Noncommutative Calculus

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Abstract

We give a brief introduction to the generalization of the classical differential calculus to non-commutative algebras based on operations on cyclic complexes extending the classical Cartan calculus. We mention some of the basic analytic and algebraic applications of this calculus.

Introduction

A natural language for calculus on a manifold or an algebraic variety is in terms of the corresponding algebras of functions, be it smooth or algebraic. The goal of non-commutative calculus is to extend the calculus to the case of associative algebras. While most of what is done below works in the case of A_∞ algebras, we will, at least until the end, avoid the ∞-language.

Let A be an algebra. Let D be a derivation of A. When A is commutative, say $A = C^\infty(M)$ for a smooth manifold M, D is a vector field on M and as such it acts on the de Rham complex $(\Omega^\bullet_M, d)$ by Lie derivative, as it does on any natural tensor construction applied to A. We will denote this action by L_D.

One also defines the contraction ι_D by D and the classical calculus is summarized by the "change of variables" identities

$$[d, \iota_D] = L_D; \ \ \iota_D^2 = 0$$

Introducing a formal variable u of degree -2, and working over power series $\mathbb{C}[[u]]$, one can combine these into a single identity

$$(ud + \iota_D)^2 = uL_D \in End_{\mathbb{C}[[u]]}(\Omega^\bullet(M)[[u]]) \tag{1}$$

Now let A be an associative algebra and D a derivation of A. The commonly used replacement for de Rham complex is the periodic cyclic complex (Loday, 1998) which we recall in the section "Noncommutative Differential Forms" where, in the first approximation, the differential graded space of differential forms get replaced by the Hochschild chain complex $(C_\bullet(A), b)$ and one can ask whether something similar to (1) exists there. The answer is yes, but with modifications. First, we relax (1) and ask for an operator $\mathcal{J}(D)$ which is no more linear in D but rather a formal combination

$$\mathcal{J}(D) = \sum_{n=1}^{\infty} \frac{J_{D^n}}{n!} \tag{2}$$

We are looking for $\mathcal{J}(D)$ satisfying

$$(b + uB + \mathcal{J}(D))^2 = uL_D \tag{3}$$

These formulas appear in Alekseev and Ševera (2012) and in Bonechi *et al.* (2023). What we find instead is an exponential series $\mathscr{I}(D)$ satisfying

$$(b + uB + \mathscr{I}(D))^2 = u(e^D - 1) \tag{4}$$

These appear in Gerasimov (1993). All the homogenous components I_{D^n} of $\mathscr{I}(D)$ are defined over $\mathbb{Z}$.

In characteristic zero, one can indeed pass from the operator $\mathscr{I}(D)$ to a more classical $\mathcal{J}(D)$, but the procedure is somewhat awkward. One way of saying this is that the operators $\mathscr{I}(D)$ on Hochschild chains of a commutative algebra are A-linear modulo u, and the "classical" operators $\mathcal{J}(D)$ that we get from them are not.

The detailed description of this calculus of derivations is in the subsection "The Cartan Calculus of Derivations"

The classical calculus has an immediate extension from vector fields to multi-vector fields, generalizing the standard formulas

$$\iota_f(\omega) = f\omega; L_f(\omega) = df \wedge \omega \tag{5}$$

for a function f and a form ω. The noncommutative Cartan calculus extends to the full DGA (differential graded algebra) of Hochschild cochains (section "Noncommutative Calculus of Multi-Vector Fields and Forms"), where the appropriate notion seems to be that of calculus.

Note that now, after extending to higher cochains and multivectors, comparing the two versions of Cartan calculus via HKR (Hochschild-Kostant-Rosenberg) when our algebra is commutative becomes much more difficult; when A is regular, a positive answer is given by the Kontsevich formality theorem. We include some information about this in the last section but, for more information, refer the reader to the original papers.

Other Approaches to NC-Calculus

Also, the text below does not contain any mention of alternative approaches to non-commutative calculus, so let us mention at least two of them.

- The language of Fredholm modules and spectral triples. While not really disjoint from the text below, it is an important subject by itself. A good starting point is the lecture of Alain Connes: see "Relevant Websites" section
- Braided commutative structures that originated from the study of matrix compact quantum groups, see f.ex (Beggs and Majid, 2021). and especially the references therein for a good introduction to the subject.

The Commutative Case

Let A be a *commutative* unital algebra over a ground ring k of characteristic zero. An algebraic version of a vector field on X is a derivation of A. More generally, for a graded k-algebra, a derivation of degree m is a k-linear map $D: A \to A$ of degree m satisfying

$$D(ab) = D(a)b + (-1)^{m|a|} aD(b) \tag{6}$$

for any homogeneous a and b in A.

An algebraic version of the algebra of differential forms on X is the *graded commutative* unital algebra $\Omega^\bullet_{A/k}$ generated by a unital subalgebra A and by elements da, $a \in A$, that are k-linear in a and satisfy

$$d(ab) = (da)b + adb; \quad d1 = 0 \tag{7}$$

This algebra is graded in such a way that $|a| = 0$ and $|da| = 1$. It has a graded derivation of degree 1

$$d : \Omega^\bullet_{A/k} \to \Omega^{\bullet+1}_{A/k}, \quad d^2 = 0$$

such that $d(a) = da$ and $d1 = 0$. The Lie algebra $\mathrm{D}er(A)$ acts on $\Omega^\bullet_{A/k}$ by derivations: for $D \in \mathrm{D}er(A)$, extend D to a degree zero derivation of $\Omega^\bullet_{A/k}$ that sends da to $dD(a)$. We will often write L_D instead of D. Define also $\iota_D : \Omega^\bullet_{A/k} \to \Omega^{\bullet-1}_{A/k}$ as the unique derivation of degree -1 sending da to $D(a)$ and a to zero. The following Cartan relations hold.

Theorem 2.1

$$[L_D, L_E] = L_{[D,E]}; \quad [L_D, \iota_E] = \iota_{[D,E]}; \quad [\iota_D, \iota_E] = 0; \quad [d, \iota_D] = L_D.$$

Moreover, since A is commutative, one can form the graded commutative algebra of algebraic multi-vector fields:

$$T^\bullet_{A/k} = \Lambda^\bullet_A Der(A),$$

with A in degree zero and $\mathrm{Der}(A)$ in degree 1. $T^\bullet_{A/k}[1]$ is a graded Lie algebra with the Nijenhuis-Schouten bracket $[\cdot, \cdot]$ which is a graded derivation in both variables and, for $a \in A$ and $D \in Der(A)$, satisfies

$$[D,a]=D(a),\ \ [D_1,D_2]=D_1D_2-D_2D_1.$$

In the case when A is the algebra of smooth functions $C^\infty(M)$ on a smooth manifold M, (Ω^*,d) is replaced by the de Rham complex of M and the role of $T^\bullet_{A/k}$ is played by the Lie algebra $\Gamma(M,\Lambda^\bullet TM)$ of multi-vector fields on M.

Remark 2.2 The whole package of the de Rham complex, multi-vector fields and the operations of Lie derivative, contraction and Cartan relations extends to the case of *graded commutative algebras*.

Non-Commutative Calculus

Noncommutative Differential Forms

When A is a not necessarily commutative algebra, the Lie algebra $Der(A)$ is defined but the algebra $\Omega^\bullet_{A/k}$ needs modification. One can drop the requirement that it has to be graded commutative. The result is a graded differential algebra $\Omega^{nc,\bullet}_{A/k}$ on which L_D, ι_D, and d act as above, and all the Cartan Eq. (19) hold. Unfortunately, the cohomology of d is trivial. In fact the map

$$a_0da_1\ldots da_{n+1}\mapsto a_0a_1da_2\ldots da_{n+1}+\sum_{k=1}^{n}(-1)^k a_0da_1\ldots d(a_ka_{k+1})da_{k+2}\ldots da_{n+1} \tag{8}$$

is a contracting homotopy for the complex $(\Omega^{nc,\bullet}_{A/k},d)$.

Remark 3.1 A meaningful noncommutative calculus based on noncommutative forms does exist; we discuss it briefly below. But first we are going to describe a related approach based on Hochschild and cyclic homology.

Cyclic Complexes

For an associative unital algebra A over a commutative unital ring k, define

$$C_\bullet(A)=A\otimes(A/k)^{\otimes\bullet} \tag{9}$$

$$b: C_\bullet(A)\to C_{\bullet-1}(A);\ \ B: C_\bullet(A)\to C_{\bullet+1}(A); \tag{10}$$

$$b(a_0\otimes\ldots\otimes a_n)=\sum_{i=0}^{n-1}(-1)^i a_0\otimes\ldots\otimes a_ia_{i+1}\otimes a_n+(-1)^n a_na_0\otimes\ldots\otimes a_{n-1}; \tag{11}$$

$$B(a_0\otimes\ldots\otimes a_n)=\sum_i(-1)^{i(n-i)}1\otimes a_i\otimes\ldots\otimes a_n\otimes a_0\otimes\ldots a_{i-1}. \tag{12}$$

One has

$$b^2=0;\ bB+Bb=0;\ B^2=0. \tag{13}$$

The B-differential appeared first in Rinehart (1963). The homology of the differential b is called the Hochschild homology of A and is denoted by $HH_\bullet(A)$.

Definition 3.2 *The negative cyclic homology* $HC^-_\bullet(A)$ *(resp. periodic cyclic homology* $HC^{per}_\bullet(A)$*) of an associative algebra A is the homology of the complexes:*

$$CC^-_\bullet(A)=(C_\bullet(A)[[u]],b+uB);\ \ CC^{per}_\bullet(A)=(C_\bullet(A)((u)),b+uB),$$

where u is a formal variable of degree -2.

Hochschild Cochain Complex

The role of noncommutative multi-vector fields is played by Hochschild cochains.

Definition 3.3 *Let A be a unital associative algebra over k. Set*

$$C^\bullet(A)=Hom_k(A^{\otimes\bullet},A) \tag{14}$$

with the differential $\delta: C^\bullet(A)\to C^{\bullet+1}(A)$ *given by*

$$\delta\phi(a_1,\ldots,a_{n+1})=a_1\phi(a_2,\ldots,a_{n+1})$$

$$+\sum_{k=1}^{n}(-1)^k\phi(a_1,\ldots,a_ka_{k+1}\ldots a_{n+1})+\phi(a_1,\ldots,a_n)a_{n+1}$$

The complex $(C^\bullet(A),\delta)$ *computes the groups* $Ext_{A\otimes A^{op}}(A,A)$, *usually called the Hochschild cohomology of A and denoted by* $HH^\bullet(A)$.

Note that the space of cocycles in $C^1(A)$ is $\mathrm{Der}(A)$. There is a differential graded Lie algebra structure on $C^\bullet(A)[1]$ that extends the commutator of derivations (the Gerstenhaber bracket). For later use, let us be more explicit. Let $\phi \in C^k(A)$ and $\psi \in C^l(A)$. Set

$$\phi \circ \psi = \phi \circ (\psi \otimes id \ldots \otimes id) + (-1)^{|\psi|} \phi \circ (id \otimes \psi \otimes id \ldots \otimes id) + \ldots + (-1)^{|\psi||\phi|} \phi \circ (id \otimes \ldots \otimes \psi)$$

and

$$[\phi, \psi] = \phi \circ \psi - (-1)^{|\psi||\phi|} \psi \circ \phi.$$

Here $|\cdot|$ refers to the Lie algebra degree, i. e., for $\phi \in C^k$, $|\phi| = k - 1$. $[\cdot, \cdot]$ is called the Gerstenhaber bracket and the following holds.

Theorem 3.4 *$(C^\bullet(A)[1], [\cdot, \cdot], \delta)$ is a differential graded Lie algebra.*

The commutative case

Suppose that A is a commutative algebra and k contains the rational numbers. Define the Hochschild-Kostant-Rosenberg map

$$\mathrm{HKR} : C_\bullet(A) \to \Omega^\bullet_{A/k}; \quad a_0 \otimes \ldots \otimes a_n \mapsto \frac{1}{n!} a_0 da_1 \ldots da_n, \; n \geq 0 \tag{15}$$

It is easy to see that HKR intertwines b with 0 and B with d. Moreover it extends to

$$\mathrm{HKR} : CC^-_\bullet(A) \to (\Omega^\bullet_{A/k}[[u]], ud); \quad CC^{per}_\bullet(A) \to (\Omega^\bullet_{A/k}((u)), ud) \tag{16}$$

Dually there is a HKR morphism of complexes

$$\mathrm{HKR} : \wedge T_{A/k} \to C^\bullet(A) \tag{17}$$

given by

$$\mathrm{HKR}(D_1 \ldots D_n)(a_1, \ldots, a_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} D_1(a_{\sigma(1)}) \ldots D_n(a_{\sigma(n)}) \tag{18}$$

Theorem 3.5 *Suppose that A is a commutative regular algebra. Then* HKR *induces an isomorphism* $HH_\bullet(A) \to \Omega^\bullet_{A/k}$ *and quasi-isomorphisms*

$$CC^-_\bullet(A) \to (\Omega^\bullet_{A/k}[[u]], ud)$$

$$CC^{per}_\bullet(A) \to (\Omega^\bullet_{A/k}((u)), ud)$$

$$(\wedge T^\bullet_{A/k}, 0) \to (C^\bullet(A), \delta).$$

Remark 3.6 In the case of $A = C^\infty(M)$, the same result holds after replacing algebraic tensor products with projective tensor products.

Operations on Cyclic Complexes

The Cartan Calculus of Derivations

Definition 4.1 *For a derivation D of A, set*

$$L_D(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} a_0 \otimes \ldots \otimes D(a_j) \otimes \ldots \otimes a_n$$

$$\iota_D(a_0 \otimes \ldots \otimes a_n) = a_0 D(a_1) \otimes a_2 \otimes \ldots \otimes a_n$$

$$S_{D^m}(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} \sum_{k=j}^{n} (-1)^{nj} L^m_D(a_j \otimes \ldots \otimes a_n) \otimes a_0 \otimes \ldots \otimes a_{j-1}$$

$$I_D = \iota_D + uS_D.$$

The following lemma is essentially due to Rinehart (1963)

Lemma 4.2 *The following Cartan relations are satisfied:*

$$[b + uB, L_D] = 0; \quad [L_D, L_E] = L_{[D,E]}; \quad [L_D, I_E] = I_{[D,E]}; \quad [b + uB, I_D] = uL_D \tag{19}$$

The missing Cartan relation

$$[I_D, I_E] = 0 \tag{20}$$

is true only at the level of homology.

Proposition 4.3 *For all* $n > 0$.

$$[b + uB, I_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} I_{D^k} I_{D^{n-k}} = uD^n$$

Remark 4.4 For comparison, define in the commutative case the operations

$$J_{D^n} = \iota_D,\ \ n = 1;\ J_{D^n} = 0,\ \ n > 1 \tag{21}$$

then

$$[ud, J_D] = uD;\ \ [ud, J_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} J_{D^k} J_{D^{n-k}} = 0,\ \ n > 1 \tag{22}$$

and, as a consequence, setting $\mathcal{J}(D) = \sum_{n=1}^{\infty} \frac{1}{n!} J_{D^n}$, we get

$$(ud + \mathcal{J}(D))^2 = uL_D \tag{23}$$

On the other hand, the relations from Proposition 4.3 are equivalent to

$$(b + uB + \mathscr{I}(D))^2 = u(e^D - 1) \tag{24}$$

where again

$$\mathscr{I}(D) = \sum_{n=1}^{\infty} \frac{1}{n!} I_{D^n} \tag{25}$$

This seems to be another instance of the appearance of the inverse Todd series $\frac{e^D - 1}{D}$, as it often happens in the comparison between commutative and non-commutative context.

One can pass from $\mathscr{I}(D)$ to $\mathcal{J}(D)$ as follows.

Lemma 4.5 *Define the Stirling numbers as the coefficients of the power series*

$$\sum_{k,l \geq 0} c_{k,l} x^k y^l = \sum_{n=1}^{\infty} \frac{1}{n!} y(y - x)\ldots(y - (n-1)x). \tag{26}$$

Let I_{D^n}, $n \geq 1$, *satisfy the relations from the proposition 4.3 and set*

$$\mathcal{J}(D) = \sum_{k,l \geq 0} c_{k,l} L_D^k I_{D^l}.$$

Then

$$(b + uB + \mathcal{J}(D))^2 = uL_D$$

Example 4.6 Let $\mathcal{A}$ be a differential graded algebra whose differential we denote by $d_{\mathcal{A}}$. Let α be a derivation of degree one of $\mathcal{A}$ and set

$$\mathrm{R} = (d_{\mathcal{A}} + \alpha)^2 = [d_{\mathcal{A}}, \alpha] + \frac{1}{2}[\alpha, \alpha].$$

Consider first the commutative case and the de Rham complex $\boldsymbol{\Omega}^{\bullet}_{\mathcal{A}/k}$ with the differential $d_{\mathcal{A}} + d$. Then

$$D_\alpha = d_{\mathcal{A}} + d + \iota_R \tag{27}$$

is again a differential. Equivalently,

$$D^u_\alpha = d_{\mathcal{A}} + ud - \frac{1}{u}\iota_R \tag{28}$$

is a differential on $\boldsymbol{\Omega}^{\bullet}_{\mathcal{A}/k}((u))$.

Now, more generally, assume that, in addition to the Lie algebra $\mathrm{Der}(\mathcal{A})$, a collection of J_{R^n} acts on a complex with differential $b + uB$, subject to (22). Then, formally, set

$$\mathbf{D}^u_\alpha = d_{\mathcal{A}} + ud - \Psi(R)\ \ \text{where}\ \ \Psi(R) = \sum_{n=1}^{\infty} \frac{u^{-n}}{n!} J_{R^n}. \tag{29}$$

One checks that $(\mathbf{D}^u_\alpha)^2 = 0$.

Let us now assume that instead of a collection of operators J_{R^n}, a collection of operators I_{R^n} acts subject to the equations of (22). For example, the complex could be $CC_{\bullet}^{per}(\mathcal{A})$. Looking for $\Phi(R)$ such that

$$(d_{\mathcal{A}} + b + uB - \Phi(R))^2 = 0, \tag{30}$$

we find

$$\Phi(R) = \sum_{k,l} c_{k,l} R^l I_{R^k} \tag{31}$$

where

$$\sum_{k,l} c_{k,l} y^l x^k = \left(1 + \frac{y}{u}\right)^{\frac{x}{y}} - 1 = \sum_{n=1}^{\infty} \frac{1}{n! u^n} x(x-y)\dots(x-(n-1)y) \tag{32}$$

We see that $\exp\left(\frac{x}{u}\right)$ gets replaced by $\left(1 + \frac{y}{u}\right)^{\frac{x}{y}}$.

Both operators require a convergence condition. For example, one might assume that k contains the rationals and the image of R is inside an ideal of $\mathcal{A}$. Or, one assumes that the image of R is contained in $p\mathcal{A}$ where $p > 2$ is a prime. In both cases, $b + uB + d_{\mathcal{A}} - \Phi(R)$ is a well-defined differential on the periodic cyclic complex completed with respect to the filtration induced by powers of the ideal (in the second case, this means p-adic completion).

Compatibility with the hochschild-kostant-rosenberg map

Theorem 4.7 *Suppose that A is commutative. There exists a natural (in D) $k[[u]]$-linear continuous morphism*

$$\mathrm{HKR}(D) : CC_{\bullet}^{-}(A)[[u]] \to \Omega_{A/k}^{\bullet}[[u]]$$

of the form $\mathrm{HKR}(D) = \mathrm{HKR} + \sum_{n=1}^{\infty} \mathrm{HKR}_{D^n}$, *where* HKR *is the quasi-isomorphism given in Theorem* 3.5 *and* HKR_{D^n} *are homogeneous of degree n in D and such that the following holds.*

$$(ud + \iota_D)\mathrm{HKR}(D) = \mathrm{HKR}(D)(b + uB + \mathcal{J}(D))$$

Remark 4.8 As in the commutative case, all of the above extends to (differential) graded algebras.

Noncommutative Calculus of Multi-Vector Fields and Forms

Return to the case of a commutative algebra A. Recall that $\wedge^{\bullet} T_{A/k}[1]$ carries a graded Lie algebra structure. The action by operators ι_D, $D \in Der(A)$ extends to an action by contraction of multi-vectors (multi-vector fields) so that $\Omega^{-\bullet}$ is a graded module over the graded algebra $\wedge^{\bullet} T_{A/k}$. Set, for $\alpha \in \Lambda^m T_{A/k}$

$$L_\alpha = [d, \iota_\alpha] : \Omega_{A/k}^{\bullet} \to \Omega_{A/k}^{\bullet - m+1} \tag{33}$$

Theorem 4.9 *The operation of Lie-derivative L makes $\Omega_{A/k}^{\bullet - m+1}$ into a module over the Lie algebra $\wedge^{\bullet} T_{A/k}[1]$. The following identities hold.*

(1) $[L_\alpha, L_\beta] = L_{[\alpha,\beta]}$;
(2) $[L_\alpha, \iota_\beta] = (-1)^{|\alpha|-1} \iota_{[\alpha,\beta]}$;
(3) $[\iota_\alpha, \iota_\beta] = 0$;
(4) $[d, \iota_\alpha] = L_\alpha$.

Moreover,

$$\iota_{\alpha\beta} = \iota_\alpha \iota_\beta; \quad L_{\alpha\beta} = L_\alpha \iota_\beta + (-1)^{|\alpha|} \iota_\alpha L_\beta \tag{34}$$

Suppose now that A is an associative algebra. The definitions of the operators L_D, ι_D, I_D as in 4.1, extend to the case of a general Hochschild cochain $D \in C^{\bullet}(A)$ as follows.

Definition 4.10 Let A be a graded vector space and D a Hochschild cochain on A. We set

$$L_D(a_0 \otimes \dots \otimes a_n) = D(a_0 \dots, a_d) \otimes a_{d+1} \otimes \dots \otimes a_n +$$
$$\sum_{k=0}^{n-d} \varepsilon_k a_0 \otimes \dots \otimes D(a_{k+1}, \dots, a_{k+d}) \otimes \dots \otimes a_n +$$
$$\sum_{k=n+1-d}^{n} \eta_k D(a_{k+1}, \dots, a_n, a_0, \dots) \otimes \dots \otimes a_k,$$

where the second sum in the above formula is taken over all cyclic permutations such that a_0 is inside D. The signs are given by

$$\varepsilon_k = (-1)^{(|D|+1)\sum_{i=0}^{k}(|a_i|+1)} \quad \text{and} \quad \eta_k = (-1)^{|D|+1+\sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)}$$

Definition 4.11 Let a A be a graded algebra. For $D \in C^d(A)$ we set

$$i_D(a_0 \otimes \ldots \otimes a_n) = (-1)^{|D|\sum_{i\le d}(|a_i|+1)} a_0 D(a_1,\ldots,a_d) \otimes a_{d+1} \otimes \ldots \otimes a_n$$

and

$$S_D(a_0 \otimes \ldots \otimes a_n) = \sum_{j\ge 0;\ k\ge j+d} \varepsilon_{jk} 1 \otimes a_{k+1} \otimes \ldots a_0 \otimes \ldots \otimes D(a_{j+1},\ldots,a_{j+d}) \otimes \ldots \otimes a_k$$

(The sum is taken over all cyclic permutations; a_0 appears to the left of D). The signs are given by

$$\varepsilon_{jk} = (-1)^{|D|(|a_0|+\sum_{i=1}^{n}(|a_i|+1))+(|D|+1)\sum_{j+1}^{k}(|a_i|+1)+\sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)}$$

As before, $I_D = \imath_D + uS_D$.

Proposition 4.12 (cf Daletskii *et al.*, 1990). *We set* $I_D = i_D + uS_D$. *Then*

$$[L_D, L_E] = 0 \quad [b+uB, I_D] = I_{\delta D} + (-1)^{|D|+1} L_D$$

The other Cartan relations hold at the level of pairing between $\mathrm{HH}^\bullet(A)$ *and* $\mathrm{HH}_\bullet(A)$, *but not on the level of chains and cochains (compare to Subsection "The Cartan Calculus of Derivations")*:

$$[L_D, \imath_E] = \imath_{[D,E]}; \quad [\imath_D, \imath_E] = 0$$

Definition 4.13 *For any differential graded Lie algebra* $\mathfrak{g}$, *let* $U^+(\mathfrak{g})$ *be the kernel of the augmentation* $U(\mathfrak{g}) \to k$. *Let* $\mathrm{Cobar}(U^+(\mathfrak{g}))$ *be the free associative algebra generated by* $U^+(\mathfrak{g})[1]$ *(the degree shift by one). We denote the free generator corresponding to* $x \in U^+(\mathfrak{g})$ *by* (x). *Define*

$$\partial_{\mathrm{Cobar}}(x) = \sum (-1)^{|x^{(1)}|}(x^{(1)})(x^{(2)}) \tag{35}$$

where the comultiplication is defined by

$$\Delta x = \sum x^{(1)} \otimes x^{(2)}$$

In addition, the differential $d_{\mathfrak{g}}$ *induces a differential on* $\mathrm{Cobar}(U^+(\mathfrak{g}))$. *Now define the dg algebra*

$$U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(U^+(\mathfrak{g}))[[u]] \tag{36}$$

as follows. It is an algebra over $k[[u]]$ *generated by the subalgebra* $(U(\mathfrak{g}), d_{\mathfrak{g}})$ *and the subalgebra* $\mathrm{Cobar}(U^+(\mathfrak{g}))$. *The only additional relations are*

$$[X, (x)] = [X, x], \quad X \in \mathfrak{g}, x \in U^+(\mathfrak{g}).$$

The differential acts as follows:

$$x \mapsto d_{\mathfrak{g}}x, x \in U(\mathfrak{g}); \quad (x) \mapsto (d_{\mathfrak{g}}x) + \partial_{\mathrm{Cobar}}(x) + ux, \quad x \in U^+(\mathfrak{g}). \tag{37}$$

The lemma 4.2 says that $\mathrm{Cobar}(U^+(\mathfrak{g}))$ acts naturally on $CC_\bullet^-(A)$ where $\mathfrak{g} = \mathrm{Der}(A)$. This has a straightforward extension as follows.

Proposition 4.14 *Let* A *be a unital algebra and let* $\mathfrak{g}$ *be a differential graded Lie subalgebra*

$$A[1] \rtimes Der(A) \subset C^\bullet(A)$$

(*consisting of zero-cochains and one-cocycles*). *Then* $\mathrm{Cobar}(U^+(\mathfrak{g}))$ *acts naturally on* $CC_\bullet^-(A)$.

Now let A be a $\mathbb{Z}/2\mathbb{Z}$-graded algebra. Suppose that τ is a $\mathrm{Der}(A)$-invariant supertrace on A. Let $\mathfrak{g}$ be as above. Given a Maurer-Cartan element λ of $\mathfrak{g}[u]$, i.e. an odd element satisfying the identity $d\lambda + \frac{1}{2}[\lambda, \lambda] = 0$, the composition $\#_\lambda = \tau \circ e^{\frac{\lambda}{u}}$ is defined as an element of $\mathrm{Hom}_{k[u]}(C_\bullet(A)[u], k[u, u^{-1}])$.

Proposition 4.15 *One has*

$$\#_\lambda \circ (b + uB) = 0.$$

Example 4.16 Let $\not{D}$ be an odd element of A and $R = \not{D}^2$. Set $\lambda = uR - u^{\frac{1}{2}}\mathrm{ad}\not{D}$. (This is a very slightly more general situation than the one above). The corresponding cyclic periodic cocycle, restricted to the subalgebra of even elements of A, is the Jaffe-Lesniewski-Osterwalder (JLO) cocycle (Jaffe *et al.*, 1988; Getzler and Szenes, 1989) of the form

$$(a_0, \ldots, a_{2n}) \mapsto \int_{\Delta^{2n}} \tau(a_0 e^{-t_0 R}[\not{D}, a_1] \ldots e^{-t_{2n-1}R}[\not{D}, a_{2n}]e^{-t_{2n}R})dt_1 \ldots dt_{2n}$$

where Δ^{2n} is the standard $2n$-simplex

$$\{(t_0 \ldots, t_{2n}) | \sum_i t_i = 1 \text{ and } t_i \geq 0,\ i = 0, \ldots, 2n.$$

Remark 4.17 The JLO cocycle is an infinite series that is not defined on periodic cyclic chains because those are infinite series themselves. It is defined on chains that are finite series in u, but the complex of those is homologically trivial (over the rationals). However, it is defined on suitable completions of the latter complex when A is a topological algebra (Connes, 1988; Meyer, 2007; Puschnigg, 1993).

Remark 4.18 Let F be an odd element of A satisfying $F^2 = 1$. A version of the proposition 4.15 can be applied to deduce the Connes cocycle (Connes, 1985)

$$\#\tau(F^{\wedge(n+1)})(a_0, \ldots, a_n) = \tau(Fa_0[F, a_1]\ldots[F, a_n])$$

Theorem 4.19 *The action described in the proposition 4.14 extends to a natural* A_∞ *action of* $\mathrm{Cobar}(U^+(\mathfrak{g}_A))$ on $CC_\bullet^-(A)$*where* $\mathfrak{g}_A = C^\bullet(A)[1]$ *is the differential graded Lie algebra of Hochschild cochains with the Gerstenhaber bracket.*

Remark 4.20 Over rational numbers there exist explicit formulas implementing this action.

Formality

Theorem 4.21 *(*Kontsevich, 2003; Dolgushev, 2006; Shoikhet, 2003, Tamarkin, 1999). *There exists an* L_∞ *quasi-isomorphism*

$$K : \wedge^{\bullet+1} T_M \to C^{\bullet+1}(\mathcal{O}_M, \mathcal{O}_M) \tag{38}$$

and a compatible quasi-isomorphism of L_∞ *modules over* $\wedge^{\bullet+1}(T_M)$

$$S : CC_\bullet^-(\mathcal{O}_M, \mathcal{O}_M) \to (\Omega_M^\bullet[[u]], ud) \tag{39}$$

Calculi

A part of non-commutative calculus involves the analog of the wedge product on multi-vector fields. While it does exist, it depends on a choice of an associator and is, in contrast to the above, non-canonical.

Definition 4.22 A Gerstenhaber algebra is a graded commutative algebra $\mathcal{A}$ together with a graded Lie algebra structure on $\mathcal{A}[1]$ such that

$$[\alpha, \beta\gamma] = [\alpha, \beta]\gamma + (-1)^{(|\alpha|-1)|\beta|}\beta[\alpha, \gamma]$$

for any homogeneous α, β, γ in $\mathcal{A}$.

Definition 4.23 (Tamarkin and Tsygan, 2000; Tamarkin and Tsygan, 2005; Dolgushev *et al.*, 2009). A calculus is a pair $(\mathcal{A}, \mathcal{M})$ together with the following data.

(1) a Gerstenhaber algebra structure on $\mathcal{A}$;
(2) a linear map

$$\iota : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$$

which gives $\mathcal{M}$ a structure of a module over the graded commutative algebra $\mathcal{A}$;

(3) a linear map

$$L : \mathcal{A}[1] \otimes \mathcal{M} \to \mathcal{M}$$

which gives $\mathcal{M}$ a structure of a module over the graded Lie algebra $\mathcal{A}[1]$;

(4) a linear map $d : \mathcal{M} \to \mathcal{M}$ of degree -1 such that

 (a) $L_{\alpha\beta} = L_\alpha \iota_\beta + (-1)^{|\alpha|}\iota_\alpha L_\beta$, $[L_\alpha, \iota_\beta] = (-1)^{|\alpha|-1} L_{[\alpha,\beta]}$;
 (b) $[d, \iota_\alpha] = L_\alpha$, $d^2 = 0$.

Example 4.24 *For a commutative algebra A, the pair* $(\wedge T_{A/k}, \Omega^{-\bullet}_{A/k})$ *is a calculus. For an associative algebra A, the pair* $(\mathrm{HH}^{\bullet}(A), \mathrm{HH}_{-\bullet}(A))$ *is a calculus* (Daletskii *et al.*, 1990).

There is a homotopy version of the calculus structure which we will denote by $Calc_\infty$ (cf. Tamarkin and Tsygan, 2005). The main formality result is the following.

Theorem 4.25 *There is a natural* Calc_∞ *structure on the pair* $(C^{\bullet}(A), C_{-\bullet}(A))$ *such that:*

(1) *the underlying* L_∞ *structure of* $C^{\bullet}(A)[1]$ *is equivalent to the one given by the Gerstenhaber bracket;*
(2) *the underlying* L_∞ *module structure of* $C_{-\bullet}(A)$ *over* $C^{\bullet}(A)[1]$ *is equivalent to the action given by operators* L_ϕ*;*
(3) *the* L_∞ *module structure over the odd Abelian Lie algebra kd is given by the cyclic differential B.*

Applications

Chern Character

One of the main applications of cohomology in the commutative case is the Chern character which gives pairing of topological K-theory with cohomology. The corresponding non-commutative analog is the Connes-Karoubi Chern character for topological algebras

$$K^{top}_*(A) \to HC^{per}_*(A)$$

and its algebraic counterpart from algebraic K-theory to negative cyclic homology. The natural map $K^{alg}(A) \to K^{top}(A)$ is a fibration and hence comes with an associated half-infinite exact sequence. The Chern character is a natural transformation between the two sequences,

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & K^{rel}_k(A) & \longrightarrow & K^{alg}_k(A) & \longrightarrow & K^{top}_k(A) & \longrightarrow & K^{rel}_{k-1}(A) & \longrightarrow \ldots . \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\ldots \longrightarrow & \mathrm{HC}^{\lambda}{}_{k-1}(A) & \longrightarrow & \mathrm{HC}^{-}_k(A) & \longrightarrow & \mathrm{HC}^{\mathrm{per}}_k(A) & \longrightarrow & \mathrm{HC}^{\lambda}{}_{k-2}(A) & \longrightarrow \ldots
\end{array}$$

As usual, homology is easier to compute then K-theory and the Chern character provides computational tools for getting information about K-theory. An important example is provided by the index theorems, see below. In this context one should note that the infinite cochains constructed in the proposition 4.15, while not in the dual of the cyclic periodic homology, have a well defined pairing to the image of Chern character.

The Gauss-Manin connection

Let S be an algebraic variety and $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$ -algebras. Then $\mathrm{CC}^{\mathrm{per}}_{\bullet}(\mathcal{A}/\mathcal{O}_S)$ (the periodic cyclic complex of $\mathcal{A}$ where the ring of scalars is $\mathcal{O}_S$) is a sheaf of complexes of $\mathcal{O}_S$-modules. In Getzler (1992), generalizing the classical results of (Grothendieck, 1968) and (Katz and Oda, 1968), Getzler constructed a flat connection on the homology sheaf $\mathrm{HC}^{per}_{\bullet}(\mathcal{A}/\mathcal{O}_S)$. (Literally speaking, one needs an assumption that the $\mathcal{O}_S$-module $\mathcal{A}$ admits a connection; this connection does not need to be flat or to preserve the product).

The Getzler-Gauss-Manin connection can be lifted to the level of complexes. Namely, there is a flat superconnection

$$\nabla^{\mathrm{GM}} : \Omega^{\bullet}_S \otimes_{\mathcal{O}_S} \mathrm{CC}^{per}_{\bullet}(\mathcal{A}/\mathcal{O}_S) \to \Omega^{\bullet}_S \otimes_{\mathcal{O}_S} \mathrm{CC}^{per}_{\bullet}(\mathcal{A}/\mathcal{O}_S)[1] \tag{40}$$

Cf. Dolgushev *et al.* (2011), Tsygan (2007) and Nest and Tsygan,. The construction combines the contents of Example 4.6 and Theorem 4.19.

Examples can be found in Yashinski (2017) and Yamashita (2017). Other approaches to and versions of the Gauss-Manin connection in noncommutative geometry are contained in Ginzburg and Schedler (2012), Kaledin (2009) and Petrov *et al.* (2018). For some examples see Yashinski (2017), Yashinski (2012)

Remark 5.1 Ginzburg and Schedler (2012), the periodic cyclic complex is shown to be quasi-isomorphic to a complex $\Omega^{\bullet,\mathrm{nc}}(A)((u)), d + \iota_\Delta$ where ι_Δ is the Ginzburg-Schedler differential. The differential ι_Δ commutes wirh Lie derivatives and contractions by derivations (because it is some sort of a contraction itself). Therefore the Cartan calculus of derivations extends to this version of the periodic cyclic complex. We do not know how to extend it to a Cartan calculus of higher Hochschild cochains (although this can be helped by passing to a semi-free resolution of A), nor to the Ginzburg and Schedler versions of the Hochschild and negative cyclic complexes. For a different approach to Gauss-Manin connection see Petrov and Vologodsky (2019)

Noncommutative Hodge Structures

In algebraic geometry, the De Rham cohomology of a smooth projective complex algebraic variety carries a pure Hodge structure. In noncommutative geometry, periodic cyclic homology replaces De Rham cohomology, and there is a notion of a smooth proper DG category. Conjecturally, the periodic cyclic homology of such a category carries a noncommutative Hodge structure. The latter is defined in Katzarkov *et al.* (2008). There are two ingredients of a (classical) pure Hodge structure: the integral (or rational) lattice and the Hodge filtration. In the noncommutative setting, what replaces the filtration is a variant of the Gauss-Manin connection (Katzarkov *et al.*, 2008; Shklyarov, 2014).

The Goodwillie Rigidity

Let $\star_1$ and $\star_2$ be two multiplication laws on the same k-module A. Assume for all a and b in A that $a\star_1 b - a\star_2 b \in I$ where I is a pro-nilpotent ideal with respect to either multiplication.

Theorem 5.2 *Let k contain the rationals. There is a natural isomorphism of complexes*

$$T(\star_1, \star_2): \quad CC_\bullet^{per}(A, \star_1)\hat{}\, \overset{\sim}{\leftarrow} CC_\bullet^{per}(A, \star_2)\hat{}$$

where $\hat{}$ *stands for the completion with respect to the ideal.*

Furthermore, $T(\star_1, \star_2)T(\star_2, \star_3)$ is homotopic to $T(\star_1, \star_3)$, and there are higher homotopies. More precisely: there is an A_∞ functor from the category whose morphisms are products on A congruent to one another modulo I, with exactly one morphism between any two objects, to the category of complexes.

Theorem 5.2 is a refinement of Goodwillie's theorem: if k contains $\mathbb{Q}$ then for a pro-nilpotent ideal I of A, the projection

$$CC_\bullet^{per}(A)\hat{}\, \to CC_\bullet^{per}(A/I)$$

is a quasi-isomorphism.

Deformation Quantization

Formality

Theorem 5.3 *Let M be a smooth manifold. There exists an* L_∞ *quasi-isomorphism*

$$K: \wedge^{\bullet+1} T_M \to C^{\bullet+1}(\mathcal{O}_M, \mathcal{O}_M) \tag{41}$$

and a compatible quasi-isomorphism of L_∞ *modules over* $\wedge^{\bullet+1}(T_M)$

$$S: CC_\bullet^-(\mathcal{O}_M, \mathcal{O}_M) \to (\Omega_M^\bullet[[u]], ud) \tag{42}$$

The existence of one such quasi-isomorphism for cochains is the formality theorem of Kontsevich (Kontsevich (2003)). The existence for chains is proven in Shoikhet (2003) and Dolgushev (2006). The fact that those quasi-isomorphisms are parametrized by Drinfeld associators follows from Tamarkin (1999) and Kontsevich (1999) (for cochains) and from the statement and proof of Theorem 4.25 (for chains).

Classification and cyclic homology of deformation quantization algebras

Definition 5.4 A formal Poisson structure on M is a formal series

$$\pi = \pi_0 + h\pi_1 + \ldots \in \wedge^2 T_M[[h]]$$

satisfying $\{\pi, \pi\} = 0$.

For a formal Poisson structure π put

$$\Pi_{\pi,\Phi} = \sum_{k=1}^{\infty} \frac{1}{2^k k!} K_{\Phi,k}(h\pi, \ldots, h\pi) \tag{43}$$

(k arguments $h\pi$).

Lemma 5.5 *Let*

$$f \star_{\pi,\Phi} g = fg + \Pi_{\pi,\Phi}(f, g)$$

Then $\star_{\pi,\Phi}$ *is a deformation quantization of* π_0.

The 2-cochain $\Pi_{\pi,\Phi}$ is an MC element of $C^2(\mathcal{O}_M, \mathcal{O}_M)[[h]]$, and we write

$$\mathcal{O}_{\pi,\Phi,M} = (\mathcal{O}_M[[h]], \star_{\pi,\Phi}) \tag{44}$$

For first appearences of deformation quantization see Bayen *et al.* (1978a,b)

Lemma 5.6

$$C^\bullet(\mathcal{O}_{\pi,\Phi,M}, \mathcal{O}_{\pi,\Phi,M}) \xrightarrow{\sim} (C^\bullet(\mathcal{O}_M, \mathcal{O}_M)[[h]], \delta + [\Pi_{\pi,\Phi},])$$

$$\mathrm{CC}^-_\bullet(\mathcal{O}_{\pi,\Phi,M}) \xrightarrow{\sim} (\mathrm{CC}^-_\bullet(\mathcal{O}_M)[[h]], b + L_{\Pi_{\pi,\Phi}} + uB)$$

Algebraic Index Theorem

Theorem 5.7

$$\begin{array}{ccc} \mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{O}_{\pi,\Phi,M}) & \xrightarrow{\mathrm{TR}_{\pi,\Phi}} & \Omega^\bullet_M[[h]]((u)) \\ \downarrow I & & \downarrow \sqrt{\hat{A}_{u,\Phi}(TM)} \\ \mathrm{CC}^{\mathrm{per}}_\bullet(\mathcal{O}_M[[h]]) & \xrightarrow{\mathrm{HKR}} & \Omega^\bullet_M[[h]]((u)) \end{array}$$

is homotopy commutative. Here I is the Goodwillie rigidity isomorphism,

$$\hat{A}_{u,\Phi}(T_M) = \sum_{k \geq 0} u^{-k} \hat{A}_{\Phi,2k}(T_M)$$

and

$$\hat{A}_\Phi(T_M) = \sum_{k \geq 0} \hat{A}_{\Phi,2k}(T_M)$$

is the characteristic class of the tangent bundle that is defined by an invariant power series $\hat{A}_\Phi$ whose restriction from $\mathfrak{g}_{2n}$ to $\mathfrak{sp}_{2n}$ *is* $\hat{A}$.

Proofs of the Poisson case can be deduced from (Kontsevich and Soibelman, 2000; Shoikhet, 2003; Dolgushev, 2005; Willwacher, 2011; Van Den Bergh *et al.*, 2012; Willwacher, 2016). When π is symplectic, this is the algebraic index theorem of Fedosov (1996). In the case when the Poisson structure has constant rank, this is the algebraic index theorem for families cf. Nest and Tsygan (1995a,b). The Atiyah-Singer index theorem can be deduced from it (Nest and Tsygan, 1996). For the case $M = T^*\mathbb{R}^n$, cf (Elliott *et al.*, 1996).

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