The Riemann-Hilbert Problem

Xinxing Tang

Abstract. This survey is an introduction to the (classical) Riemann-Hilbert problem: the problem of finding a linear differential equation of Fuchsian type with given singular points and prescribed monodromy representation.

Contents

1. Introduction

In 1850s, Riemann introduced the concept of "local system" on a punctured $\mathbb{P}^1$. He claimed that one could (and should) study the solutions of an N-th order linear differential equation via studying the corresponding rank N local system. He knew that a rank N local system on $\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}$ was just a collection of n invertible $N \times N$ matrices M_i $(i = 1, \ldots, n)$ with the constraints that the product (in a certain order) of all of them is the identity matrix. He also understood that each M_i, up to $GL_N(\mathbb{C})$ conjugation, has the effect of the analytic continuation of the solution along a small loop around the singular point.

He applied this idea to the study of the Gauss hypergeometric function via studying the corresponding rank 2 local system on $\mathbb{P}^1$ with three punctures removed and obtained a stunning success (It is pertinent to emphasize that this local system examined by Riemann is rigid).

In 1900, inspired by Riemann's work, Hilbert [29, 30] posed a problem on the existence of a Fuchsian equation with given singular points and monodromy matrices M_i. Nowadays, this problem is known as the Riemann-Hilbert problem.

To have a right understanding of the (classical) Riemann-Hilbert problem (RHP), in Section 2, we start from the theory of differential systems on $\mathbb{P}^1$ with regular or Fuchsian (which is stronger than regular) singularities and talk about the corresponding monodromy representation. The Riemann-Hilbert problem is also known as the inverse monodromy problem. In Section 3, We will give a historic review of the Riemann-Hilbert problem. In particular, of Plemelj's work on it, the first counterexample given by Bolibrukh and some asymptotic analysis ideas for this problem. Then we will talk about the asymptotic analysis of RHP via pants decomposition. Next, some related problems are briefly introduced. In Section 4, we will briefly talk about the Riemann-Hilbert correspondence which contains both the monodromy problem and the inverse monodromy problem in terms of the geometric language in a general setting. In Section 5, we will briefly talk about the parameter dependence problem which includes the isomonodromy problem and the monodromy dependence. Finally, we end with the example of $N = 2$ 4-point Fuchsan system, its middle convolution and the isomonodromy equation (Painlevé VI equation).

There are surveys on RHP and its modern applications, for example, [10, 12, 27, 55].

2. Monodromy Representation

2.1 Singular Points and Local Fundamental Solutions

Consider a linear system

$$\frac{d\mathbf{y}}{dz} = A(z)\mathbf{y}, \quad \mathbf{y} = (y_1(z), \ldots, y_N(z))^T, \tag{2.1}$$

with $A(z)$ being a $N \times N$ meromorphic map on $\mathbb{P}^1$.

Definition 2.1. Let $a \in \mathbb{C}$ be a singular point of $A(z)$. It is called a *regular singular* point if in every sectorial neighbourhood V of a containing no other singular points of (2.1), there is an analytic fundamental solution $\Phi(z)$ of (2.1) of *moderate growth*. That is, there exists λ_V, $C_V \in \mathbb{R}$ such that each matrix element $\Phi_{ij}(z)$ of $\Phi(z)$ satisfies

$$|\Phi_{ij}(z)| < C_V |z-a|^{\lambda_V}, \quad z \in V.$$

Otherwise, a is called an *irregular singular* point.

Definition 2.2 (Poincaré rank)**.** Assume $z = 0$ is a singular point of the matrix function $A(z)$ in (2.1). If $A(z)$ is of the form

$$A(z) = \frac{B(z)}{z^{r+1}}, \quad r \in \mathbb{N},$$

where $B(z)$ is holomorphic at $z = 0$ and $B(0) \neq 0$. Then, the Poincaré rank of $z = 0$ is defined to be r. If $r = 0$, then $z = 0$ is called a *Fuchsian singular point.*

Theorem 2.3. *Assume $z = 0$ is a Fuchsian singular point of (2.1), that is, we can write the system as*

$$z\frac{d\mathbf{y}}{dz} = B(z)\mathbf{y},$$

where $B(z)$ is holomorphic at $z = 0$ and $B(0) \neq 0$. Furthermore, assume that $B(0)$ has distinct eigenvalues modulo non-zero integers (non-resonance condition)[1]*, then the system (2.1) has a fundamental solution around $z = 0$ of the following form*

$$\Phi_0(z) = (I + O(z))z^{B(0)},$$

[1] In some papers, the notion of strongly non-resonant is used, i.e., $\lambda_a - \lambda_b \notin \mathbb{Z}$.

where $I + O(z)$ is a power series of z with matrix-valued coefficients and it is convergent to a holomorphic matrix-valued map in a neighborhood of $z = 0$.

If $B(0) = G_0 \operatorname{diag}(\lambda_1, \ldots, \lambda_n) G_0^{-1}$, then (2.1) has a fundamental solution around $z = 0$ of the following form

$$\Phi_0(z) = G_0(I + O(z)) z^{\operatorname{diag}(\lambda_1, \ldots, \lambda_n)}.$$

More generally, the non-resonance condition could be removed using meromorphic gauge transformations.

Theorem 2.4. *Assume $z = 0$ is a Fuchsian singular point of (2.1), then the system (2.1) has a fundamental solution around $z = 0$ of the following form*

$$\Phi_0(z) = P(z) z^E, \quad E \in gl_N(\mathbb{C}),$$

where $P(z)$ is holomorphic at $z = 0$ and the eigenvalues of E do not differ by non-zero integers.

Then, we see immediately that

Proposition 2.5. *The Fuchsian singular points are regular singular.*

The converse is not true.

Example 2.6. For the following differential system

$$\frac{d\mathbf{y}}{dz} = \begin{pmatrix} 1/z & 1 \\ 0 & 0 \end{pmatrix} \mathbf{y},$$

- $z = 0$ is a Fuchsian singular point;
- $z = \infty$ is a regular singular point but not Fuchsian (via the variable change $w = \frac{1}{z}$).

Remark 2.7. There is another equivalent way to define the regular singular points. Let $z = 0$ be a singular point of (2.1), we call $z = 0$ regular singular, if there exists $P(z) \in GL_N(\mathbb{C}(\{z\}))$[2] such that the transformed system

$$\frac{d\widetilde{\mathbf{y}}}{dz} = \widetilde{A}(z)\widetilde{\mathbf{y}}(z), \quad \widetilde{A}(z) = P^{-1}(z)A(z)P(z) - P^{-1}(z)P'(z)$$

under the meromorphic transformation

$$\widetilde{\mathbf{y}}(z) = P(z)\mathbf{y}(z)$$

has a simple pole at $z = 0$.

[2] $\mathbb{C}(\{z\})$ is the field of germs of meromorphic functions at 0.

2.1.1 Levelt Fundamental Solution

A necessary and sufficient condition for a regular singular point being Fuchsian can be also read from the fundamental solution of Levelt form and the Fuchs relation. By Theorem 2.4 and Remark 2.7, we have

Proposition 2.8. *Let $z=0$ be a regular singular point of (2.1). Then the system has a fundamental solution of the form*

$$Q(z)z^E, \quad \text{as } z\to 0,$$

where

- *$Q(z)$ is meromorphic at $z=0$;*
- *E is a matrix with distinct eigenvalues modulo non-zero integers.*

Note that for any $E\in gl_N(\mathbb{C})$, the entries of z^E are of the form

$$\left(z^E\right)_{ij}=\sum_{l\geq 1} z^{\rho_l}P_{ijl}(\log z)$$

where P_{ijl} are polynomials of degree less than N and $\{\rho_l\}$ are eigenvalues of E. Without loss of generality, in the following, one can re-express the form $Q(z)z^E$ such that the eigenvalues E $\{\rho_i\}_{i=1}^N$ of E satisfy $0\leq \operatorname{Re}\rho_i<1$.

Definition 2.9 (Levelt valuation)**.** Let Sol_0 be the space of local solutions of (2.1) near 0 and let K be the following differential field extension of $\mathbb{C}(\{z\})$:

$$K=\mathbb{C}(\{z\})\left(z^{\rho_1},\cdots,z^{\rho_N},\log z\right).$$

(1) Define the valuation $v: K\to\mathbb{Z}\cup\{\infty\}$ on K as follows:

1. $v(0)=\infty$;
2. $v\left(f(z)z^{\rho_k}(\log z)^m\right):=d$ if $f(z)=\sum_{i\geq d} a_i z^i\in\mathbb{C}(\{z\})$, $a_d\neq 0$.

(2) The Levelt valuation on the local solution space Sol_0 is defined to be

$$\begin{aligned} v:\ & \mathrm{Sol}_0\to\mathbb{Z}\cup\{\infty\} \\ & v((\mathbf{y}))=\min_{1\leq j\leq N} v(y_j), \quad \mathbf{y}=(y_1,\ldots,y_N)^T\in \mathrm{Sol}_0\,. \end{aligned}$$

Similarly, if $\Phi(z)$ is a fundamental solution near $z=0$, then one can also define $v(\Phi)=\min_{ij}\{v(\Phi_{ij}(z))\}$.

Theorem 2.10 (Levelt [48])**.**

(1) If $z=0$ is a regular singular point of (2.1), then there exists a solution of the form

$$\Phi(z)=U(z)z^B z^E, \tag{2.2}$$

where

- *U is holomorphic at $z=0$;*
- $B=\operatorname{diag}(\nu(e_1),\nu(e_2),\ldots,\nu(e_N))$*, $\{e_i\}$ is a basis of the solution and $\nu(e_i)\geq\nu(e_{i+1})$, $i=1,\ldots,N-1$;*
- *E is an upper block triangular matrix with eigenvalues $\{\rho_i\}$, $0\leq\operatorname{Re}\rho_i<1$, $i=1,\ldots,N$.*

(2) The regular singular point $z=0$ is Fuchsian if and only if $U(0)$ is invertible.

A fundamental solution of (2.1) around $z=0$ of the form (2.2) is called a Levelt fundamental solution. One can further define

$$\beta_k=\nu(e_k)+\rho_k,\quad k=1,\ldots,N,$$

which are called the Levelt exponents at $z=0$.

If all the singular points of $A(z)$ are Fuchsian, i.e., all the poles of $A(z)$ are simple poles, then we obtain the Fuchsian system of the form

$$\frac{d\mathbf{y}}{dz}=\sum_{i=1}^{n}\frac{A_i}{z-a_i}\mathbf{y},\quad A_i\in gl_N(\mathbb{C}). \tag{2.3}$$

In particular, if $A_\infty:=-\sum_{i=1}^{n}A_i\neq0$, then ∞ is also a Fuchsian singular point of this system. In the Fuchsian case, the local exponents at $z=a_i$ are the eigenvalues of A_i.

Theorem 2.11 (Fuchs relation)**.** *Let $\{a_1,a_2,\ldots,a_n\}\subset\mathbb{P}^1$ be the set of singular points of (2.1) and all assumed to be regular singular. Let $\{\beta_{ij}\}_{j=1}^{N}$ be the Levelt exponents at $z=a_i$, $i=1,\ldots,n$. Then*

$$\sum_{i=1}^{n}\sum_{j=1}^{N}\beta_{ij}\leq0.$$

In particular, $\sum_{i=1}^{n}\sum_{j=1}^{N}\beta_{ij}=0$ if and only if (2.1) is a Fuchsian system.

Remark 2.12. If the system (2.1) has an irregular singularity $z=a$, then around $z=a$, there exists a unique formal fundamental solution $\widehat{\Phi}(z)$ of the form

$$\widehat{\Phi}(z)=G_a(I+O(z-a))e^{T_a(z)},$$

where $T=T_0\log(z-a)+\sum_{k=1}^{r_a}\frac{T_{-k}(z-a)^k}{-k}$ is generically diagonal, r_a is the Poincaré rank of a. However, the power series part $I+O(z-a)$ is in general divergent. Thus, the formal solution just represents the asymptotic behavior of actual solutions $\Phi^{(j)}(z)$ in some sectors $S^{(j)}$ around $z=a$. The actual solutions in different sectors are related by so-called (constant) Stokes matrices. The monodromy representation and the corresponding inverse problem in this case is complicated and we will not discuss irregular singularities in this survey.

2.2 Fuchsian System and Monodromy Representation

Now, let us focus on the Fuchsian system (2.3) and assume that $A_\infty \neq 0$. Then by Theorem 2.4, there is a local fundamental solution around each singular point $z = a_i$ given by

$$\Phi_i(z) = \begin{cases} P_i(z-a_i)(z-a_i)^{E_i}, & a_i \neq \infty; \\ P_\infty(z^{-1})z^{-E_\infty}, & a_i = \infty. \end{cases}$$

Globally, we want to see how the local fundamental solutions change when it is analytically continued around the singular points. Choose a path l_i from ∞ to each a_i. Then, we can continue Φ_∞ along the path l_i and compare the result to Φ_i. This defines a connection matrix C_i. If we fix $z = \infty$ to be the reference point, the global solution $\Phi(z)$, obtained by analytic continuation, has the following asymptotic expansion around the singular points $z = a_i$ and $z = \infty$

$$\Phi(z) = \begin{cases} P_i(z-a_i)(z-a_i)^{E_i}C_i, & z \to a_i; \\ P_\infty(z^{-1})z^{-E_\infty}, & z \to \infty. \end{cases} \tag{2.4}$$

In particular, if $A_i = G_i \Theta_i G_i^{-1}$, with Θ_i being a diagonal matrix and its eigenvalues satisfy the non-resonance condition, then the fundamental solution near a_i can also be written as

$$\Phi_i(z) = \begin{cases} G_i(I + O(z-a_i))(z-a_i)^{\Theta_i}, & a_i \neq \infty; \\ G_\infty(I + O(z^{-1}))z^{-\Theta_\infty}, & a_i = \infty. \end{cases}$$

Via analytic continuation along l_i, we can write $\Phi(z)$ as

$$\Phi(z) = \begin{cases} G_i(I + O(z-a_i))(z-a_i)^{\Theta_i}C_i, & z \to a_i; \\ G_\infty(I + O(z^{-1}))z^{-\Theta_\infty}, & z \to \infty. \end{cases} \tag{2.5}$$

Let γ be a based loop, and Φ^{γ_i} be the analytic continuation of Φ around γ, then there is an invertible matrix M_γ such that

$$\Phi^\gamma = \Phi M_\gamma.$$

M_γ is called the monodromy matrix associated to γ. In particular, it only depends on γ via its homotopy class.

Let z_0 be a based point and γ_i be the based simple loop around a_i, then the corresponding monodromy M_i of $\Phi(z)$ in (2.4) (respectively in (2.5)) is given by

$$M_i = C_i^{-1} e^{2\pi i E_i} C_i, \; i = 1, \dots, n; \quad M_\infty = e^{2\pi i E_\infty}.$$
$$(M_i = C_i^{-1} e^{2\pi i \Theta_i} C_i, \; i = 1, \dots, n; \quad M_\infty = e^{2\pi i \Theta_\infty} \text{ corresponding to (2.5).})$$

Since the path around all singularities is contractible, that is

$$\gamma_1 \cdot \gamma_2 \cdots \gamma_n \cdot \gamma_\infty = 1^3,$$

these matrices are subjected to the relation:

$$M_\infty M_n \cdots M_2 M_1 = I.$$

More precisely, we get an anti-representation

$$\begin{aligned} \rho : \pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n, \infty\}, z_0) &\longrightarrow GL_N(\mathbb{C}), \\ \gamma_i &\longmapsto M_i. \end{aligned} \tag{2.6}$$

Remark 2.13. In some papers, one defines the product of γ_1 and γ_2 in an inverse way, or writes $\Phi(z-a_i) = \Phi((z-a_i)e^{2\pi i})T_i$, and then obtains a representation $\pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n, \infty\}, z_0) \to GL_N(\mathbb{C})$. In this survey, we do not distinguish these cases and just call them the monodromy representations.

2.2.1 Singular Data and Monodromy Data

Furthermore, let us consider the constant gauge transformation of the fundamental matrix solution

$$\widetilde{\Phi} = K\Phi K^{-1}, \quad K \in GL_N(\mathbb{C}).$$

Then $\widetilde{\Phi}$ satisfies

$$\frac{d\widetilde{\Phi}}{dz} = \widetilde{A}(z)\widetilde{\Phi}(z), \quad \widetilde{A}(z) = KA(z)K^{-1}, \ i.e.\ \widetilde{A}_i = KA_iK^{-1},$$

and for each i, the monodromy matrix $\widetilde{M}_i(z)$ of $\widetilde{\Phi}(z)$ around γ_i is given by

$$\widetilde{M}_i = KM_iK^{-1}.$$

Thus, we consider the two following moduli space associated to the $N \times N$ Fuchsian systems with n singular points. The first one is

$$\begin{aligned} \mathcal{A} &= \left\{ (A_1, \ldots, A_n) \mid A_j \in gl_N(\mathbb{C}), \sum_{j=1}^n A_j = 0 \right\} / GL_N(\mathbb{C}) \\ &\approx \left\{ (G_j, \Theta_j)_{j=1}^n \mid G_j \in GL_N(\mathbb{C}), \Theta_j \in \mathrm{diag}_N^{nr}(\mathbb{C}), \sum_{j=1}^n G_j \Theta_j G_j^{-1} = 0 \right\} / \sim \end{aligned}$$

[3] The product of the based loops γ_1, γ_2 is defned by

$$\gamma_1 \cdot \gamma_2 = \begin{cases} \gamma_1(2t), & \text{if } 0 \le t \le 1/2; \\ \gamma_2(2t-1), & \text{if } 1/2 \le t \le 1. \end{cases}$$

where $\approx$ means that generically, when $A_j = G_j \Theta_j G_j^{-1}$, with $\Theta_j \in \mathrm{diag}_N^{nr}(\mathbb{C})$ and $\mathrm{diag}_N^{nr}(\mathbb{C})$ denotes the set of diagonal matrices with eigenvalues satisfying non-resonance condition, we have equality. The equivalence condition is given by the $GL_N(\mathbb{C})$-action $G_j \mapsto KG_j$.

The second one is

$$\begin{aligned}\mathcal{M} &= \left\{(M_1,\ldots,M_n) \,\middle|\, M_j \in GL_N(\mathbb{C}), M_n \cdots M_1 = I\right\}/GL_N(\mathbb{C})\\ &\approx \left\{(C_j,\Theta_j)_{j=1}^n \,\middle|\, C_j \in GL_N(\mathbb{C}), \Theta_j \in \mathrm{diag}_N^{nr}(\mathbb{C}), C_n e^{2\pi i \Theta_n} C_n^{-1} \cdots C_1 e^{2\pi i \Theta_1} C_1^{-1} = I\right\}/\sim,\end{aligned}$$

where the equivalence is given by the $GL_N(\mathbb{C})$-action $C_j \mapsto SC_j$.

The monodromy representation gives us a map from $\mathcal{A}$ to $\mathcal{M}$. If $\{\lambda_{j,\alpha}\}_\alpha$ are eigenvalues of A_j, then $\{\exp(2\pi i \lambda_{j,\alpha})\}_\alpha$ are the eigenvalues of M_j. In particular, under the non-resonance condition, M_j is conjugate to $\exp^{2\pi i A_j}$, and the corresponding two spaces have the same dimension. Thus one can expect an one-to-one correspondence between $\mathcal{A}$ and $\mathcal{M}$. The inverse monodromy problem is known as the Riemann-Hilbert space, see the details in the next section.

Remark 2.14. We say that the corresponding irreducible linear system has rigid monodromy, if the subset with fixed local exponents of $\mathcal{M}$ is a point. This means that the global behaviour of the solutions under analytic continuation is determined by the local behaviour at the singularities. Most local systems are not rigid, but there are typical examples:

- Rank N irreducible linear systems on $\mathbb{P}^1 \setminus \{a_1, a_2, a_3\}$, whose monodromies M_i has N distinct eigenvalues, are rigid only if $N = 1, 2$;
- The classical functions such as ${}_nF_{n-1}$ [5] and the Pochhammer hypergeometric functions [28] solve the corresponding higher order differential equations for rigid local systems.

2.3 Basic Example

Now, let us consider the following simple example,

$$\frac{d\mathbf{y}}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right)\mathbf{y}, \quad A_0, A_1 \in gl_2(\mathbb{C}), \tag{2.7}$$

where

- The matrix A_0 has eigenvalue $\{0, 1-\gamma\}$, $\gamma \notin \mathbb{Z}$;
- The matrix A_1 has eigenvalue $\{0, \gamma-\alpha-\beta-1\}$ with $\gamma-\alpha-\beta \notin \mathbb{Z}$;
- $A_\infty = -A_0 - A_1 = \mathrm{diag}(\alpha, \beta)$ with $\alpha - \beta \notin \mathbb{Z}$.

The system (2.7) is uniquely determined by the above constraints up to a diagonal conjugation

$$A_0 \to T^{-1} A_0 T, \quad A_1 \to T^{-1} A_1 T, \quad T = \mathrm{diag}(1, r),\ r \neq 0.$$

Lemma 2.15. *Let* $\mathbf{y} = (y_1, y_2)^T$ *be a solution of* (2.7). *Then* y_1 *satisfies the Gauss hypergeometric differential equation*

$$z(1-z)\frac{d^2y_1}{dz^2} + [c-(a+b+1)z]\frac{dy_1}{dz} - aby_1 = 0, \tag{2.8}$$

where $a = \alpha$, $b = \beta+1$, $c = \gamma$. *And* y_2 *can be solved via the component equation of* (2.7) *and* y_1.

Thus, the local solution of (2.7) around $z = 0, 1, \infty$ is given in terms of the differential expansion of hypergeometric function ${}_2F_1(z)$, and the connection matrix is given by the Kummer relations. The fundamental solution normalized at ∞ is given by

$$\begin{pmatrix} F(\alpha,\alpha-\gamma+1,\alpha-\beta|\frac{1}{z}) & \frac{\beta(\beta-\gamma+1)}{(\beta-\alpha)(\beta-\alpha+1)}\frac{1}{z}F(\beta+1,\beta-\gamma+2,\beta-\alpha+2|\frac{1}{z}) \\ \frac{\alpha(\alpha-\gamma+1)}{(\alpha-\beta)(\alpha-\beta+1)}\frac{1}{z}F(\alpha+1,\alpha-\gamma+2,\alpha-\beta+2|\frac{1}{z}) & F(\beta,\beta-\gamma+1,\beta-\alpha|\frac{1}{z}) \end{pmatrix}$$
$$\times z^{-\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}}.$$

The global solution can be written as

$$\Phi(z) = \begin{cases} G_0(I+O(z))z^{\mathrm{diag}(1-\gamma,0)}C_0, & z\to 0;\\ G_1(I+O(z-1))(z-1)^{\mathrm{diag}(\gamma-\alpha-\beta-1,0)}C_1, & z\to 1;\\ (I+O(z^{-1}))z^{-\mathrm{diag}(\alpha,\beta)}, & z\to\infty \end{cases}$$

where

$$G_0 = \frac{1}{\beta-\alpha}\begin{pmatrix}\beta-\gamma+1 & \beta\\ \alpha-\gamma+1 & \alpha\end{pmatrix}, \qquad G_1 = \frac{1}{\beta-\alpha}\begin{pmatrix}1 & \beta(\beta-\gamma+1)\\ 1 & \alpha(\alpha-\gamma+1)\end{pmatrix},$$
$$C_0 = \begin{pmatrix} e^{-\pi i(\alpha-\gamma+1)}\frac{\Gamma(\gamma-1)\Gamma(\alpha-\beta+1)}{\Gamma(\gamma-\beta)\Gamma(\alpha)} & -e^{-\pi i(\beta-\gamma+1)}\frac{\Gamma(\gamma-1)\Gamma(\beta-\alpha+1)}{\Gamma(\gamma-\alpha)\Gamma(\beta)} \\ e^{-\pi i\alpha}\frac{\Gamma(1-\gamma)\Gamma(\alpha-\beta+1)}{\Gamma(1-\gamma)\Gamma(\alpha-\gamma+1)} & -e^{-\pi i\beta}\frac{\Gamma(1-\gamma)\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)\Gamma(\beta-\gamma+1)} \end{pmatrix},$$
$$C_1 = \begin{pmatrix} -\frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(\alpha-\beta+1)}{\Gamma(\alpha-\gamma+1)\Gamma(\alpha)} & \frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\gamma+1)\Gamma(\beta)} \\ -e^{-\pi i(\gamma-\alpha-\beta-1)}\frac{\Gamma(\gamma-\alpha-\beta-1)\Gamma(\alpha-\beta+1)}{\Gamma(1-\beta)\Gamma(\gamma-\beta)} & e^{-\pi i(\gamma-\alpha-\beta-1)}\frac{\Gamma(\gamma-\alpha-\beta-1)\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)\Gamma(\gamma-\alpha)} \end{pmatrix}.$$

Remark 2.16.

1. The monodromy group acts reducibly on the space of solutions, if and only if at least one of a, b, $c-a$, $c-b$ is an integer.
2. When the monodromy representation is irreducible, the equivalent class of the representation is determined by the eigenvalues of M_0, M_∞ and M_0M_∞, here it depends only on $a, b, c \mod \mathbb{Z}$.

Remark 2.17 (The case of $a, b, c \in \mathbb{R}$). In this case the eigenvalues of the monodromy matrices lie in S^1.

1. Suppose the monodromy group is irreducible. Let F be the invariant hermitian form for M_0, M_∞. If $\{c\}$ is between $\{a\}$ and $\{b\}$, then F is positive definite.

2. Let W be the group generated by the reflections in the edge of Schwartz's triangle (the image of the upper half plane under Schwartz map[4]). Then the above monodromy map is a subgroup of W generated by product of an even number of reflections.

3. The Riemann-Hilbert Problem

3.1 On the Origins of the Riemann-Hilbert Problem

In his 1857 paper Riemann suggested to study the problem of constructing a system of functions with regular singularities that has the prescribed monodromy properties. At the ICM 1900, Hilbert included it as Problem no. 21 on his list of "Mathematical Problems". It was formulated as follows:

> ***"Prove that there always exist a linear differential equation of Fuchsian type with given singular points and with a given monodromy group."***

In Hilbert's original formulation: *the problem requires the vector-function of the variable z that is regular on the z-plane except at the given singular points; at these points the functions may become infinite of only finite order.* He mentioned the singularities, nowadays called regular singularities (i.e. Definition 2.1), but did not speak of Fuchsian linear systems.

More explicitly, we can think of two problems as follows: given the monodromy group with encoded singularities, can it be realized by

1. a linear $N\times N$ system on $\mathbb{P}^1$ having only regular singularities?
2. a $N\times N$ Fuchsian system on $\mathbb{P}^1$?

The answer to the first problem is positive, which is proved by Plemelj in 1908. While in 1989, Bolibrukh constructed the first counterexample for the second one, see, more explanation in the following subsections. The question 2. is known as the Riemann-Hilbert problem (RHP).

Remark 3.1 (The N-th order Fuchsian linear differential equation). The linear differential equation of order N

$$y^{(N)}+q_1(z)y^{(N-1)}+\cdots+q_N(z)y=0 \tag{3.1}$$

defined on the complex plane with coefficients $q_i(z)$ being rational functions is called of Fuchsian type, if $q_i(z)$ has at most a pole of order i. One can translate

[4] Let f and g be two linearly independet solutions of the Gauss hypergeometric equation ($0\le |1-c|,|c-a-b|,|a-b|<1$), then Schwartz map is a map from $\mathbb{H}\cup\mathbb{R}$ to $\mathbb{P}^1$, given by

$$z_0\mapsto f(z_0)/g(z_0)$$

.

the N-th order Fuchsian linear differential equation into a (first order) Fuchsian linear differential equation of size N with $\mathbf{y} = (y, y', y'', \dots, y^{(N-1)})^T$.

One can also ask the question: is there a Fuchsian linear N-th order differential equation having a prescribed monodromy group with encoded singularities? The answer is negative since the number of parameters in (3.1) with singularities $a_1, \dots, a_n$ is less than the dimension of the space of the monodromy representations.

About the formulation of the problem, Anosov and Bolibrukh [2] argued as follows:

> "*In mathematical literature, Hilbert's 21st problem is often called the Riemann-Hilbert problem, although Riemann never spoke exactly of something like it. This was well-known: Klein in his "Lectures on the development of the mathematics in 19th century" said that "Riemann speaks in such a careless way as if existence of functions* $(y_1(z), \dots, y_p(z))$ *(having the given singularities and monodromy) is self-evident and one has only to study their properties". However, Hilbert mentioned that "presumably Riemann was thinking on this problem", and Röhrl [62] made a final step in this mythological direction and distinctly attributed Hilbert's 21st problem to Riemann. As well as the majority of the mathematicians who have dealt with the problem, we prefer to say that the Riemann-Hilbert problem (Hilbert's 21st problem) is close to the sphere of Riemann's ideas and it has arisen in the course of research stimulated by him.*"

3.2 PLEMELJ'S METHOD

In 1908, Plemelj[5] [61] published a solution to the RHP that was widely accepted until the early 1980s. In his work, he reduced the RHP to a Hilbert boundary value problem in the theory of singular integral equations. The idea is as follows: join all singularities $a_1, \dots, a_n \in \mathbb{C}$ by a simple closed oriented contour Γ, and define a piecewise constant invertible matrix valued function $M(z)$

$$M(z) = [M_i M_{i-1} \cdots M_1]^{-1}, \quad z \in [a_i, a_{i+1}),\ i = 1, \dots, n;\ a_{n+1} = a_1,$$

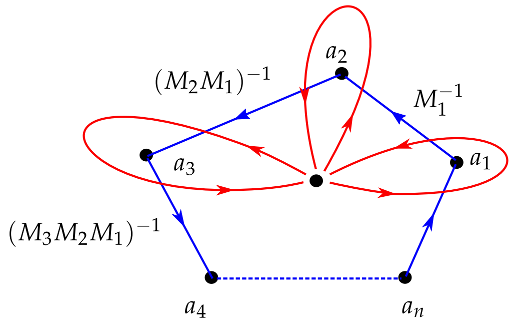
where M_i is the matrix of monodromy corresponding to a small loop around a_i. Denote by Γ_+ and Γ_- the interior and exterior of Γ in $\mathbb{CP}^1$ respectively. Then one can consider the following boundary value problem.

Hilbert Boundary Value Problem. Find all vector-valued functions $\mathbf{\Phi}_+ = \mathbf{\Phi}_+(z), \mathbf{\Phi}_- = \mathbf{\Phi}_-(z) \in \mathbb{C}^{1\times N}$ such that

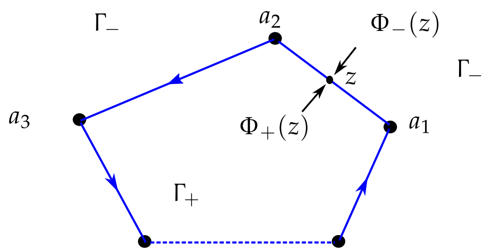
1. $\mathbf{\Phi}_+(z)$ is holomorphic in Γ_+ and $\mathbf{\Phi}_-(z)$ is holomorphic in Γ_- and of finite order at $z = \infty$;
2. $\mathbf{\Phi}_\pm$ are continuous right up to the contour Γ with the exception of points $a_1, \dots, a_n$, and on (a_i, a_{i+1}),

$$\mathbf{\Phi}_-(z) = \mathbf{\Phi}_+(z) M(z).$$

[5] Josip Plemelj (December 11, 1873 – May 22, 1967) was a Slovene mathematician, best known for his work on analytic functions and the application of integral equations to potential theory.



3. $\Phi_\pm(z)(z-a_i)^\epsilon$ tends to zero for some $0 \le \epsilon < 1$ as $z \to a_i$ over Γ_+ and Γ_-, respectively.



Such a problem can be solved by the methods of the theory of singular integral equations.

Next, Plemelj applied a procedure to proceed from the constructed system to another one with the same monodromy and the same singular points, which is a Fuchsian system at all except for perhaps one of the points. The remaining proof to obtain an actual Fuchsian system was shown to have a serious gap by Kohn [43]. But Kohn also proved that Plemelj's argument is valid, if one of the monodromy matrices is diagonalizable, or if *apparent* singularities (this means that $A(z)$ has poles outside a_i, but the associated monodromy matrix is the identity) are permitted.

More details and applications on various integrable systems can be found in [10].

3.3 A Brief History

Here are some major results after Plemelj's work:

- In 1913, Revisiting Plemelj's paper [61], Birkhoff [6] refined the argument grounded in the method of successive approximations. Concurrently, he pioneered the extension of RHP to incorporate certain difference equations.
- In the late 1920s, Lappo-Danilevskiĭ [47] solved the RHP in a constructive way in the case where all generators M_i are sufficiently close to the identity matrix. His method expressed solutions of a Fuchsian system and their

associated monodromy in the form of convergent series of the matrix coefficients A_i. Then the RHP reduces to the problem of inverting the series and studying its convergence.

- In 1956, Krylov [46] constructed an effective solution for RHP for all 2×2 systems with 3 singular points. His work made crucial use of the Gauss hypergeometric functions.
- In 1957, Röhrl [62] introduced the algebro-geometric ideas in the analysis of the RHP.[6] He constructed a principal bundle on $\mathbb{P}^1\setminus\{a_1,\dots,a_n\}$ with structure group $GL_N(\mathbb{C})$. In terms of geometric language, $\mathbb{P}^1$ may be replaced by an arbitrary Riemann surface and then, in higher dimensions, Riemann surfaces are replaced by complex manifolds. Nowadays, such a generalization are summarized under the umbrella of Riemann-Hilbert correspondences. Important early contributions to this field were achieved by Deligne [18], Kashiwara [39] and Mebkhout [53, 54].
- In 1979, Dekkers [17] proved the solvability of the RHP for $N=2$ with an arbitrary number of singularities. More explicitly, he solved the case when all monodromy matricies are not diagonalizable via appropriate meromorphic gauge transformations.
- In 1982, Erugin [22] considered the RHP for all 2×2 systems with 4 singular points. In particular, he elucidated a striking connection of this problem and the Painlevé VI equation (see Subsubsection 5.3.3).
- In 1983, Kohn [43] showed that if one of the given monodromy matrices M_i is diagonalizable, then ρ is realizable as the monodromy representation of a Fuchsian system.
- In 1992, Kostov [45] and Bolibrukh [8] showed that if the representation ρ is irreducible, the RHP has a positive answer.
- In 1998, Kitaev-Korotkin [42] and in 1999, Deift-Its-Kapaev-Zhou [16] solved the case $N=2$ with off-diagonal monodromy matrices in terms of Theta functions. In 2004, Korotkin [44] and Enolski-Grava [21] solved the RHP for general N with quasi-permutation monodromy matrices by means of the Szegö kernel on Riemann surfaces (realized as a N-fold branched covering of $\mathbb{P}^1$).
- For $N=3,4$, a complete characterization was given by Bolibrukh [9] (1999, $N=3$) and Gladyshev [26] (2000, $N=4$).
- In 2002 Malek [50] studied the special case of reducible representations and he showed how to produce new families of reducible counterexamples.

3.4 The First Counterexample Given by Bolibrukh

In general, the RHP is not solvable. There are monodromy representations which are not given by solutions of Fuchsian systems. This surprising fact came

[6] Röhrl was thought of as the first one to apply the methods of fiber bundles to the solution of the problem, but the original ideas goes back to Birkhoff. It is a pity that an adequate geometric language was not available at that time to describe it appropriately.

to light in the important works of Bolibrukh in 1989, see [7]. In more details, Bolibrukh's first counterexample concerns the following 3×3 system with $n=4$ singular points:

$$\frac{d\mathbf{y}}{dz} = A(z)\mathbf{y} = \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\widetilde{A}(z)$}} \\ 0 & & \end{pmatrix} y \tag{3.2}$$

where

$$\begin{aligned} a_{12}(z) &= \frac{1}{z^2} + \frac{1}{z+1} - \frac{1}{z-1/2}, \qquad a_{13}(z) = \frac{1}{z-1} - \frac{1}{z-1/2}, \\ \widetilde{A}(z) &= \frac{1}{z}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{6(z+1)}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \frac{1}{2(z-1)}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \\ &\quad + \frac{1}{3(z-1/2)}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}. \end{aligned} \tag{3.3}$$

3.4.1 Basic Observations

This system is not Fuchsian, but we have the following basic observations.

1. This system has singularities at $a_1=0$, $a_2=-1$, $a_3=1/2$, $a_4=1$. In particular,
   - a_2, a_3, a_4 are Fuchsian singularities;
   - a_1 is a regular singular point but not Fuchsian: Consider the transformation

     $$\Phi = P\Psi = \begin{pmatrix} 0 & 1/z & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Psi,$$

     then 0 is the Fuchsian singular point of the resulting system

     $$\frac{d\Psi}{dz} = (P^{-1}AP - P^{-1}P')\Psi.$$

2. Expanding (3.2) for $\mathbf{y}=(y_1,y_2,y_3)^T$, we obtain

   $$\frac{dy_1}{dz} = a_{12}(z)y_2 + a_{13}(z)y_3, \tag{3.4}$$

   $$\frac{d\widetilde{\mathbf{y}}}{dz} = \widetilde{A}(z)\widetilde{\mathbf{y}}, \quad \widetilde{\mathbf{y}} = (y_2,y_3)^T. \tag{3.5}$$

   Then, we have
   - $e_1 := (1,0,0)^T$ is a solution of the first equation;
   - All a_i, $i=1,2,3,4$ are Fuchsian singular points of the second equation.

We want to compute the Levelt fundamental solutions of (3.2). Based on the second basic observation, we will solve (3.5) first and then substitute its solution to (3.4).

3.4.2 The Second Order System

Write $\widetilde{A}(z) = \sum_{i=1}^{4} \frac{\widetilde{A}_i}{z-a_i}$, we have

- A_1 has eigenvalues $\{1, -1\}$;
- A_2, A_3, A_4 are nilpotent.

By Theorem 2.10, there exists a local fundamental solution around a_i, $i = 2, 3, 4$ of the Levelt form

$$Q_i(z)(z-a_i)^{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)},$$

where $Q_i(z)$ is a holomorphic invertible matrix at $z = a_i$.

Bolibrukh proved that the system (3.2) has non-trivial monodromy, and there does not exist a Fuchsian system with the same monodromy group and encoded singularity locations.

Two key points of Bolibrukh counterexample are:

1. In this example, the monodromy representation is reducible with sub-representation given by $\mathbb{C}e_1$. In particular, each monodromy matrix is similar to a Jordan form with only one block.
2. This example has sensitive dependence on the location of the singular points: once slightly perturbed, the RHP with the same monodromy might be solvable.

The general answer for $N = 3$ is as follows:

Theorem 3.2 ([7]). *The RHP for $N = 3$ has a negative answer if and only if the following three conditions hold:*

1. *the representation ρ is reducible;*
2. *each matrix $M_i = \rho(\gamma_i)$ can be reduced to a Jordon normal form, consisting of only one block;*
3. *the corresponding two dimensional subrepresentation of the quotient representation ρ_2 is irreducible, and the Fuchsian weight of the canonical extension for ρ_2 is greater than zero.*

3.5 Pants Decomposition and CFT Approach to the RHP

Even though there are some positive answers for the existence of solutions to the RHP, only few constructive results exist. The following two cases admit an explicit construction of the solution:

1. Let $\rho : \pi_1(\mathbb{P}^1 \setminus \{a_1, \dots, a_n\}) \to GL_N(\mathbb{C})$ is a quasi-permutation representation satisfying

- ρ cannot decompose into two sub-representations which are both quasi-permutation representations in the same basis of $\mathbb{C}^N$ (this implies that the compact Riemann surface corresponding to the associated permutation representation is connected).
- Non-triviality condition: the monodromy matrices cannot be simultaneously diagonalizable (otherwise, we obtain N scalar RHPs).

Korotkin [44] constructed the solution in terms of modified Szegö kernels and prime forms.

2. The rigid case. Katz [41] introduced a middle convolution functor MC_λ on the categories of perverse sheaves and showed that any irreducible rigid local system on the punctured affine line is connected to a one-dimensional local system by an iterative application of middle convolutions and scalar multiplications. Since these steps can be inverted, this leads to an existence algorithm for rigid local systems in terms of the local monodromies. Dettweiler-Reiter [19] gave a purely algebraic analogue of Katz middle convolution as well as an additive version acting on tuples of residue matrices $\{A_i\}$ of Fuchsian systems. We will review Dettweiler-Reiter's construction on the $N=2$ 4-point Fuchsian system in Subsubsection 5.3.2.

There are also some ideas involving the asymptotic analysis of the solutions. For simplicity, let us focus on the situation where all monodromy matrices are diagonalizable and have distinct eigenvalues. First, let us give a refined boundary value problem which is closely related to linear ODEs.

3.5.1 A Refinement of the Boundary Value Problem

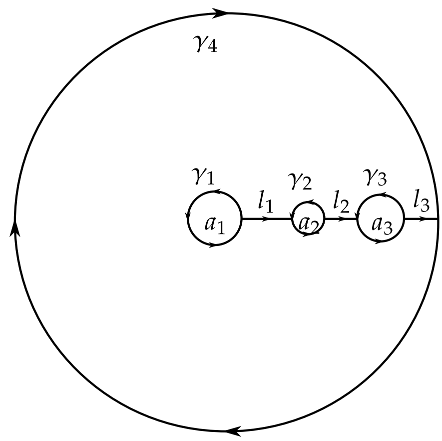
For simplicity, we fix the n singular points $a=(a_1=0,a_2\ldots,a_{n-1},a_n=\infty)$ with the radial ordering condition $0=|a_1|<|a_2|<\ldots<|a_{n-1}|$. Or even more conveniently, assume $a_2,\ldots,a_{n-1}\in\mathbb{R}_+$. Inspired by the asymptotic expansions of the fundamental solution near the singular point, the Riemann-Hilbert problem is also transformed as the boundary value problem consisting in finding an analytic invertible matrix $\Psi:\mathbb{P}^1\setminus\Gamma\to GL_N(\mathbb{C})$ with the following contour Γ and the jump matrix $J(z;a)$. The uniqueness of the solution is ensured by an appropriate normalization condition. The contour Γ is chosen to be a collection of circles γ_k and segments $l_k\subset\mathbb{R}$

$$\Gamma=\left(\bigcup_{k=1}^{n}\gamma_k\right)\cup\left(\bigcup_{k=1}^{n-1}l_k\right).$$

More explicitly,

- γ_k is a counter-clockwise oriented circles centered at a_k such that its interior doesn't contain other singular points;
- l_k is the segment joining the circles γ_k and γ_{k+1}.

For example, when $n=4$, the contour Γ is drawn as follows:



To define the jump matrix, we use the (C,Θ)-space for monodromy matrices consisting of

- a n-tuple of $N\times N$ diagonal matrices Θ_j, $j=1,\ldots,n$ satisfying the Fuchs relation $\sum_{j=1}^n \operatorname{tr}\Theta_j = 0$ and the non-resonance condition.
- a collection of $2n$ matrices $C_{k,\pm}\in GL_N(\mathbb{C})$ satisfying

$$\begin{aligned} M_{k\to 1} &:= C_{k,+}^{-1} e^{2\pi i\Theta_k} C_{k,-} = C_{k+1,+}^{-1} C_{k+1,-} \qquad \text{for } k=1,\ldots,n-2\\ M_{n-1\to 1} &:= C_{n-1,+}^{-1} e^{2\pi\Theta_{n-1}} C_{n-1,-} = C_{n,+}^{-1} e^{-2\pi i\Theta_n} C_{n,-}\\ M_{n\to 1} &:= I = C_{n,+}^{-1} C_{n,-} = C_{1,+}^{-1} C_{1,-}. \end{aligned}$$

In fact, there are only n independent matrices.

The jump matrix $J(z)$ is then given by

$$\begin{aligned} J(z)\big|_{l_k} &= M_{k\to 1}^{-1} && k=1,\ldots,n-1,\\ J(z)\big|_{\gamma_k} &= C_{k,\pm}^{-1}(a_k-z)^{-\Theta_k} && \Im z \gtrless 0,\quad k=1,\ldots,n-1,\\ J(z)\big|_{\gamma_n} &= C_{n,\pm}^{-1}(-z)^{\Theta_n} && \Im z \gtrless 0. \end{aligned}$$

The above RHP has a more straightforward connection to the linear ODE with rational coefficients: define a matrix function $\Phi(z)$ by

$$\Phi(z) = \begin{cases} \Psi(z), & z \text{ outside } \gamma_1,\ldots,\gamma_n,\\ \Psi(z)(a_k-z)^{\Theta_k} C_k, & z \text{ inside } \gamma_1,\ldots,\gamma_{n-1},\\ \Psi(z)(-z)^{-\Theta_n} C_n, & z \text{ inside } \gamma_n. \end{cases}$$

One can check that $\Phi(z)$ has the constant jump $M_{k\to 1}^{-1}$ on (a_k,a_{k+1}). Define

$$A(z) := \partial_z\Phi\cdot\Phi^{-1},$$

then $A(z)$ is a meromorphic matrix valued map on $\mathbb{P}^1$ with possible simple poles only at $a_1,\ldots,a_n$. Thus,

$$\frac{d\Phi}{dz}=\sum_{k=1}^{n-1}\frac{A_k}{z-a_k}\Phi,\quad A_k=\Psi(a_k)\Theta_k\Psi(a_k)^{-1}.$$

In particular, the monodromy matrices of the resulting Fuchsian system are uniquely determined by the jump matrices:

$$M_1=M_{1\to 1},\quad M_{k+1}=M_{k+1\to 1}M_{k\to 1}^{-1}.$$

3.5.2 Pants Decomposition for Monodromy Data

For simplicity, let $A_0,A_1,A_\infty\in sl_N(\mathbb{C})$ satisfy $A_0+A_1+A_\infty=0$, and each A_i has distinct eigenvalues modulo $\mathbb{Z}$. Then we consider Fuchsian systems defined on a 3-punctured sphere and the corresponding monodromy space

$$\begin{aligned}\mathcal{M}_3^{sl_N}=\Big\{&(M_0,M_1,M_\infty)\in SL_N(\mathbb{C})^3: M_\infty M_1 M_0=\mathrm{I},\\ &M_j\sim e^{2\pi i\Theta_j},\ j=0,1,\infty\Big\}/SL_N(\mathbb{C}).\end{aligned}$$

One starts with (M_0,M_1,M_∞) with fixed non-degenerated distinct eigenvalues, then there are $3(N^2-N)$ parameters. They are constrained by

$$M_\infty M_1 M_0=\mathrm{I}$$

and considered up to an overall $SL_N(\mathbb{C})$ conjugation, which decreases the number of parameters by $2(N^2-1)$. Thus,

$$\dim\mathcal{M}_3^{sl_N}=3(N^2-N)-2(N^2-1)=(N-1)(N-2).$$

In particular, $\dim\mathcal{M}_3^{sl_2}=0$.

Remark 3.3. In general, for any Lie algebra $\mathfrak{g}$ and $A_i\in\mathfrak{g}$, with fixed generic local exponents, one has

- $\dim\mathcal{M}_3^{\mathfrak{g}}=\dim\mathfrak{g}-3\,\mathrm{rank}\,\mathfrak{g}$;
- $\dim\mathcal{M}_n^{\mathfrak{g}}=(n-2)\dim\mathfrak{g}-n\,\mathrm{rank}\,\mathfrak{g}$.

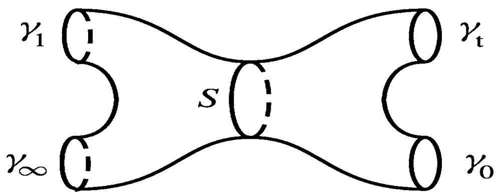
Let us look at the $n=4$ case carefully, that is,

$$\begin{aligned}\mathcal{M}_4^{sl_N}=\Big\{&(M_0,M_t,M_1,M_\infty)\in SL_N(\mathbb{C})^4: M_\infty M_1 M_t M_0=\mathrm{I},\\ &M_j\sim e^{2\pi i\Theta_j},\ j=0,t,1,\infty\Big\}/SL_N(\mathbb{C}).\end{aligned}$$

Applying the computation above, we have

$$\dim\mathcal{M}_4^{sl_N}=2(N-1)^2,\quad \dim\mathcal{M}_4^{sl_2}=2.$$

In the case of 4 (or more) poles, a convenient parameterization is suggested by **pants decompositions.**



Introduce $M_{t0} = M_t M_0$ (assumed to be diagonalizable) and consider the two triples

$$\{(M_0, M_t, M_{t0}^{-1}), (M_{t0}, M_1, M_\infty)\}.$$

Now let us choose the submanifold with fixed eigenvalues of $M_{0,t,1,\infty}$, M_{t0}.

$$M_{t0} = e^{2\pi i \mathfrak{S}}, \quad \mathfrak{S} \in \mathfrak{h} \text{ (Cartan subalgebra)}.$$

Then, we obtain a submanifold

$$\mathcal{M}_4^{sl_N}(\mathfrak{S}) = \left\{(M_0, M_t, e^{-2\pi i \mathfrak{S}}), (e^{2\pi i \mathfrak{S}}, M_1, M_\infty)\right\}/H \subset \mathcal{M}_4^{sl_N}.$$

It is similar to

$$\mathcal{M}_3^{sl_N} = \{(M_1, M_2, M_3) : \ldots\}/G = \left\{(M_1, M_2, e^{2\pi i \Theta_3}) : \ldots\right\}/H,$$

except that the conjugation is simultaneous for both triples. To relax this condition, one can consider the twist action on $\mathcal{M}_4^{sl_N}$ by the Cartan group $H \subset SL_N(\mathbb{C})$:

$$h : \{(M_0, M_t, e^{-2\pi i \mathfrak{S}}), (e^{2\pi i \mathfrak{S}}, M_1, M_\infty)\} \mapsto \{(M_0, M_t, e^{-2\pi i \mathfrak{S}}), h^{-1}(e^{2\pi i \mathfrak{S}}, M_1, M_\infty)h\}.$$

Thus, roughly one can say that

$$\mathcal{M}_4^{sl_N}(\mathfrak{S})/H = \mathcal{M}_3^{sl_N} \times \mathcal{M}_3^{sl_N}.$$

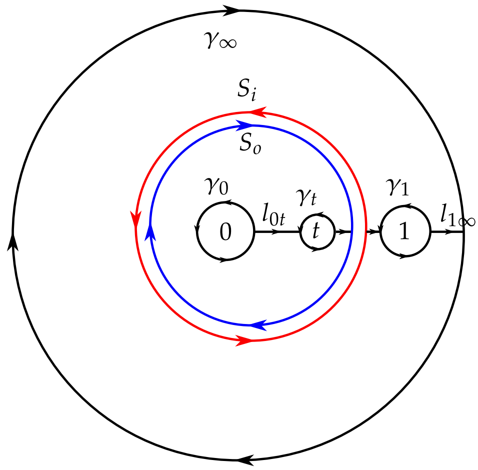
Such a parametrization (by pants decomposition) of monodromy data associated to 4 (or more) poles implies that we can reduce a corresponding RHP (RH boundary value problem) into a sequence of simpler RHP (3-point case) that can be solved exactly or asymptotically. This is the idea of the Riemann nonlinear steepest descend method of Deift-Zhou [15].

3.5.3 The Decomposed 3-Point RHPs

For example, when $n = 4$, the contour Γ is drawn as follows:

Then the original 4-punctured sphere is decomposed into 2 pairs of pants $\mathcal{T}^{0t}$, $\mathcal{T}^{1\infty}$ and one annulus $\mathcal{A}$. Now we can associate two 3-point RHPs corresponding to $\mathcal{T}^{0t}$, and $\mathcal{T}^{1\infty}$, respectively.

(1) The 3-point RHP corresponding to $\mathcal{T}^{0t}$ is given as follows.



- The contour Γ^{0t} is given by the union of γ_0, γ_t, S_o and two segments in the above picture in $\mathcal{T}^{0t}$;
- Fix a matrix $C_S \in GL_N(\mathbb{C})$ such that

$$M_t M_0 = C_S^{-1} e^{2\pi i \mathfrak{S}} C_S, \quad \mathfrak{S} = \mathrm{diag}(\sigma_1, \cdots, \sigma_N);$$

  the jump matrix function $J^{0t}(z)$ is defined by

$$J^{0t}(z)\big|_{\Gamma^{0t}\cap\Gamma} = J(z)\big|_{\Gamma^{0t}\cap\Gamma},$$
$$J^{0t}(z)\big|_{S_o} = (-z)^{\mathfrak{S}} C_S.$$

The solution $\Psi^{0t}(z)$ gives the fundamental matrix solution Φ^{0t} of a Fuchsian system with 3 singular points $0, t, \infty$ characterized by appropriate monodromies. Furthermore, write

$$\Phi^{0t}(z) = \widetilde{\Phi}^{0t}(z/t),$$

then the rescaled matrix $\widetilde{\Phi}^{0t}(z)$ solves a Fuchsian system characterized by the same monodromy as $\Phi^{0t}(z)$ but the singular points are at $z = 0, 1, \infty$. Adding the normalization condition $\widetilde{\Phi}^{0t}(\infty) \simeq (-z)^{\mathfrak{S}}$, as $z \to \infty$ (which implies $\Psi^{0t}(\infty) = t^{-\mathfrak{S}}$), we can write the asymptotic expansion of $\widetilde{\Phi}^{0t}(z)$ as

$$\widetilde{\Phi}^{0t}(z) = \begin{cases} G_0^{0t}(z)(-z)^{\Theta_0} C_0 C_S^{-1}, & \text{as } z \to 0, \\ G_1^{0t}(z)(1-z)^{\Theta_t} C_t C_S^{-1}, & \text{as } z \to 1, \\ G_\infty^{0t}(z)(-z)^{\mathfrak{S}}, & \text{as } z \to \infty, \end{cases}$$

where

$$G_0^{0t}(z) = G_0^{0t}\big(I + \sum_{m\geq 1} g_{0,m}^{0t} z^m\big),$$
$$G_1^{0t}(z) = G_1^{0t}\big(I + \sum_{m\geq 1} g_{1,m}^{0t} (z-1)^m\big),$$

$$G^{0t}_{\infty}(z) = I + \sum_{m\geq 1} g^{0t}_{\infty,m} z^{-m}.$$

(2) The 3-point RHP corresponding to $\mathcal{T}^{1\infty}$ is given as follows.

- The contour $\Gamma^{1\infty}$ is given by the union of S_i, γ_1, γ_∞ joined by the two segments in $\mathcal{T}^{1\infty}$ as shown in the above picture;
- The jump matrix function $J^{1\infty}(z)$ is defined by

$$\begin{aligned} J^{1\infty}(z)\big|_{\Gamma^{1\infty}\cap\Gamma} &= J(z)\big|_{\Gamma^{1\infty}\cap\Gamma}, \\ J^{1\infty}(z)\big|_{S_i} &= C_S^{-1}(-z)^{-\mathfrak{S}}. \end{aligned}$$

Now the solution $\Psi^{1\infty}(z)$ gives a solution of Fuchsian system with 3 singular points $0, 1, \infty$ characterized by the monodromies $M_t M_0$, M_1, M_∞. Furthermore, we normalize such a solution at $z \to 0$ by $\Phi^{1\infty}(z) \simeq (-z)^{\mathfrak{S}}$, i.e., the asymptotic expansion is given by

$$\Phi^{1\infty}(z) = \begin{cases} G_0^{1\infty}(z)(-z)^{\mathfrak{S}}, & \text{as } z \to 0, \\ G_t^{1\infty}(z)(1-z)^{\Theta_1} C_1 C_S^{-1}, & \text{as } z \to 1, \\ G_\infty^{1\infty}(z)(-z)^{-\Theta_\infty} C_\infty C_S^{-1}, & \text{as } z \to \infty, \end{cases}$$

where

$$\begin{aligned} G_0^{1\infty}(z) &= I + \sum_{m\geq 1} g^{1\infty}_{0,m} z^m, \\ G_1^{1\infty}(z) &= G_1^{1\infty}\big(I + \sum_{m\geq 1} g^{1\infty}_{1,m}(z-1)^m\big), \\ G_\infty^{1\infty}(z) &= G_\infty^{1\infty}\big(I + \sum_{m\geq 1} g^{1\infty}_{\infty,m} z^{-m}\big). \end{aligned}$$

The solution $\Psi(z)$ of the original 4-point RHP can be approximated by Ψ^{0t} and $\Psi^{1\infty}$ by shrinking the length of the annulus $\mathcal{A}$ to be the circle S, and by considering the matrix function

$$\Psi^S(z) = \begin{cases} \Psi(z)\big(\Psi^{0t}(z)\big)^{-1}, & z \in \mathcal{T}^{0t}, \\ \Psi(z)\big(\Psi^{1\infty}(z)\big)^{-1}, & z \in \mathcal{T}^{1\infty}. \end{cases}$$

Then $\Psi^S(z)$ is holomorphic on $\mathbb{P}^1 \setminus S$ and has no jumps along Γ. The normalization and the nonconstant jump of $\Psi^S(z)$ on S are given by

$$J^S(z) = \Psi^S_-(z)^{-1}\Psi^S_+(z), \quad \Psi^S(\infty) = G_\infty \Psi^{1\infty}(\infty)^{-1}.$$

More explicitly, along the contour S we can compute

$$\begin{aligned} J^S(z) = \Psi^S_-(z)^{-1}\Psi^S_+(z) &= \Psi^{1\infty}_-(z)^{-1}\Psi^{0t}_+(z) \\ &= \Big[G_0^{1\infty}(z) z^{\mathfrak{S}}\Big]\Big[G_\infty^{0t}(z/t)(z/t)^{\mathfrak{S}}\Big]^{-1} \\ &= G_0^{1\infty}(z) t^{\mathfrak{S}} G_\infty^{0t}(z/t)^{-1} \end{aligned}$$

$$= \left\{ G_0^{1\infty}(z) \left[I + \sum_{m=1}^{\infty} t^m t^{\mathfrak{S}} g_{\infty,m}^{0t} t^{-\mathfrak{S}} z^{-m} \right]^{-1} G_0^{1\infty}(z)^{-1} \right\} G_0^{1\infty}(z) t^{\mathfrak{S}}.$$

Let us consider the following assumption (almost without lose of generality)[7] for the diagonal matrix $\mathfrak{S} = \mathrm{diag}(\sigma_1, \dots, \sigma_N)$:

$$|\mathrm{Re}(\sigma_j - \sigma_k)| < 1, \qquad \sigma_j \neq \sigma_k, \text{ for } j \neq k.$$

Then we have

$$t^m t^{\mathfrak{S}} g_{\infty,m}^{0t} t^{-\mathfrak{S}} = O(t^{m-s}), \quad s = \max_{j,k} |\mathrm{Re}(\sigma_j - \sigma_k)| < 1.$$

Thus, we arrive at the estimate for $J^S(z)$:

$$J_S(z) = (I + O(t^{1-s})) G_0^{1\infty}(z) t^{\mathfrak{S}}.$$

The solution of RHP generated by the jump matrix $G_0^{1\infty}(z) t^{\mathfrak{S}}$ is given by

$$\Psi_0^S(z) = \begin{cases} G_\infty \Psi^{1\infty}(\infty) G_0^{1\infty}(z) t^{\mathfrak{S}}, & z \in \mathcal{T}^{0t}, \\ G_\infty, & z \in \mathcal{T}^{1\infty}. \end{cases}$$

By Deift-Zhou's argument [15] for the singular integral operator associated with RHP for the ratio $\Psi^S(z)\Psi_0^S(z)^{-1}$, one can obtain the estimate

$$\Psi^S(z) = \left(I + O\left(\frac{t^{1-s}}{1+|z|} \right) \right) \Psi_0^S(z), \quad t \to 0, z \in \mathbb{C}.$$

Now the error term is of order $O(t^{1-s})$, a better result up to the error $o(t)$ is obtained by Its-Lisovyy-Prokhorov [31].

Proposition 3.4 ([31])**.** *A uniform approximation for $\Psi(z)$ as $t \to 0$ is given by*

$$\Psi(z) = \begin{cases} G_\infty \Psi^{1\infty}(\infty)^{-1} (I + \frac{\mathfrak{E}(t)}{z} + O(\frac{t^{2-s}}{|z|})) \Psi^{1\infty}(z), & z \in \mathcal{T}^{1\infty}, \\ G_\infty \Psi^{1\infty}(\infty)^{-1} (I - q(z,t) + O(t^{2-s}) G_0^{1\infty}(z) t^{\mathfrak{S}} \Psi^{0t}(z), & z \in \mathcal{T}^{0t}. \end{cases}$$

Here, the functions $\mathfrak{E}(t)$ and $q(z,t)$ can be determined using the t-expansion of $J^S(z)$.

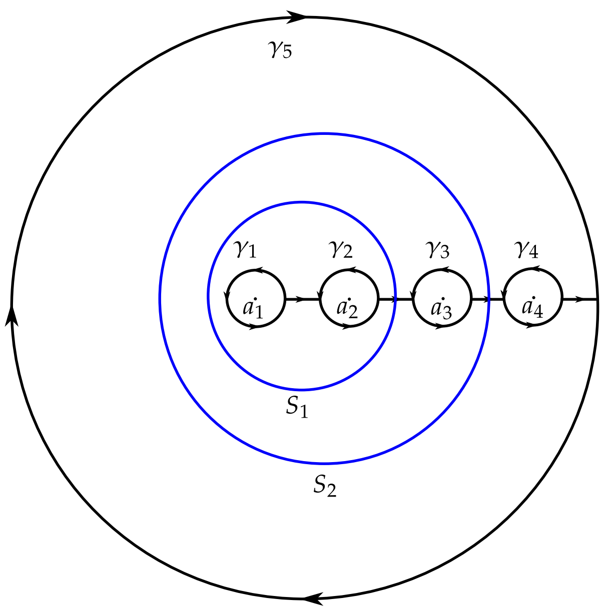
3.5.4 Linear Pants Decomposition

For general n, one can consider the following pants decomposition of $\mathbb{P}^1 \setminus \{a_1, \dots, a_n\}$ into $n-2$ pairs of pants $\mathcal{T}^1, \dots, \mathcal{T}^{n-2}$, with boundaries

$$\partial\mathcal{T}^1 = \{\gamma_1, \gamma_2, S_1\}, \quad \partial\mathcal{T}^{n-2} = \{S_{n-2}, \gamma_{n-1}, \gamma_n\},$$
$$\partial\mathcal{T}^k = \{S_{k-1}, \gamma_{k+1}, S_k\}, \ k = 2, \dots, n-3,$$

[7] The values with $\mathrm{Re}(\sigma_j - \sigma_k) = \pm 1$ are excluded for technical reason.

where the interior of the circle S_k contains exactly the cycles $\gamma_1, \dots, \gamma_{k+1}$ and $S_1, \dots, S_{k-1}$. Such a decomposition is also known as the linear pants decomposition (every pair of pants has at least one external boundary component γ_i). The following picture shows the decomposition for $n = 5$.



Assume $M_{k\to 1}$ with $k = n-1, \dots, 2$ are diagonalizable

$$M_{k\to 1} = C_{S_k} e^{2\pi i \mathfrak{S}_k} C_{S_k}^{-1}, \quad \mathfrak{S}_k = \mathrm{diag}(\sigma_{k,1}, \dots, \sigma_{k,N}),$$

and identity $\mathfrak{S}_1 = \Theta_1$, $\mathfrak{S}_{n-1} = -\Theta_{n-1}$.

Then one can pose a 3-point RHP for each pair (Γ_k, J_k), $k = 1, \dots, n-2$ such that

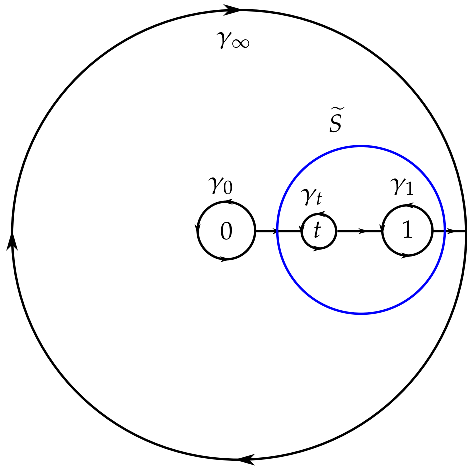
- Γ_k is the union of the boundary of $\mathcal{T}^k$ and the corresponding two segments;
- J_k is defined by the rule:
  - $J_k\big|_{\Gamma\cap\Gamma_k} = J\big|_{\Gamma\cap\Gamma_k}$;
  - $J_k\big|_{S_k} = C_{S_k}^{-1} z^{-\mathfrak{S}_k}$.

3.5.5 Connection Problem

One can also consider another pants decomposition of $\mathbb{P}^1 \setminus \{0, t, 1, \infty\}$, e.g.,

Again there are two 3-point RHPs corresponding to this decomposition into two pants. After gluing, one obtains the asymptotic expansion of $\Psi(z)$ or $\Phi(z)$ as $t \to 1$.

The different pants decompositions correspond to the asymptotic expansions of $\Psi(z)$ or $\Phi(z)$ in different regions around different singular points.



3.5.6 CFT Approach to RHP: Monodromy Fields

Inspired by the idea in [35, 57], Gamayun-Iorgov-Lisovyy [23] constructed a formal QFT solution of the Riemann-Hilbert problem for the following ansatz of $\mathbf{\Phi}$:

$$\mathbf{\Phi}_{jk}(z,w) = (z-w)^{2\Delta}\frac{\langle V_{\Theta_1}(a_1)\dots V_{\Theta_n}(a_n)\varphi_j(z)\bar{\varphi}_k(w)\rangle}{\langle V_{\Theta_1}(a_1)\dots V_{\Theta_n}(a_n)\rangle}, \quad j,k=1,\dots,N. \tag{3.6}$$

Here $\{V_{\Theta_i}(a_i)\}$, $\{\varphi_j,\bar{\varphi}_k\}$ are assumed to be primary fields in a 2d CFT characterized by some central charge c and the leading term of OPE of $\varphi_j(z)$ and $\bar{\varphi}_k(w)$ is of the form

$$\varphi_j(z)\bar{\varphi}_k(w) = (z-w)^{-2\Delta}\delta_{jk}.$$

The other fields $V_{\Theta_i}(a_i)$ are called monodromy fields (or twist fields), which are defined by the following OPE

$$V_{\Theta_i}(a_i)\varphi_k(z) = \sum_{j=1}^{n}\left((z-a_i)^{C_i^{-1}\Theta_i C_i}\right)_{jk}\sum_{l\geq 0}V_{\Theta_i,l}(a_i)(z-a_i)^l.$$

Then the row vector $(\varphi_1(z),\dots,\varphi_n(z))$ should be multiplied by M_i when continued around $V_{\Theta_i}(a_i)$. Thus, (3.6) gives the solution of the corresponding RHP.

One possible way to realize the above realization CFT is through the N component free fermions $\psi_j(z)$, $\bar{\psi}_k(z)$ and Fock representations [25]. The monodromy field

$$V_\Theta(a):\mathcal{H}_\mu\to\mathcal{H}_\nu$$

is the operator between the bosonic Fock spaces $\mathcal{H}_\mu$ and $\mathcal{H}_\nu$ with different charges μ, ν. By conformal Ward identity, one only needs to define $V_\Theta(1)$, which is characterized by

1. $V_\Theta(1)\mathcal{H}_\mu V_\Theta^{-1}(1)\subset\mathcal{H}_\nu$, $V_\Theta(1)^{-1}\mathcal{H}_\nu V_\Theta(1)\subset\mathcal{H}_\mu$,

2. the normalization condition: $\langle \nu | V_\Theta(1) | \mu \rangle = 1$,
3. the 2-fermionic correlators in the different regions

$$\begin{aligned} &\langle \nu | V_\Theta(1) \psi(z) \bar{\psi}(w) | \mu \rangle, && |z|, |w| \le 1, \\ &-\langle \nu | \bar{\psi}(w) V_\Theta(1) \psi(z) | \mu \rangle, && |z| \le 1, |w| \ge 1, \\ &\langle \nu | \psi(z) V_\Theta(1) \bar{\psi}(w) | \mu \rangle, && |z| \ge 1, |w| \le 1, \\ &\langle \nu | \psi(z) \bar{\psi}(w) V_\Theta(1) | \mu \rangle, && |z|, |w| \ge 1, \end{aligned}$$

give solutions to the 3-point RHP in the different regions.

The first condition ensures that one can use the Wick theorem, combined with the third condition, one can compute all matrix elements of $V_\Theta(1)$. Then, via the insertion of resolution of the unity between $V_{\Theta_i}(a_i)$ and $V_{\Theta_{i+1}}(a_{i+1})$, $i = 1, \ldots, n-1$, one can show that (3.6) gives the desired solution for the n-point RHP.

We have seen that the n-point Riemann-Hilbert problem can be approximated by $(n-2)$ 3-point Riemann-Hilbert problems. Unfortunately, finding 3-point solutions remains an open problem in general. However, in a number of cases the 3-point inverse monodromy problem can be solved in terms of generalized hypergeometric functions.

3.6 Some Related Problems

3.6.1 Deligne-Simpson Problem

Find the necessary and sufficient conditions for the choice of the conjugacy classes $C_j \subset GL_N(\mathbb{C})$ or $c_j \subset gl_N(\mathbb{C})$ so that there exist irreducible (or, respectively with trivial centralizer) n-tuples of matrices $M_j \in C_j$ or $A_j \in c_j$ satisfying

$$M_n \cdots M_2 M_1 = I, \quad \text{or} \quad A_1 + A_2 + \cdots + A_n = 0.$$

This is the Deligne-Simpson problem (DSP). Here "irreducible" means "with no common proper invariant subspace". In technical terms, this means that it is impossible to simultaneously conjugate the tuple to a block upper-triangular form with the same sizes of the diagonal blocks for all matrices M_j or A_j. The name of the problem is due to the fact that the multiplicative version (i.e. for M_j) it was stated by Deligne in 1980s and C. Simpson [64] was the first to obtain important results towards its resolution.

Theorem 3.5 (Simpson [64])**.** *For generic eigenvalues and when one of the matrices M_j has n distinct eigenvalues, the DSP is solvable for matrices M_j if and only if the following two inequalities hold:*

1. $d_1 + \cdots + d_n \ge 2N^2 - 2$;
2. $r_1 + \cdots + \hat{r}_j + \cdots + r_n \ge N$, *for* $j = 1, 2, \ldots, n$,

where

- $d_j = \dim(C_j)$ *is the dimension of the conjugacy class* C_j *in* $GL_N(\mathbb{C})$, *and*
- $r_j := \min_{\lambda \in \mathbb{C}} \operatorname{rank}(Y_j - \lambda I)$ *for a matrix* Y_j *from* C_j.

Crawley-Boevey [13] solved the additive version using quiver theory.

3.6.2 The Inverse Problem in Differential Galois Theory

We have seen that the monodromy representation of a Fuchsian system is given by the analytic continuation of the fundamental solution. There is another important action coming from automorphisms of objects that do not depend on the equations but only on the base field.

In the Galois theory, the automorphisms are those of a separable closure of the base field from which the coefficients of the equation are taken. In the theory of differential geometry, the analogous role is played by a universal covering of the base domain.

Let (k, ∂) be a differential field of characteristic zero with $C_k := \ker \partial$ an algebraically closed field. Consider the differential system

$$\partial \mathbf{y} = A\mathbf{y}, \quad A \in gl_N(k), \ \mathbf{y} = (y_1, \ldots, y_N)^T. \tag{3.7}$$

We call it a system over k.

Definition 3.6. A differential extension (L, ∂) of (k, ∂) is a Picard-Vessiot extension for (3.7), if there is a fundamental solution $\mathbf{\Phi}$ in L:

$$\partial \mathbf{\Phi} = A\mathbf{\Phi}, \quad \mathbf{\Phi} \in GL_N(L)$$

such that

1. L is generated as a field extension of k, by the entries of $\mathbf{\Phi}$;
2. $C_L = C_k$.

Theorem 3.7. *For every differential field* (k, ∂) *of characteristic zero with* C_k *algebraically closed, there exist a Picard-Vessiot extensions of* k *for every system* (3.7). *Two Picard-Vessiot extensions of* k *with respect to the same system* (3.7) *are* k*-isomorphic.*

Definition 3.8. The differential Galois group of (3.7) over k is the group of differential k-automorphisms of L.

The dimension of the differential Galois group G is related to the transcendence of the solutions.

Proposition 3.9. *Let* G *be the differential Galois group of a Picard-Vessiot extension* $k \subset L$. *Then the dimension of* G *as an algebraic variety over* C_k *is equal to the transcendence degree of* L *over* k.

The inverse problem asks whether G can be realized as the differential Galois group of some system (3.7).

Theorem 3.10 (Tretkoff-Tretkoff [65])**.** *The inverse problem in differential Galois group has a solution over $k = \mathbb{C}(z)$, where $\mathbb{C}(z)$ denotes the field of rational functions.*

3.6.3 Grothendieck-Katz p-Curvature Conjecture

It involves a hypothetical criterion for a linear algebraic differential equation with coefficients in $\overline{\mathbb{Q}}(z)$ to have all solutions in algebraic functions. For example, we can look at a linear algebraic differential equation as follows,

$$y^{(N)}(z) + q_1(z)y^{(N-1)}(z) + \cdots + q_N(z)y(z) = 0, \quad q_i(z) \in \mathbb{Q}(z).$$

First, one can reduce it to a linear differential equation for $\mathbf{y} = (y, y^{(1)}, \ldots, y^{(N-1)})^T$ as

$$\frac{d\mathbf{y}}{dz} + A(z)\mathbf{y} = 0, \quad A(z) = (A_{ij}(z))_{i,j=1,\ldots,N} \in gl_N(\mathbb{Q}(z)). \tag{3.8}$$

For a large enough prime number $p \geq 2$, one can further obtain a well-defined reduction modulo p of (3.8), that is,

$$A_{ij}(z) \mod p \in \mathbb{F}_p(z), \quad \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}.$$

Definition 3.11. For the reduced operator $\partial_z + A_{[p]}$, where

$$A_{[p]} := (A_{ij}(z) \mod p)_{i,j=1,\ldots,N},$$

the following $\mathbb{F}_p(z)$ operator

$$\mathrm{Curv}_p := \left(\partial_z + A_{[p]}\right)^p : \mathbb{F}_p(z)^N \to \mathbb{F}_p(z)^N$$

is called the p-curvature operator.

If the solutions of (3.8) are algebraic, then the p-curvature vanishesfor all sufficiently large prime numbers p.

A. Grothendieck conjectured that the inverse is also true: the vanishing of p-curvature for $p \gg 2$ implies the algebraicity of all solutions.

Katz [40] applied Tannakian category techinques to show that this conjecture is the same as saying that the differential Galois group G can be determined by mod p information, for a certain wide class of differential equations. In 1982, he also proved that this conjecture holds for the Gauss-Manin connection. This conjecture is now known as Grothendieck-Katz p-curvature conjecture. A highly non-trivial arithmetic result is obtained by D. Chudnovsky-G. Chudnovsky in 1984 to confirm this conjecture in rank 1.

Theorem 3.12 (Chudnovsky-Chudnovsky [14])**.** *Grothendieck-Katz p-curvature conjecture holds for connections on line bundles.*

4. RIEMANN-HILBERT CORRESPONDENCE

4.1 MEROMORPHIC CONNECTION ON $\mathbb{P}^1$

Let X be a connected one dimensional analytic complex variety. We can define the meromorphic connection on its holomorphic bundles as follows.

Definition 4.1. Let F be a holomorphic vector bundle of rank N over X, and $\mathcal{U} = \big(U_i, f_i\big)$ be a trivializing atlas with local coordinate z_i on U_i. A meromorphic connection ∇ on F is a family of meromorphic differential systems $\{(S_i)\}_i$:

$$\frac{d\mathbf{y}}{dz_i} = A_i(z_i)\mathbf{y} \tag{4.1}$$

of rank N such that on each U_{ij}, the systems on U_i and U_j are gauge transformations of each other: using the defining cocycle $g = (g_{ij})$ corresponding to $\mathcal{U}$,

$$A_i = \frac{dg_{ij}}{dz_i} g_{ij}^{-1} + g_{ij} A_j g_{ij}^{-1}.$$

Then singular points of ∇ are understood as follows.

Definition 4.2. Let ∇ be a meromorphic connection on a holomorphic bundle F over X. A point $a \in X$ is a singular point for ∇ if for a trivializing covering $\mathcal{U} = \big(U_i\big)_{i\in I}$ and $a \in U_i$, a is a singular point of the differential system (S_i) defined by ∇ on U_i.

1. a is called a regular (or irregular) singular point if it is a regular (or irregular) singular point of (S_i) on U_i;
2. a is called a logarithmic pole of ∇ if it is a Fuchsian singular point of (S_i) on U_i.

It is known that the rank N holomorphic vector bundle over $\mathbb{P}^1$ is determined by a cocycle

$$g_{0\infty} : U_{0\infty} = \mathbb{C}^* \to GL_N(\mathbb{C})$$

for $U_0 = \mathbb{C}$, $U_\infty = \mathbb{P}^1 \setminus \{0\}$. More completely, Birkhoff-Grothendieck describes all holomorphic vector bundles on $\mathbb{P}^1$ as follows. Any rank N holomorphic vector bundle F over $\mathbb{P}^1$ is a direct sum of line bundles

$$F \simeq \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_N),$$

i.e. $g_{0\infty} = \mathrm{diag}(z^{k_1}, \ldots, z^{k_N})$, $k_i \in \mathbb{Z}$. The degree of F is defined to be

$$\deg(F) = \sum_{i=1}^{N} k_i,$$

which determines the associated determinant line bundle $\det(F)$ by the cocycle $z^{\deg(F)}$, that is, $\det(F) = \mathcal{O}(\deg(F))$. This line bundle is also endowed with the induced connection $\mathrm{tr}\nabla$.

By the above definition, a meromorphic connection on a holomorphic vector bundle F over $\mathbb{P}^1$ is equivalent to a differential system on $\mathbb{C}$ with rational function coefficients (2.1). Furthermore, using the Fuchsian relation 2.11, one can show that if F can be endowed with a Fuchsian connection, then $\deg(F) = 0$.

4.2 Röhlr's Work

In 1957, Röhrl applied methods from the theory of fibre bundles to study the Riemann-Hilbert problem. From the monodromy representation ρ, Röhrl constructed a principal bundle over $\mathbb{P}^1 \setminus \{a_i\}_{i=1}^n$ with the structure group $GL_N(\mathbb{C})$.

Consider the universal covering $\pi : S \to \mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}$. Denote the points of S by $\tilde{z}$. Let $\tilde{z}_0$ be a point such that $\pi(\tilde{z}_0) = z_0$. Then one can identify the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}, z_0)$ with the group Δ of deck transformations of the universal covering. An element σ of Δ acts from the left on the points of S. We define the action of Δ from the right on S as follows: $\tilde{z} \cdot \sigma = \sigma^{-1}\tilde{z}$. For any function $f(\tilde{z})$ on S we have

$$\sigma^* f(\tilde{z}) = f(\tilde{z} \cdot \sigma).$$

Thus, the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}, z_0)$ acts on the space of functions $f(\tilde{z})$ by the above identification.

The triple $(S, \pi, \pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}, z_0))$ can be considered as a principal bundle $\tilde{P}$ with the discrete structure group Δ. Consider the principal bundle

$$P = (\pi_P : P_E \to \mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}, GL_N(\mathbb{C}), \pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}, z_0))$$

associated to $\tilde{P}$ via the representation

$$\rho : \Delta \cong \pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}, z_0) \to GL_N(\mathbb{C}).$$

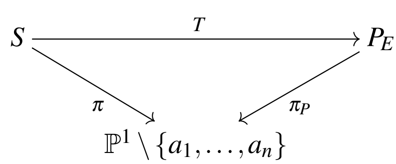
The total space P_E is holomorphically equivalent to the space $S \times GL_N(\mathbb{C})/\sim$, with equivalence relation

$$(\tilde{z} \cdot \sigma, \rho(\sigma^{-1})G) \sim (\tilde{z}, G), \quad \sigma \in \Delta.$$

We denote a point of P_E by $\langle \tilde{z}, G \rangle$ and consider the holomorphic map

$$T : S \to P_E, \quad T(\tilde{z}) = \langle \tilde{z}, I \rangle,$$

where I is the $N \times N$ identity matrix. Then the following diagram is commutative:



It is well known that every holomorphic bundle over non-compact Riemann surfaces (here, punctured Riemann spheres) is holomorphically trivial. Thus, there exists a holomorphic section $U : \mathbb{P}^1 \setminus \{a_1, \dots, a_n\} \to P_E$. For every $\tilde{z} \in S$, we find that $T(\tilde{z})$ and $U(\pi(\tilde{z}))$ lie in the same fiber of P and get

$$U(z) = T(\tilde{z})\widetilde{\Phi}(\tilde{z})^{-1}, \quad z = \pi(\tilde{z}), \tag{4.2}$$

where $\widetilde{\Phi}(\tilde{z})$ is a holomorphic function from S to $GL_N(\mathbb{C})$. Then, one can see that

$$\sigma^*\widetilde{\Phi}(\tilde{z}) = \widetilde{\Phi}(\tilde{z})\rho(\sigma), \quad \sigma \in \Delta.$$

That is, $\widetilde{\Phi}(\tilde{z})$ has the monodromy given by ρ. Thus, the matrix differential one form

$$\omega = d\widetilde{\Phi}(\tilde{z})\widetilde{\Phi}(\tilde{z})^{-1}$$

is single valued on $\mathbb{P}^1 \setminus \{a_1, \dots, a_n\}$, and the system defined by $\nabla = d - \omega$ has the given monondromy ρ leading to the following result.

Theorem 4.3 (Röhrl [62])**.** *There is a one-to-one correspondence between the set of all systems $d\Phi = \omega\Phi$ with the prescribed monodromy ρ and the set $H^0(\mathbb{P}^1 \setminus \{a_1, \dots, a_n\}, \mathcal{O}(P))$ of holomorphic sections of the principal bundle P.*

Similarly, one can construct a vector bundle F over $\mathbb{P}^1 \setminus \{a_1, \dots, a_n\}$ by

$$F = (S \times \mathbb{C}^N)/\sim,$$

and the principal bundle P is the frame-bundle of F whose fibers over U_{ij} are the $\mathbb{C}$-linear isomorphisms between the fibers of F over U_i and F over U_j respectively. Röhrl also extended F to $\mathbb{P}^1$ using the section $\widetilde{\Phi}$. The extended bundle has a meromorphic section that is holomorphically invertible everywhere except at the singular points $a_1, \dots, a_n$. The corresponding system has the given monodromy and $a_1, \dots, a_n$ are regular singular points. Röhrl's method works also well for any non-compact Riemann surface. That is,

Theorem 4.4. *Let X be a non-compact Riemann surface, for any representation $\rho : \pi_1(X) \to GL_N(\mathbb{C})$, there exists a linear differential equation on X with the monodromy representation ρ.*

Remark 4.5 (About extension). We have seen that in the Fuchsian system, at the neighborhood V_i of a_i (with local coordinate $x = z - a_i$), there exists a Levelt fundamental solution (2.2) of the form $U_i(x)x^{B_i}x^{E_i}$. By Levelt Theorem 2.10, U_i is holomoprhically invertible at a_i. Thus, there exists an invertible matrix $X_i \in GL_N(\mathbb{C})$ such that nearl a_i

$$\Phi(x) = X_i \left(U_i(x)x^{B_i}x^{E_i}\right) X_i^{-1}$$

and such that $X_i^{-1}M_iX_i$ is an upper block triangular matrix.

onversely, one can first use the set of pairs (X_i, B_i) satisfying the conditions

1. $X_i^{-1}M_iX_i$ is upper block triangular,

2. $B_i = \text{diag}(\lambda_{i1}, \ldots, \lambda_{iN})$, $\lambda_{ij} \in \mathbb{Z}$ for all i, j and $\lambda_{ij} \geq \lambda_{i(j+1)}$,

to reconstruct $\widetilde{\Phi}(\widetilde{x})$ on the universal covering S by

$$\widetilde{\Phi}(\tilde{x}) = X_i \left(x^{B_i} (\tilde{x} \cdot \sigma)^{E_i} \right) X_i^{-1} \rho(\sigma), \quad E_i = \frac{1}{2\pi i} \log(X_i^{-1} M_i X_i),$$

where x is the local coordinate centered at a_i. Then, one can apply formula (4.2) to simultaneously extend F and P to all a_i, and the corresponding extended connection at a_i is defined by the differential system satisfied by $\widetilde{\Phi}$.

This way one can recover Plemelj and Kohn's result:

1. there exists a differential system with regular singular points $\{a_1, \ldots, a_n\}$ and monodromy representation ρ, where all but one singular point are Fuchsian.
2. if at least one of $M_i = \rho(\gamma_i)$ is diagonalizable, then there exists a Fuchsian system with singular points $\{a_1, \ldots, a_n\}$ and monodromy representation ρ.

The detailed proof can be found in [9] or Chapter 4 of [55].

4.3 Generalization

With the geometric language of bundles and connection, one can easily generalize the correspondence to higher genus Riemann surfaces and even the higher dimensional complex connected varieties.

Theorem 4.6. *Let X be a connected complex variety. Then the following categories are equivalent:*

1. *the category of rank N flat holomorphic vector bundles over X;*
2. *the category of rank N locally constant sheaves (i.e. local systems) over X;*
3. *the category of N-dimensional representations of $\pi_1(X)$.*

In terms of the modern language, one can consider certain D_X-modules on a smooth variety X instead of differential systems, and consider constructible sheaves on the complex manifold X^{an} associated to X instead of representations of the fundamental group and obtain

Theorem 4.7. *The de Rham functor DR_X gives an equivalence of categories:*

$$DR_X : D^b_{rh}(D_X) \xrightarrow{\sim} D^b_c(X),$$

where

- *$D^b_{rh}(D_X)$ is the bounded derived category of D_X-modules consisting of complexes whose cohomology sheaves are regular holonomic D_X-modules;*
- *$D^b_c(X)$ is the bounded derived category of $C_{X^{an}}$-modules whose cohomology sheaves are constructible.*

5. ISOMONODROMIC DEFORMATION AND MONODROMY DEPENDENCE

5.1 ISOMONODROMIC DEFORMATION AND ISOMONODROMIC TAU FUNCTION

In Riemann's paper, he also studied the desired functions as functions of the branch points when the monodromy is kept invariant. The question was first studied by Schlesinger [63].

Theorem 5.1. *The monodromy matrices of* $\Phi(z;a)$ *are independent of deformation parameters* $\{a_i\}$ *if and only if* $\Phi(z;a)$ *satisfy*

$$\partial_{a_i}\Phi = \Omega_i \Phi, \quad i = 1,\dots,n,$$

where each Ω_i *is a rational matrix function of* z.

More explicitly, for the Fuchsian system (2.3), monodromy preserving requirement implies

$$\frac{\partial \Phi}{\partial a_i} = \left(-\frac{A_i}{z-a_i} + R_i\right)\Phi. \tag{5.1}$$

The compatibility of (2.3) and (5.1) gives the zero curvature equations

$$\begin{aligned} &dA_i + \left[\sum_{j=1}^{n}\left(A_j d\ln(a_i - a_j) + R_j da_j\right), A_i\right] = 0 \\ &\frac{\partial}{\partial a_i}R_j - \frac{\partial}{\partial a_j}R_i + [R_j, R_i] = 0. \end{aligned} \tag{5.2}$$

In particular, choose $R = \sum_i R_i da_i = 0$, we obtain the **Schlesinger equations**

$$\begin{aligned} \frac{\partial A_i}{\partial a_j} &= \frac{[A_i, A_j]}{a_i - a_j}, \quad i \neq j, \\ \frac{\partial A_i}{\partial a_i} &= -\sum_{j\neq i}\frac{[A_i, A_j]}{a_i - a_j}. \end{aligned} \tag{5.3}$$

Remark 5.2 (Discrete isomonodromic deformation). Assume that A_i is similar to a diagonal matrix Θ_i, $i = 1,\dots,n,\infty$, then the following transformation

$$\Theta_i \to \Theta_i + L_i, \quad L_i \in \mathrm{diag}_N(\mathbb{Z}), \quad \sum_{i=1}^{n}\mathrm{tr}\, L_i + L_\infty = 0,$$

preserves the monodromy, which is known as Schlesinger transformation.

Jimbo-Miwa-Ueno [36, 37, 38] developed the theory of the isomonodromic deformation in a general setting. In particular, if A_i's are solutions of Schlesinger equations (5.3), one can check that

$$\omega_{\mathrm{JMU}} = \sum_{i\neq j}\mathrm{tr}\, A_i A_j d\ln(a_i - a_j)$$

is a closed 1-form on $\mathcal{T}$ (the space of the singular points $a_1,\ldots,a_n$, also known as the isomonodromic times). Then, locally there exists a function τ_{JMU} such that

$$d_{\mathcal{T}} \ln \tau_{\text{JMU}} = \omega_{\text{JMU}}.$$

It is called the isomonodromic tau function.

A remarkable property [51, 56] of this tau function is that it admits an analytic continuation as an entire function to the whole universal covering $\widetilde{\mathcal{T}}$.

Riemann-Hilbert problems and isomonodromic deformations play an important role in mathematical physics, in particular, in the theory of integrable systems and theory of random matrices. The main object of interest is the isomonodromic tau function, which has several other striking properties:

- the zeros of tau function correspond to the points in $\mathrm{Conf}_n(\mathbb{C})$ where the inverse monodromy problem (RHP) is not solvable for given monodromy data.
- geometrically, the isomonodromic deformation could be explained as an integrable deformation (E,∇) of the trivial bundle (E^0,∇^0) over $\mathbb{P}^1$. Let

$$\vartheta := \left\{ a \in \mathrm{Conf}_n(\mathbb{C}) \;\middle|\; E\big|_{\mathbb{P}^1\times\{a\}} \text{ is non-trivial} \right\}.$$

  ϑ is called the Malgrange divisor which also corresponds to the zeros of the corresponding tau function.
- the symplectic/Hamiltonian formalism: one can explain the Fuchsian systems (under gauge equivalence) as points of the phase space, and its isomonodromic deformations are trajectories of the corresponding Hamiltonian systems

$$\frac{d(A_j)_{st}}{da_i} = \{(A_j)_{st}, H_i\},$$

  where

  - the Hamiltonian H_i is given by $H_i = \sum_{j\neq i} \frac{\mathrm{tr}(A_iA_j)}{a_i-a_j}$,
  - the Poisson structure is given in terms of r-matrix.

  Then τ_{JMU} becomes a generating function of Hamiltonians.
- determinant representation: isomonodromic tau function can be extended to general isomonodromic deformations (for non-Fuchsian system). In different theories, such a tau function can be interpreted as determinant of the Töplitz operator, the Fredholm operator, or the Cauchy-Riemann operator.
- it is related to quantum field theory developed by Sato, Miwa and Jimbo, where the isomonodromic tau function is defined to be the correlator of monodromy fields (also known as twist fields, which can be seen in the operator formalism as Bogoliubov transformations of the fermion algebra ensuring the required monodromy properties).

- in concrete examples, the isomonodromic tau function is related to the tau function in the applications of Painlevé equations, the representing gap probabilities in random matrix theory, the correlation functions of Ising model and the sine-Gordon field theory at the free-fermion point etc.

5.2 Constant Problem and Monodromy Dependence

Note that τ_{JMU} is defined by

$$d \ln \tau_{\mathrm{JMU}} = \omega_{JMU},$$

where d is taken over the singular points. Then the tau function is defined up to a multiplicative constant which may be related to the monodromy data. In rather special examples, such a constant problem was handled in terms of Fredholm determinant or Töplitz determinant.

Its-Lisovyy-Prokhorov [31] developed the mononodromy-dependent method to solve such constant problem, which is inspired by [4, 51]. The key idea is to find an extension of the Jimbo-Miwa-Ueno differential form ω_{JMU} to a closed 1-form on the whole space

$$\widehat{\mathcal{A}} \simeq \widetilde{\mathcal{T}} \times \mathcal{M},$$

where

- $\widehat{\mathcal{A}}$ denotes the variety of the rational matrix-valued function $A(z)$ of Fuchsian type;
- $\mathcal{M}$ denotes the space of monodromy data.

This means the construction of a differential 1-form $\hat{\omega} = \hat{\omega}(A) = \hat{\omega}(a;M)$ such that

- $d\hat{\omega} = d_{\mathcal{T}}\hat{\omega} + d_{\mathcal{M}}\hat{\omega} = 0$;
- $\hat{\omega}(\partial_{a_i}) = \omega_{\mathrm{JMU}}(\partial_{a_i})$, for any $a_i \in \mathcal{T}$.

They show how this 1-form $\hat{\omega}$ can be used to solve the connection formulae for the isomonodromic tau functions which would include an explicit computation of the relevant constant factors.

5.3 $N=2$ 4-Point Fuchsian System and Painlevé VI Equation

5.3.1 $N=2$ 4-Point Fuchsian System

We consider the $N=2$ Fuchsian system

$$\frac{d\mathbf{y}}{dz} = A(z)\mathbf{y} = \left(\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}\right)\mathbf{y}, \quad A_{0,t,1} \in gl_2(\mathbb{C}) \tag{5.4}$$

with four regular singular points $0, t, 1, \infty$ on $\mathbb{P}^1$.

By the transformation

$$\mathbf{y} \to z^{l_0}(z-t)^{l_t}(z-1)^{l_1}\mathbf{y},$$

the system (5.4) is replaced as

$$A(z) \to A(z) + \left(\frac{l_0}{z} + \frac{l_t}{z-t} + \frac{l_1}{z-1}\right) I.$$

Without loss of generality, we can assume that one of the eigenvalues of A_i ($i = 0, t, 1$) is zero, and by a constant gauge transformation

$$A_\infty = -(A_0 + A_t + A_1) = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

By eliminating y_2 in $\mathbf{y} = (y_1, y_2)^T$ of (5.4), we obtain a 2nd order linear differential equation

$$\begin{aligned} \frac{d^2 y_1}{dz^2} &+ \left(\frac{1-\theta_0}{z} + \frac{1-\theta_t}{z-t} + \frac{1-\theta_1}{z-1} - \frac{1}{z-q}\right)\frac{dy_1}{dz} \\ &+ \left(\frac{\kappa_1(\kappa_2+1)}{z(z-1)} + \frac{q(q-1)p}{z(z-1)(z-q)} - \frac{t(t-1)H}{z(z-1)(z-t)}\right) y_1 = 0, \end{aligned} \tag{5.5}$$

where

- θ_i is the other eigenvalue of A_i, $i = 0, t, 1$;
- q is the zero of the (12)-entry $A_{12}(z)$ of the matrix $A(z)$;
- $p = A_{11}(q)$;
- the function H is given by

$$\begin{aligned} H = \frac{1}{t(t-1)} &\{q(q-1)(q-t)p^2 - \{\theta_0(q-1)(q-t) + \theta_1 q(q-t) \\ &+ (\theta_t - 1)q(q-1)]p + \kappa_1(\kappa_2+1)(\lambda - t)\}. \end{aligned}$$

The equation (5.5) has regular singularities at $z = 0, t, 1, q, \infty$, but the local exponents of $z = q$ are 0,2, thus it is an apparent singularity.

5.3.2 Middle Convolution

Dettweiler-Reiter [19] presented both multiplicative (MC_λ) and additive (mc_μ) versions of Katz's algebraic analogue and reproduce Katz's main result. Here MC_λ is a functor of category of finite dimensional $\mathbb{C}[F_r] - \text{mod}$ of free group F_r to iteself; and the additive one mc_μ is related to MC_λ via Riemann-Hilbert corespondence. They [20] also gave a cohomological interpretation and applyed their theory for the construction of explicit algebraic solution of Painleé VI equation.

Now we review their theory for the present setting. For the above 2×2 matrices A_0, A_t, A_1, and $\mu \in \mathbb{C}$, one can define the following 6×6 convolution matrices

B_0, B_t, B_1:

$$B_0 = \begin{pmatrix} A_0+\mu & A_t & A_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_t = \begin{pmatrix} 0 & 0 & 0 \\ A_0 & A_t+\mu & A_1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_0 & A_t & A_1+\mu \end{pmatrix}.$$

Proposition 5.3. *Let γ_a $(a \in \mathbb{C})$ be a cycle in $\mathbb{C} \setminus \{0,t,1,z\}$ with a base point z_0 and $[\gamma_z, \gamma_a] = \gamma_z \gamma_a \gamma_z^{-1} \gamma_a^{-1}$ be the Pochhammer countor. Assume that $\mathbf{y} = (y_1, y_2)^T$ be a solution of (5.4), then for $a = 0,t,1,\infty$, the vector function*

$$\mathbf{u} = \begin{pmatrix} \int_{[\gamma_z,\gamma_a]} w^{-1} y_1(w)(z-w)^{\mu} dw \\ \int_{[\gamma_z,\gamma_a]} w^{-1} y_2(w)(z-w)^{\mu} dw \\ \int_{[\gamma_z,\gamma_a]} (w-t)^{-1} y_1(w)(z-w)^{\mu} dw \\ \int_{[\gamma_z,\gamma_a]} (w-t)^{-1} y_2(w)(z-w)^{\mu} dw \\ \int_{[\gamma_z,\gamma_a]} (w-1)^{-1} y_1(w)(z-w)^{\mu} dw \\ \int_{[\gamma_z,\gamma_a]} (w-1)^{-1} y_2(w)(z-w)^{\mu} dw \end{pmatrix}$$

satisfies the differential equation

$$\frac{d\mathbf{u}}{dz} = \left(\frac{B_0}{z} + \frac{B_t}{z-t} + \frac{B_1}{z-1} \right) \mathbf{u}.$$

Note that there exist invariant subspaces of column vector space $\mathbb{C}^6$:

1. $\mathcal{L}_0 = (\ker(A_0), 0, 0)^T$, $\mathcal{L}_t = (0, \ker(A_t), 0)^T$, $\mathcal{L}_1 = (0, 0, \ker(A_1))^T$;
2. $\mathcal{K} = \ker(B_0) \cap \ker(B_t) \cap \ker(B_1) = \ker(B_0 + B_t + B_1)$.

Let $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_t \oplus \mathcal{L}_1$, and fix an isomorphism between $\mathbb{C}^6/(\mathcal{L}+\mathcal{K})$ and $\mathbb{C}^m$ for some m. The matrices

$$mc_{\mu}(A_0, A_t, A_1) := (\widetilde{B}_0, \widetilde{B}_t, \widetilde{B}_1) \in (gl_m(\mathbb{C}))^3,$$

where $\widetilde{B}_{0,t,1}$ are induced by $B_{0,t,1}$ under the isomorphism $\mathbb{C}^6/(\mathcal{L}+\mathcal{K}) \cong \mathbb{C}^m$, are called the additive version of the middle convolution $A_{0,t,1}$ with parameter μ.

Since the eigenvalues of A_i are 0 and θ_i, $i = 0,t,1$, we can express A_i as follows:

$$A_i = \begin{pmatrix} z_i + \theta_i & -u_i z_i \\ u_i^{-1}(z_i + \theta_i) & -z_i \end{pmatrix}$$

Then we can compute

$$\mathcal{L}_0 = (u_0 z_0/(\theta_0 + z_0), 1, 0, 0, 0, 0)^T,$$
$$\mathcal{L}_t = (0, 0, u_t z_t/(\theta_t + z_t), 1, 0, 0)^T,$$
$$\mathcal{L}_1 = (0, 0, 0, 0, u_1 z_1/(\theta_1 + z_1), 1)^T.$$

If $\mu \neq 0$, $\mathcal{K}$ is spanned by $(l_1,l_2,l_1,l_2,l_1,l_2)^T$, where $(l_1,l_2)^T \in \ker(A_0+A_t+A_1+\mu)$. Then $\mathcal{K} \neq \emptyset$ in case μ coincides with one of the eigenvalues of A_∞.

Lemma 5.4. *If* $\mu = 0, \kappa_1, \kappa_2$*, then* $\dim \mathbb{C}^6/(\mathcal{L}+\mathcal{K}) = 2$*, i.e.* $m = 2$*; otherwise,* $m = 3$*. In particular, if* $\mu = 0$*, the middle convolution is the identity.*

If $\mu = \kappa_1, \kappa_2$, $mc_\mu(A_0,A_t,A_1)$ give the non-trivial transformations on the 2×2 Fuchsian system with 4 singularties $0,t,1,\infty$. The corresponding transformations of the resulting isomonodromic defomration equations coincide with Okamoto's birational canonical transformations [60].

Remark 5.5. Cecotti [11] investigated the relation between the theory of the Fuchsian system and the theory of Seiberg dualities for quiver SUSY gauge theories in the spirit of Physical Mathematics. In particular, he explained Katz's middle convolution as the Seiberg duality at the central node of the squid quiver.

5.3.3 Painlevé VI Equation

The simplest isomonodromic deformation is associated to the $N = 2$ 4-point Fuchsian system. The Schlesinger equations for the isomonodromic deformations of (5.4) are

$$\frac{dA_0}{dt} = \frac{[A_t,A_0]}{t}, \quad \frac{dA_1}{dt} = \frac{[A_t,A_1]}{t-1}. \tag{5.6}$$

Let q be the zero point of the (12)-entry $A_{12}(z)$ as above, that is,

$$A_{12}(z) = \frac{k(z-q)}{z(z-1)(z-t)}.$$

Then the Schlesinger equation (5.6) implies that $q = q(t)$ satisfies the Painlevé VI equation:

$$\begin{aligned} \frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt} \\ + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{q^2}+\gamma\frac{t-1}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right], \end{aligned} \tag{5.7}$$

where the parameters $(\alpha,\beta,\gamma,\delta)$ are related to the eigenvalues of $A_{0,t,1,\infty}$ as follows:

$$\alpha = \frac{1}{2}(\theta_\infty-1)^2, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1}{2}-\frac{\theta_t^2}{2}.$$

This is the most general ODE of the form $q'' = F(t,q,q')$, with F rational in q, q' and t, whose general solution has no movable branch points and essential singularities. It can therefore be analytically continued to a meromorphic function on the universal covering of $\mathbb{P}^1 \setminus \{0,1,\infty\}$.

Remark 5.6. In some papers, one also considers the 4-point sl_2-Fuchsian system, where A_i $(i=0,t,1)$ have eigenvalues $\pm\theta_i$ and $A_\infty = -(A_0+A_t+A_1) = \mathrm{diag}(\theta_\infty, -\theta_\infty)$. Then one can also obtain Painlevé VI equation (5.7) with

$$\alpha = \frac{1}{2}(2\theta_\infty - 1)^2, \quad \beta = -2\theta_0^2, \quad \gamma = 2\theta_1^2, \quad \delta = \frac{1}{2} - 2\theta_t^2.$$

In the following, we use the sl_2 set-up.

There are many special solutions of Painlevé VI equation with important applications:

- Riccati solutions: these solutions appear when the monodromy representation is equivalent to an upper triangular one.
- Chazy solutions: these solutions can be obtained by Bäcklund transformation of $\mathrm{PVI}_{\alpha,0,0,0}$ from the singular solutions with $\alpha = 0$. In the conformal field, such solutions also correspond to the insertion of singular vectors.
- Picard solutions: these solutions correspond to some orbit of $\theta_{0,t,1,\infty} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. Up to Bäcklund transformations, this is the only case where the general two-parameter solution of PVI is available [52].
- Algebraic solutions: these solutions correspond to finite orbits of the braid/modular group action on monodromy of the associated linear system. Lisovyy-Tykhyy [49] classified all algebraic solutions of the general Painlevé VI equation up to parameter equivalence. In particular, known examples of algebraic solutions turn out to be related to various mathematical structures, including e.g. Frobenius manifolds, symmetry groups of regular polyhedra, complex reflections, Grothendieck's dessins d'enfants and their deformations.

5.3.4 Isomonodromy/CFT/Gauge Theory Correspondence

In 1982, Jimbo [34] gave the asmyptotic expansion of the tau function for Painlevé VI.

In 2012, Gamayun-Its-Lisovyy [23] conjectured an amazing complete series expansion for the tau function for Painlevé VI via the AGT correspondence [1]. Near $t=0$,

$$\tau_{PVI} = \mathrm{const} \sum_{n\in\mathbb{Z}} C_n(\boldsymbol{\theta}, \boldsymbol{\sigma}) t^{(\sigma_{0t}+n)^2 - \theta_0^2 - \theta_t^2} \mathcal{B}(\boldsymbol{\theta}, \sigma_{0t}+n; t), \tag{5.8}$$

where

- $\boldsymbol{\theta} = (\theta_0, \theta_t, \theta_1, \theta_\infty)$;
- $\boldsymbol{\sigma} = (\sigma_{0t}, \sigma_{1t})$, where σ_{ij} are defined via $\mathrm{tr} M_i M_j = 2\cos 2\pi\sigma_{ij}$, and $0 < |\mathrm{Re}\,\sigma_{ij}| < 1/2$;
- the structure constants $C_n(\boldsymbol{\theta}, \boldsymbol{\sigma})$ can be solved recursively and given in terms of Barnes G-function;

- $\mathcal{B}(\boldsymbol{\theta},\sigma;t)$ is a power series in t which coincides with the $c=1$ 4-point Liouville conformal block. More explcitly, it admits an expression via configuration space of double Young diagrams

$$\mathcal{B}(\boldsymbol{\theta},\sigma;t)=(1-t)^{\theta_t\theta_1}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta},\sigma)t^{|\lambda|+|\mu|},$$

where

$$\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta},\sigma)=\prod_{(i,j)\in\lambda}\frac{\left[(\theta_t+\sigma+i-j)^2-\theta_0^2\right]\left[(\theta_1+\sigma+i-j)^2-\theta_\infty^2\right]}{h_\lambda^2(i,j)(\lambda_j'-i+\mu_i-j+1+2\sigma)^2}$$
$$\times\prod_{(i,j)\in\mu}\frac{\left[(\theta_t-\sigma+i-j)^2-\theta_0^2\right]\left[(\theta_1-\sigma+i-j)^2-\theta_\infty^2\right]}{h_\mu^2(i,j)(\mu_j'-i+\lambda_i-j+1-2\sigma)^2}.$$

This conjecture was proved via two different ways [3, 33]. The first one is to check such an expansion satisfies the Painlevé bilinear relation. The second one is to consider the operator-valued monodromy of conformal blocks with additional level 2 degenerate insertions. At $c=1$, the Fourier transform of such conformal blocks reduces the operator-valued monodromy to ordinary 2×2 matrices, which can be used to construct the solution of the RHP and tau function.

Similarly, one can obtain a series expansion of τ_{PVI} near $t=1$. In 2018, Ito-Lisovyy-Prokhorov [31] used the idea of monodromy denpendence and solved the conjectural formula for the constant problem proposed by Iorgov-Lisovyy-Tykhyy [32].

In the general case of n Fuchsian singular points on $\mathbb{P}^1$, the isomonodromic tau function admits a Fredholm determinant representation [24]. One can compute the Fredholm determinant via the von Koch's formula, where the combinatorial structure coincides with the one in Nekrasov-Okounkov instanton partition function [58, 59], as sums over tuples of Young diagrams.

Acknowledgements

I express my deep gratitude to Lynn Heller and Nathan Thomas Carruth for the meticulous perusal and the insightful suggestions. I also appreciate Prof. Shing-Tung Yau for his propose, encouragement and support. The work of the author is supported by research funding at BIMSA.

References

[1] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories.* Lett. Math. Phys. 91 (2010), 167–197. MR2586871

[2] D. V. Anosov and A. A. Bolibruch, *The Riemann-Hilbert Problem.* (Aspects of Mathematics, E22), 1994. MR1276272

[3] M. A. Bershtein and A. I. Shchechkin, *Bilinear equations on Painlevé τ functions from CFT.* Commun. Math. Phys. 339 (2015), 1021–1061. MR3385990

[4] M. Bertola, *The dependence on the monodromy data of the isomonodromic tau function*. Comm. Math. Phys. 294 (2010), 539–579. MR2579465

[5] F. Beukers and G. Heckman, *Monodromy for the hypergeometric function* ${}_nF_{n-1}$. Invent. Math. 95 (1989), 325–354. MR0974906

[6] G. D. Birkhoff, *The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations*. Proc. Am. Acad. Arts Sci. 49 (1913), 521–568.

[7] A. A. Bolibrukh, *The Riemann-Hilbert problem*. Uspekhi Mat. Nauk 45 (1990), 3–47. MR1069347

[8] A. A. Bolibrukh, *On sufficient conditions for the positive solvability of the Riemann-Hilbert problem*. Mat. Zametki. 51 (1992), 110–117. MR1165460

[9] A. A. Bolibrukh, *On sufficient conditions for the existence of a Fuchsian equation with prescribed monodromy*. J. Dynam. Control Systems 5 (1999), 453–472. MR1722011

[10] T. Bothner, *On the origins of Riemann-Hilbert problems in mathematics*. Nonlinearity, 34 (2021), 1–73. MR4246443

[11] S. Cecotti, *Fuchsian ODEs as Seiberg dualities*. Adv. Math. Phys. 27(8) (2023), 2429–2490. arXiv:2212.09370v4. MR4788093

[12] G. Chênevert, *The Riemann-Hilbert Problem*. Final project for the course, Vector Bundles on Curves by Prof. Eyal Z. Goren.

[13] W. Crawley-Boevey, *On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero*. Duke Math. J. 118(2) (2003), 339–352. MR1980997

[14] D. V. Chudnovsky and G. V. Chudnovsky, *Applications of Padé approximations to the Grothendieck conjecture on linear differential equations*. In: Chudnovsky, D.V., Chudnovsky, G.V., Cohn, H., Nathanson, M.B. (eds) Number Theory. Lecture Notes in Mathematics, vol. 1135, pp. 52–100. Springer, Berlin, Heidelberg. MR0803350

[15] P. A. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems*. Asymptotics for the MKdV equation Ann. Math. 137 (1993), 295–368. MR1207209

[16] P. A. Deift, A. R. Its, A. Kapaev and X. Zhou, *On the algebro-geometric integration of the Schlesinger equations*. Commun. Math. Phys. 203 (1999), 613–633. MR1700154

[17] W. Dekkers, *The Matrix of a Connection Having Regular Singularities on a Vector Bundle of Rank 2 on* $\mathbb{P}^1(\mathbb{C})$. Lecture Notes in Mathematics vol. 712, 1970. MR0548141

[18] P. Deligne, *Équations Différentielles à Points Singuliers Réguliers*. Lecture Notes in Mathematics, vol. 163. Springer, Berlin. MR0417174

[19] M. Dettweiler and S. Reiter, *Middle convolution of Fuchsian systems and the construction of rigid differential systems*. Journal of Algebra 318 (2007), 1–24. MR2363121

[20] M. Dettweiler and S. Reiter, *Painlevé equations and the middle convolution*. Adv. Geom. 7 (2007), 317–330. MR2334802

[21] V. Enolski and T. Grava, *Singular* $\mathbb{Z}_N$*-curves and the Riemann-Hilbert problem*. Internat. Math. Res. Notices 32 (2004), 1619–1683. MR2035223

[22] N. P. Erugin, *Problema Rimana (The Riemann Problem)*. Lecture Notes in Mathematics, vol. 936. Nauka i Tekhnika, Minsk. MR0665587

[23] O. Gamayun, N. Iorgov and O. Lisovyy, *Conformal field theory of Painlevé VI*. J. High Energy Phys. 2012 (2012), No. 038. MR3033813

[24] P. Gavrylenko and O. Lisovyy, *Fredholm determinant and Nekrasov sum*

representations of isomonodromic tau functions. Commun. Math. Phys. 363 (2018), 1–58. MR3849982

[25] P. Gavrylenko and A. Marshakov, *Free fermions, W-algebras and isomonodromic deformations*. Theor. Math. Phys. 187 (2016), 649–677. MR3507535

[26] A. I. Gladyshev, *On the Riemann-Hilbert problem in dimension 4*. J. Dynam. Control Systems 6(2) (2000), 219–264. MR1746973

[27] R. R. Gontsov and V. A. Poberezhnyi, *Various versions of the Riemann-Hilbert problem for linear differential equations*. Russian Math. Surveys 63(4) (2008), 603–639. MR2483198

[28] Y. Haraoka, *Finite monodromy of Pochhammer equation*. Ann. Inst. Fourier, Grenoble 44 (1994), 767–810. MR1303884

[29] D. Hilbert, *Mathematische Probleme*. In: Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse. Heft 3, 1900, S. 253–297.

[30] D. Hilbert, *Mathematical problems*. Bull. Amer. Math. Soc. 8 (1902), 437–479. MR1557926

[31] A. R. Its, O. Lisovyy and A. Prokhorov, *Monodromy dependence and connection formulae for isomonodromic tau functions*. Duke Mathmatical Journal 167(7) (2018), 1347–1432. MR3799701

[32] A. R. Its, O. Lisovyy and Yu. Tykhyy, *Connection problem for the sine-Gordon/Painlevé III tau function and irregular conformal blocks*. Int. Math. Res. Not. IMRN 18 (2015), 8903–8924. MR3417698

[33] N. Iorgov, O. Lisovyy and J. Teschner, *Isomonodromic tau-functions from Liouville conformal blocks*. Commun. Math. Phys. 336 (2015), 671–694. MR3322384

[34] M. Jimbo, *Monodromy problem and the boundary condition for some Painlevé equations*. Publ. Res. Inst. Math. Sci. 18 (1982), 1137–1161. MR0688949

[35] M. Jimbo, T. Miwa and M. Sato, *Holonomic quantum fields I–V*. Publ. RIMS Kyoto Univ. 14 (1978), 223–267; 15 (1979), 201–278; 15 (1979), 577–629; 15 (1979), 871–972; 16 (1980), 531–584. MR0499666

[36] M. Jimbo, T. Miwa and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I*. Phys. D 2 (1981), 306–352. MR0630674

[37] M. Jimbo and T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II*. Phys. D 2 (1981), 407–448. MR0625446

[38] M. Jimbo and T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III*. Phys. D 2 (1981–1982), 26–46. MR0636469

[39] M. Kashiwara, *The Riemann-Hilbert problem for holonomic systems*. Publ. Res. Inst. Math. Sci. 20 (1984) 319–365. MR0743382

[40] N. Katz, *A conjucture in the arithmetic theory of differential equations*. Bulletin de la S. M. F., 110 (1982), 203–239. MR0667751

[41] N. Katz, *Rigid Local Systems*. Annals of Mathematics Studies, vol. 139. Princeton University Press (1995). MR1366651

[42] A. V. Kitaev and D. A. Korotkin, *On solutions of the Schlesinger equations in terms of* Θ*-functions*. Inter. Math. Res. Not. 17 (1998), 877–905. MR1646648

[43] A. T. Kohn, *Un résultat de Plemelj*. Progr. Math. 37 (1983), 307–312. MR0728426

[44] D. Korotkin, *Solution of matrix Riemann-Hilbert problems with quasi-permutation monodromy matrices*. Mathematische Annalen 329 (2004),

335–364. MR2060367

[45] V. P. Kostov, *Fuchsian linear systems on $\mathbb{C}\mathbb{P}^1$ and the Riemann-Hilbert problem*. C. R. Acad. Sci. Paris Sér. I Math. 315 (1992), 143–148. MR1197226

[46] B. L. Krylov, *The solution in a finite form of the Riemann problem for a Gauss system*. Trudy Kazan. Aviats. Inst. 31 (1956), 203–445.

[47] I. A. Lappo-Danilevskiĭ, *Application of Matrix Functions to the Theory of Linear Systems of Ordinary Differential Equations*. Gosudarstv. Izdat. Tehkn. Teor. Lit., Moscow (1957). MR0090529

[48] A. H. M. Levelt, *Hypergeometric functions*. Nederl. Akad. Wet., Proc. Ser. A. 64 (1961), 361–403. MR0137856

[49] O. Lisovyy and Yu. Tykhyy, *Algebraic solutions of the sixth Painlevé equation*. Journal of Geometry and Physics 85 (2014), 124–163. MR3253555

[50] S. Malek, *Fuchsian systems with reducible monodromy are meromorphically equivalent to Fuchsian systems*. Proc. Steklov Inst. Math. 3(326) (2002), 468–477. MR1931047

[51] B. Malgrange, *Sur les déformations isomonodromiques, I: Singularités régulières*. Mathematics and Physics (Paris, 1979/1982), Progr. Math., vol. 37, pp. 401–426. Birkhäuser, Boston (1983). MR0728431

[52] M. Mazzocco, *Picard and Chazy solutions to the Painlevé VI equation*. Math. Ann. 321 (2001), 157–195. MR1857373

[53] Z. Mebkhout, *Sur le problème de Hilbert-Riemann*. In: Iagolnitzer, D. (eds) Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory. Lecture Notes in Physics, vol. 126. pp. 90–110. Springer, Berlin, Heidelberg (1980). MR0579742

[54] Z. Mebkhout, *Une autre équivalence de catégories*. Compositio Math 51 (1984), 63–88. MR0734785

[55] C. Mitschi and D.Sauzin, *Divergent Series, Summability and Resurgence I. Monodromy and Resurgence*. Lecture Notes in Mathematics, vol. 2153. MR3526111

[56] T. Miwa, *Painlevé property of monodromy preserving deformation equations and the analyticity of τ functions*, Publ. Res. Inst. Math. Sci. 17 (1981), 703–712. MR0642657

[57] G. Moore, *Geometry of the string equations*. Comm. Math. Phys. 133 (1990), 261–304. MR1090426

[58] N. Nekrasov, *Seiberg-Witten prepotential from instanton counting*. Adv. Theor. Math. Phys. 7 (2004), 831–864. MR2045303

[59] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*. “The unity of mathematics”, 525–596. MR2181816

[60] K. Okamoto, *Studies on the Painlevé equations. I. Sixth Painlevé equation PVI*. Ann. Mat. Pura Appl. (4) 146 (1987), 337–381. MR0916698

[61] J. Plemelj, *Riemannsche Funktionenscharen mit gegebener Monodromiegruppe*. Monatsh. f. Mathematik und Physik 19 (1908), 211–245. MR1547764

[62] H. Röhrl, *Das Riemann-Hilbertsche problem der theorie der linearen differentialgleichungen*. Math. Ann. 133 (1957), 1–25. MR0086958

[63] L. Schlesinger, *Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten*. Journal für die reine und angewandte Mathematik 141 (1912), 96–145. MR1580847

[64] C. T. Simpson, *Products of matrices, Department of Mathematics, Princeton University, NJ*. In: Differential Geometry, Global Analysis and Topology, Proceedings of the Halifax Symposium, June 1990. Canad. Math. Soc.

Conf. Proc., vol. 12, pp. 157–185. Amer. Math. Soc., Providence, RI, 1991. MR1158474

[65] C. Tretkoff and M. Tretkoff, *Solution of the inverse problem of differential Galois theory in the classical case* Amer. J. Math. 101 (1979), 1327–1332. MR0548884

Xinxing Tang
tangxinxing@bimsa.cn
Beijing Institute of Mathematical Sciences and Applications
Huairou District, Beijing
China