal{T}_*\colon \widehat{H}_n^h(Q) \to H_n^{\Delta h}(Q)$$

are well defined and functorial with respect to quiver (digraph) homomorphisms, where $\mathcal{I}$ and $\mathcal{T}$ are inclusion homomorphisms in (4.1) and (4.2), respectively.

Theorem 5.15. *Two h-homotopic quiver maps $f \doteq g\colon Q \to Q'$ induce the same homomorphism*

$$f_* = g_*\colon H_*^h(Q) \to H_*^h(Q')$$

of homology groups. Hence, homology groups H_*^h are h-homotopy invariant on the categories $\mathbb{D} \subset \mathbb{Q}$.

Proof. It is sufficient to consider the case of a one-step homotopy. Let $F\colon I\Box Q \longrightarrow Q'$ be a one-step h-homotopy that is a homomorphism on the bottom map f and the top map g. Any h-singular n-cube $\phi\colon I^n \to Q$ defines an h-singular $(n+1)$-cube $\mathcal{D}(\phi)\colon I^{n+1} = I\Box I^n \to Q'$ in Q' which is given by the composition

$$I^{n+1} = I\Box I^n \overset{\mathrm{Id}\,\Box\phi}{\longrightarrow} I\Box Q \overset{F}{\longrightarrow} Q'.$$

The same line of argument used in the proof of Theorem 5.8 finishes the proof. □

Theorem 5.16. *The homology groups $\widehat{H}_*^h$ and $H_*^{\Delta h}$ are not h-homotopy invariant on the categories $\mathbb{D} \subset \mathbb{Q}$.*

Proof. The digraph $G = \left[0 \underset{b}{\overset{a}{\rightleftarrows}} 1 \underset{d}{\overset{c}{\rightleftarrows}} 2\right]$ and the digraph $G' = \left[0' \underset{b'}{\overset{a'}{\rightleftarrows}} 1'\right]$ from Example 3.9, ii) are h-homotopy equivalent. There are only zero-dimensional and one-dimensional singular transitive h-cubes (h-simplexes) in these digraphs. Now it is easy to compute using the definitions that

$$\widehat{H}^h_n(G,\mathbb{Z}) = H^{\Delta h}_n(G,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n=0, \\ \mathbb{Z}^2 & \text{for } n=1, \\ 0 & \text{for } n\geq 2, \end{cases}$$

$$\widehat{H}^h_n(G',\mathbb{Z}) = H^{\Delta h}_n(G',\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n=0, \\ \mathbb{Z} & \text{for } n=1, \\ 0 & \text{for } n\geq 2 \end{cases}$$

and the statement of the theorem follows. □

Theorem 5.17. *Two sh-homotopic maps $f \approx g\colon Q \to Q'$ induce the same homomorphisms*

$$f_* = g_*\colon H^h_*(Q) \to H^h_*(Q'),$$
$$f_* = g_*\colon \widehat{H}^h_*(Q) \to \widehat{H}^h_*(Q'),$$
$$f_* = g_*\colon H^{\Delta h}_*(Q) \to H^{\Delta h}_*(Q').$$

Hence, homology groups $H^h_*, \widehat{H}^h_*, H^{\Delta h}_*$ are sh-homotopy invariant.

Proof. Similar to Theorem 5.8 and Theorem 5.15. □

We summarize the differences and similarities of various homologies introduced above in the following table.

	$H_*(Q)$	$\widehat{H}_*(Q)$	$H^{\Delta}_*(Q)$	$H^h_*(Q)$	$\widehat{H}^h_*(Q)$	$H^{\Delta h}_*(Q)$
invariance under homotopy	Yes	No	No			
invariance under s-homotopy	Yes	Yes	Yes			
invariance under h-homotopy				Yes	No	No
invariance under sh-homotopy				Yes	Yes	Yes
isomorphic with		$H^{\Delta}_*(Q)$	$\widehat{H}_*(Q)$			

6. Inclusion homology groups and isotopy

In this section we give a few additional remarks about the singular homology theories based on inclusions.

We now provide one more way to define singular homology on digraphs and quivers. In the category $\mathbb{Q}$ (and, hence, in $\mathbb{D}$) any h-singular n-simplex in a quiver Q is given by an inclusion $\phi\colon \Delta^n \to Q$. Thus, a natural way to define an *inclusion singular cubical homology theories* for categories $\mathbb{D} \subset \mathbb{Q}$ is as follows.

Definition 6.1. Let Q be a quiver or a digraph.

i) An *i-singular n-cube* in Q is an inclusion $\phi\colon I^n \to Q$.
ii) A *transitive i-singular n-cube* in Q is an inclusion $\phi\colon \widehat{I^n} \to Q$.
iii) An *i-singular n-simplex* in Q is an inclusion $\phi\colon \Delta^n \to Q$.

Let Q be a quiver. Similarly to the definition of h-homology groups $H^h_*(Q,R)$, $\widehat{H}^h_*(Q,R)$ and $H^{\Delta h}_*(Q,R)$ in Definition 5.9, we define i-singular homology groups $H^i_*(Q,R)$, $\widehat{H}^i_*(Q,R)$, and $H^{\Delta i}_*(Q,R)$ corresponding to cases i, ii, iii of Definition 6.1 respectively. In the category $\mathbb{Q}$ the homology theory $H^{\Delta i}_*(Q,R)$ is functorial and coincides with the homology theory $H^{\Delta h}_*(Q,R)$. We introduce the category $\mathbb{Q}^i$ of quivers ($\mathbb{D}^i$ of digraphs) for which the set of objects is the same as $\mathbb{Q}$ ($\mathbb{D}$) and morphisms are given by inclusions. Then, the homologies $H^i_*(Q,R)$, $\widehat{H}^i_*(Q,R)$ and $H^{\Delta i}_*(Q,R)$ are functorial in the category $\mathbb{Q}^i$ ($\mathbb{D}^i$). An *isotopy* between two inclusions is given by a homotopy which is an inclusion. It is an easy exercise to describe relation between i-homology groups up to isotopy in the categories $\mathbb{D}^i \subset \mathbb{Q}^i$. The next proposition shows that the notion of isotopy is not generated.

Proposition 6.2. *Let $I_n (n \geq 1)$ be a directed line digraph with the set of vertices $V_{I_n} = \{0,\dots,n\}$ and $\tau_k\colon I_1 \to I_n\ (0 \leq k \leq n-1)$ be an inclusion given on the set of vertices by $\tau_k(0) = k, \tau_k(1) = k+1$. Then any two inclusions $\tau_k, \tau_l\ (0 \leq k,l \leq n-1)$ are isotopic.*

Proof. Consider the case $n=2$ and $k=0, l=1$. A one-step homotopy

$$F\colon \begin{pmatrix} (1,0) & \to & (1,1) \\ \uparrow & & \uparrow \\ (0,0) & \to & (0,1) \end{pmatrix} \longrightarrow [0 \to 1 \to 2]$$

is given by $F(0,0) = 0, F(0,1) = F(1,0) = 1, F(1,1) = 2$. The restriction of F to the bottom boundary coincides with τ_0 and the restriction of F to the top boundary coincides with τ_1. The general case follows now by induction. $\square$

Using the category $\mathbb{Q}^i$ we can define the following homotopy categories.

Definition 6.3. i) Let $\mathbb{H}\mathbb{Q}^i$ be a category in which objects are the same as $\mathbb{Q}^i$ and morphisms are given by classes of h-homotopic maps between two inclusions as in Definition 3.4, ii).

ii) Let $\mathbb{H}^i\mathbb{Q}^i$ be a category in which objects are the same as $\mathbb{Q}^i$ and morphisms are given by classes of h-homotopic maps between two inclusions as in Definition 3.4, ii) such that the homotopy F in (3.1) is an inclusion.

iii) Let $\mathbb{H}^s\mathbb{Q}^i$ be a category in which objects are the same as $\mathbb{Q}^i$ and morphisms are given by classes of s-homotopic maps between two inclusions as in Definition 3.13 such that the homotopy F in (3.6) is an h-homotopy.

iv) Let $\mathbb{H}^{si}\mathbb{Q}^i$ be a category in which objects are the same as $\mathbb{Q}^i$ and morphisms are given by classes of s-homotopic maps between two inclusions as in Definition 3.13 such that the homotopy F in (3.6) is an inclusion.

It is an exercise for readers to check that the categories introduced in Definition 6.3 are pairwise distinct. Similarly, we can define homotopy categories in the category $\mathbb{D}^i$ of digraphs and inclusions. The relation between these categories are given by the commutative diagram

$$\begin{array}{ccccccccc} \mathbb{H}^s\mathbb{D}^i & \to & \mathbb{H}^s\mathbb{D} & \to & \mathbb{H}\mathbb{D} & \leftarrow & \mathbb{H}\mathbb{D}^i & \leftarrow & \mathbb{H}^i\mathbb{D}^i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^s\mathbb{Q}^i & \to & \mathbb{H}^s\mathbb{Q} & \to & \mathbb{H}\mathbb{Q} & \leftarrow & \mathbb{H}\mathbb{Q}^i & \leftarrow & \mathbb{H}^i\mathbb{Q}^i, \end{array}$$

where morphisms are the natural inclusions. Moreover, we have the commutative diagram in which all morphisms are natural inclusions

$$\begin{array}{ccccc} \mathbb{H}^s\mathbb{D}^i & \leftarrow & \mathbb{H}^{si}\mathbb{D}^i & \to & \mathbb{H}^i\mathbb{D}^i \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^s\mathbb{Q}^i & \leftarrow & \mathbb{H}^{si}\mathbb{Q}^i & \to & \mathbb{H}^i\mathbb{Q}^i. \end{array}$$

In conclusion, we note that it is possible to introduce mixed homotopy categories by varying morphisms and homotopies. For example, we can consider a category in which morphisms are given by classes of homotopic maps between two homomorphisms. Similarly, we can consider a category in which morphisms are given by classes of homotopic maps between inclusions.

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