MODULAR TENSOR CATEGORIES ARISING FROM CENTRAL EXTENSIONS AND RELATED APPLICATIONS

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ABSTRACT. A modular tensor category is a non-degenerate ribbon finite tensor category and a ribbon factorizable Hopf algebra is a Hopf algebra whose finite-dimensional representations form a modular tensor category. In this paper, we provide a method of constructing ribbon factorizable Hopf algebras using central extensions. We then apply this method to n-rank Taft algebras, which are considered finite-dimensional quantum groups associated with abelian Lie algebras (see Section 2 for the definition), and obtain a family of non-semisimple ribbon factorizable Hopf algebras E_q, thus producing non-semisimple modular tensor categories using their representation categories. And we provide a prime decomposition of $\mathrm{Rep}(E_q)$ (the representation category of E_q). By further studying the simplicity of E_q (whether it is a simple Hopf algebra or not), we conclude that

(1) there exists a twist J of $u_q(sl_2^{\oplus 3})$ such that $u_q(sl_2^{\oplus 3})^J$ is a simple Hopf algebra,
(2) there is no relation between the simplicity of a Hopf algebra H and the primality of $\mathrm{Rep}(H)$,
(3) there are many ribbon factorizable Hopf algebras that are distinct from some known ones, i.e., not isomorphic to any tensor products of trivial Hopf algebras (group algebras or their dual), Drinfeld doubles, and small quantum groups.

1. INTRODUCTION

Originally, a modular tensor category meant a ribbon fusion category whose S-matrix is invertible. Modular tensor categories have been widely researched in connection with conformal field theories, topological quantum field theories, and quantum computing (e.g., [27, 35, 21]). For example, a modular tensor category provides a topological quantum field theory in dimension 3, and in particular, invariants of links and 3-manifolds [25]. As pointed out in [12], a large source of modular tensor categories is provided by the representation categories of Hopf algebras. In [37]-[39], Lyubashenko studied the "non-semisimple" generalization of a modular tensor category. Kerler and Lyubashenko used the term "modular tensor category" to mean a non-degenerate ribbon finite tensor category [34]. Moreover, they showed that a "non-semisimple" modular tensor category also yields an invariant of closed 3-manifolds and a projective representation of the mapping class group of a closed surface as in the semisimple case in [34].

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Given a Hopf algebra H, it is known that $\mathrm{Rep}(H)$ (the representation category of H) is a modular tensor category if and only if H is a ribbon factorizable Hopf algebra (see, e.g., [18]). Modular tensor categories arising from ribbon factorizable Hopf algebras have many ideal properties. For example, they have Verlinde formula even in non-semisimple cases [20]. Perhaps the most well-known examples of ribbon factorizable Hopf algebras come from quantum doubles $D(H)$, which were introduced by Drinfeld in [36]. In [19], Kauffman and Radford characterized the conditions for $D(H)$ to be a ribbon Hopf algebra, i.e., all the ribbon factorizable Hopf algebras coming from quantum doubles were determined abstractly. Using this result, many ribbon factorizable Hopf algebras can be constructed concretely. For example, Hu and Wang obtained ribbon factorizable Hopf algebras by studying restricted two-parameter quantum groups in this way [24]. Aside from considering quantum doubles directly, Gelaki and Westreich determined when a quantum group $U_q(sl_n)'$ has ribbon elements and when it is a factorizable Hopf algebra in [29], hence providing ribbon factorizable Hopf algebras. Another celebrated work related to ribbon factorizable Hopf algebras was that by Laugwitz and Waton in 2022 [28]. They mainly used Shimizu's result [17] on relative Muger centers to obtain modular tensor categories. In particular, they recovered some small quantum groups by applying this method and constructed a family of non-semisimple modular tensor categories in [28]. These modular tensor categories are also representation categories of ribbon factorizable Hopf algebras. Inspired by this method and the fact (pointed out in [12]) that representation categories of Hopf algebras provide a significant source of modular tensor categories, we will explore the construction of ribbon factorizable Hopf algebras in this work, thus providing modular tensor categories through their representations.

It has been shown in many previous works (such as [18, 9, 27, 34]) that typical and known ribbon factorizable Hopf algebras are tensor products of trivial Hopf algebras (group algebras or their dual), quantum doubles, and small quantum groups. In this paper, we will see that there are many ribbon factorizable Hopf algebras that do not belong to these cases. In addition, from the result of [22, 28], we know that every modular tensor category is a finite direct product of prime modular tensor categories. In [22], Muger studied the prime decomposition of $\mathrm{Rep}(D(G))$ concretely, where G is a finite abelian group. Using his decomposition, he showed that there is no relation between the simplicity of G and the primality of $\mathrm{Rep}(D(G))$. This fact motivates us to further determine if there is any relation between the simplicity of H itself and the primality of $\mathrm{Rep}(H)$ when H is a ribbon factorizable Hopf algebra. Since there has been little work on the prime decomposition of modular tensor categories (see, e.g., [22, 7, 31]), this problem has not yet been addressed. In this paper, we will study how Hopf algebras can be used to perform a prime decomposition and try to answer this question. Beyond considering the above relationship, we will also study whether the simplicity of H is determined by its tensor category of representations when H is a ribbon factorizable Hopf algebra. This exploration is mainly inspired by the work of Galindo and Natale [10]. In that paper, they showed that the notion of simplicity of a semisimple Hopf algebra is not determined by its tensor category of representations. We will further see that the notion of simplicity of a Hopf algebra H

is not determined by its tensor category of representations even if H is assumed to be a ribbon factorizable Hopf algebra.

This paper is organized as follows. In Section 1, we first recall some preliminary knowledge, which mainly includes the definitions of a modular tensor category and a ribbon factorizable Hopf algebra, central extensions, n-rank Taft algebras, small quantum groups. Section 2 is devoted to giving a general way to construct ribbon factorizable Hopf algebras. To do this, we employ the double centralizer theorem of modular tensor categories to central extensions. In the final section, we apply our method to n-rank Taft algebras, yielding a family of ribbon factorizable Hopf algebras denoted as E_q. We proceed to describe E_q using generators and relations, and explicitly provide its universal $\mathcal{R}$-matrix and ribbon element. Utilizing these descriptions, we offer a prime decomposition of $\mathrm{Rep}(E_q)$. By investigating the simplicity of E_q, we address previous inquiries. Furthermore, we compare E_q with known ribbon factorizable Hopf algebras, including those presented in [28, Example 5.18] in 2022.

Convention 1.1. Throughout this paper we work over an algebraically closed field $\mathbb{k}$ of characteristic 0. All Hopf algebras in this paper are finite-dimensional. For the symbol δ, we mean the classical Kronecker's symbol. Our references for the theory of Hopf algebras are [11, 33]. For a Hopf algebra H, the set of group-like elements in H will be denoted by $G(H)$.

2. Preliminaries

We collect some necessary notions and results in this section.

2.1. Modular tensor categories and ribbon factorizable Hopf algebras.

A finite tensor category $(\mathcal{C}, \otimes, \mathbb{1})$ [27] is defined as a rigid monoidal category where $\mathcal{C}$ is a finite abelian category, the tensor product of $\mathcal{C}$ is $\mathbb{k}$-linear in each variable, and the unit object of $\mathcal{C}$ is a simple object. If $(\mathcal{C}, \otimes, \mathbb{1})$ is equipped with a braiding c, then it is termed a braided finite tensor category [27]. In such a case, the Muger center $\mathcal{C}'$ is the full subcategory on the objects $\mathrm{Ob}(\mathcal{C}') = \{X \in \mathcal{C} | \ c_{Y,X}c_{X,Y} = \mathrm{Id}_{X\otimes Y} \text{ for all } Y \in \mathcal{C}\}$.

Denote the tensor category of finite-dimensional vector spaces over $\mathbb{k}$ as $\mathbf{vect}_{\mathbb{k}}$. Recall the following equivalent definition of a non-degenerate braided finite tensor category, as provided by [17, Theorem 1.1].

Definition 2.1. *A braided finite tensor category $(\mathcal{C}, \otimes, \mathbb{1}, c)$ is called non-degenerate if the Muger center $\mathcal{C}'$ is equal to $\mathbf{vect}_{\mathbb{k}}$.*

To review the definition of a modular tensor category, we need ribbon tensor category. A braided tensor category $(\mathcal{C}, \otimes, \mathbb{1}, c)$ is ribbon if it is equipped with a natural isomorphism $\theta_X : X \xrightarrow{\sim} X$ (a twist) satisfying $\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}$ and $(\theta_X)^* = \theta_{X^*}$ for all $X, Y \in \mathcal{C}$. Now we recall:

Definition 2.2. [34, Definition 5.2.7], [17, Theorem 1] *A braided finite tensor category is called modular if it is non-degenerate and ribbon.*

Recall that a full subcategory of an abelian category is called *topologizing subcategory* [3, 17] if it is closed under finite direct sums and subquotients. By a *braided tensor subcategory* of a braided tensor category $(\mathcal{C}, \otimes, \mathbb{1}, c)$ we mean a subcategory of $\mathcal{C}$ containing the unit object of $\mathcal{C}$, closed under the tensor product of $\mathcal{C}$, and containing the braiding isomorphisms. Let S be a subset of objects of a braided category $(\mathcal{C}, \otimes, \mathbb{1}, c)$, the Muger centralizer $C_{\mathcal{C}}(S)$ [22, Definition 2.6] of S in $\mathcal{C}$ is defined as the full subcategory of $\mathcal{C}$ with objects

$$\mathrm{Ob}(C_{\mathcal{C}}(S)) := \{X \in \mathcal{C} |\ c_{Y,X} c_{X,Y} = \mathrm{Id}_{X\otimes Y} \ \text{ for all } Y \in S\}.$$

The following theorem was implied by [28, Theorem 4.3].

Theorem 2.3. *Let $(\mathcal{D}, \otimes, \mathbb{1}, c)$ be a non-degenerate finite braided tensor category, let $\mathcal{E}$ be a topologizing braided tensor subcategory of $\mathcal{D}$. Then $C_{\mathcal{D}}(\mathcal{E})$ is a non-degenerate braided tensor category if and only if $\mathcal{E}$ is a non-degenerate braided tensor category.*

Now we recall general prime modular tensor categories (including non-semisimple cases).

Definition 2.4. [22, 28] *A modular tensor category $\mathcal{C}$ is* prime *if every topologizing non-degenerate braided tensor subcategory is equivalent to either $\mathcal{C}$ or $\mathbf{vect}_{\Bbbk}$.*

A fact that every modular tensor category is equivalent to a finite Deligne tensor product of prime modular tensor categories is shown in [28, Corollary 4.20]. In Section 4, we will use following result to study prime decomposition.

Theorem 2.5. *Let $\mathcal{D}$ be a modular tensor category, with a topologizing non-degenerate braided tensor subcategory $\mathcal{E}$. Then there is an equivalence of ribbon categories:*

$$\mathcal{D} \simeq \mathcal{E} \boxtimes C_{\mathcal{D}}(\mathcal{E}).$$

To construct modular tensor categories utilizing Hopf algebras, we need ribbon factorizable Hopf algebras. Recall that a quasitriangular Hopf algebra is a pair (H, R) where H is a Hopf algebra over $\Bbbk$ and $R = \sum R^{(1)} \otimes R^{(2)}$ is an invertible element in $H \otimes H$ such that

$$(\Delta \otimes \mathrm{Id})(R) = R_{13}R_{23},\ (\mathrm{Id} \otimes \Delta)(R) = R_{13}R_{12},\ \Delta^{op}(h)R = R\Delta(h),$$

for $h \in H$. Here by definition $R_{12} = \sum R^{(1)} \otimes R^{(2)} \otimes 1$ and similarly for R_{13} and R_{23}. A quasitriangular Hopf algebra (H, R) is a ribbon Hopf algebra if there exists a central element θ in H satisfying the relations:

$$\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta),\ \epsilon(\theta) = 1,\ S(\theta) = \theta.$$

For a quasitriangular Hopf algebra (H, R), there are linear maps $f_{R_{21}R} : H^{*cop} \to H$ and $g_{R_{21}R} : H^{*op} \to H$, given respectively by

$$f_{R_{21}R}(a) := (a \otimes \mathrm{Id})(R_{21}R),\quad g_{R_{21}R}(a) := (\mathrm{Id} \otimes a)(R_{21}R),\ a \in H^*. \tag{2.1}$$

A factorizable Hopf algebra is a quasitriangular Hopf algebra (H, R) such that $f_{R_{21}R}$, or equivalently $g_{R_{21}R}$, is a linear isomorphism. Let us introduce following lemma.

Lemma 2.6. [11, Lemma 12.4.6] *Suppose that (H, R) is a factorizable Hopf algebra. Then (H^{cop}, R^{-1}) is also a factorizable Hopf algebra.*

If a quasitriangular Hopf algebra is both a ribbon Hopf algebra and a factorizable Hopf algebra, then it is called a *ribbon factorizable* Hopf algebra.

Let (H, R) be a quasitriangular Hopf algebra. Denote the category of finite-dimensional representations of H as $\mathrm{Rep}(H)$. Let $(\mathrm{Rep}(H), \otimes, \Bbbk, c)$ be the finite braided tensor category with braiding structure given by $\tau \circ R$, where τ is the flip map. Then $\mathrm{Rep}(H)$ is non-degenerate if and only if (H, R) is a factorizable Hopf algebra. Moreover, it is modular if and only if (H, R) is a ribbon factorizable Hopf algebra (see [18, Section 2.5] for example).

2.2. Hopf exact sequence, n-rank Taft algebras, small quantum groups, and the Drinfeld double.

Definition 2.7. *A short exact sequence of Hopf algebras is a sequence of Hopf algebras and Hopf algebra maps*

$$K \xrightarrow{\iota} H \xrightarrow{\pi} \overline{H} \tag{2.2}$$

such that

(i) ι *is injective,*
(ii) π *is surjective,*
(iii) $\mathrm{Ker}(\pi) = HK^+$, K^+ *is the kernel of the counit of* K,

here we view K as a sub-Hopf algebra of H through the map ι. Take an exact sequence (2.2), then K is a normal Hopf subalgebra of H, i.e. $h_{(1)}kS(h_{(2)}) \in K$ for all $k \in K, h \in H$. Conversely, if K is a normal Hopf subalgebra of a Hopf algebra H, then the quotient coalgebra $\overline{H} = H/HK^+ = H/K^+H$ is a quotient Hopf algebra and H fits into an extension (2.2), where ι and π are the canonical maps. In particular, if $K \subseteq Z(H)$ (center of H) is a sub-Hopf algebra and let $\overline{H} = H/HK^+$, then there is a exact sequence (2.2) corresponding to K which is called *central extension*. Central extensions are extensively employed in the study of Hopf algebras (see e.g, [26, 32, 5]). For instance, central extensions play important roles in Natale's classification work [32].

If H has no non-trivial normal Hopf subalgebras, then it is called a *simple* Hopf algebra. If H fits into an extension (2.2), then $\dim(H) = \dim(K)\dim(\overline{H})$ (see [15] for example).

To obtain non-semisimple ribbon factorizable Hopf algebras, we will employ the n-rank Taft algebras as delineated in Hu's work [23, Section 5]. These Hopf algebras, which can be regarded as finite-dimensional quantum groups associated with abelian Lie algebras, have been extensively investigated by various researchers (for instance, [8, 40, 13]). Assume $n \in \mathbb{N}^*$ and let $M = \{(i, j) | \ 1 \leq i, j \leq n\}$. Define the map $\theta : M \times M \longrightarrow \Bbbk$ as follows

$$\theta(i, j) = \begin{cases} q & i > j \\ 1 & i = j \\ q^{-1} & i < j \end{cases}.$$

Suppose $l \in \mathbb{N}^*$ and q is a primitive l-th root of unity. Then, the n-rank Taft algebra $\overline{\mathscr{A}}_q(n)$ is generated as an algebra by $x_1, ..., x_n$ and $g_1, ..., g_n$, with the relations

$$g_i g_j = g_j g_i,\ g_i^l = 1,\ g_i x_j = \theta(i,j) q^{\delta_{i,j}} x_j g_i,\ x_i x_j = \theta(i,j) x_j x_i,\ x_i^l = 0. \tag{2.3}$$

The coproduct, counit and antipode are defined as follows

$$\Delta(g_i) = g_i \otimes g_i,\ \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i, \tag{2.4}$$

$$\epsilon(g_i) = 1,\ \epsilon(x_i) = 0, \tag{2.5}$$

$$S(g_i) = g_i^{-1}, S(x_i) = -g_i^{-1} x_i, \tag{2.6}$$

where $1 \leq i, j \leq n$. If $n = 1$ then $\overline{\mathscr{A}}_q(1)$ is exactly the Taft algebra with dimension l^2.

Next, we revisit the definition of small quantum groups which will be employed in Section 4. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank l. Assume $\{\alpha_1, \cdots, \alpha_l\}$ are simple roots of $\mathfrak{g}$. Then, the Cartan matrix of $\mathfrak{g}$ is given by $(a_{ij})_{1 \leq i,j \leq l}$, where $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. Let $n \geq 2$ be an integer, not divisible by 3 if $\mathfrak{g} = G_2$ and let q be a primitive root of unity of order n. For integers $0 \leq r \leq m$, let us recall the analogue of q-binomial coefficient $\begin{bmatrix} m \\ r \end{bmatrix}_q$ which is defined by $\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[r]_q! [m-r]_q!}$, where $[r]_q! = \prod_{i=1}^r \frac{q^i - q^{-i}}{q - q^{-1}}$. Then, the small quantum group $u_q(\mathfrak{g})$ (see [4, 14] for example) is generated by e_i, f_i, k_i, $1 \leq i \leq l$ as an algebra, with the relations

$$k_i k_j = k_j k_i,\ k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j,\ k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j,$$

$$e_i f_j - f_j e_i = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},\ k_i^n = 1,\ e_i^n = f_i^n = 0,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_q e_i^{1-a_{ij}-r} e_j e_i^r = 0,\ i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_q f_i^{1-a_{ij}-r} f_j f_i^r = 0,\ i \neq j,$$

where $1 \leq i, j \leq l$ and $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$. The coproduct, counit and antipode are given by

$$\Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i,\ \Delta(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1,\ \Delta(k_i) = k_i \otimes k_i,$$

$$\epsilon(e_i) = \epsilon(f_i) = 0,\ \epsilon(k_i) = 1,$$

$$S(e_i) = -e_i k_i^{-1}, S(f_i) = -k_i f_i, S(k_i) = k_i^{-1}.$$

Lastly, let us recall the Drinfeld double of a Hopf algebgra H, denoted by $D(H)$. As a coalgebra, $D(H) = (H^*)^{cop} \otimes H$ where $(H^*)^{cop}$ means the Hopf algebra associated with H^*, which has an opposite coalgebra structure. The multiplication of $D(H)$ is defined as $(f \otimes h)(g \otimes k) = f[h_{(1)} \rightharpoonup g \leftharpoonup S^{-1}(h_{(3)})] \otimes h_{(2)} k$, where $f, g \in H^*$, $h, k \in H$ and $\langle a \rightharpoonup g \leftharpoonup b, c \rangle := \langle g, bca \rangle$ for $a, b, c \in H$. For convenience, we represent fh as $f \otimes h$ in the subsequent discussion. Let $\mathcal{R}$ be the standard universal $\mathcal{R}$-matrix of $D(H)$, i.e. $\mathcal{R} = \sum_{i=1}^n h_i \otimes h^i$, where $n = \dim(H)$ and $\{h_i\}_{i=1}^n$, $\{h^i\}_{i=1}^n$ are dual basis of H. To discuss ribbon Hopf algebras, let us recall the following result demonstrated in [19, Theorem 3].

Theorem 2.8. *Let g and α be the distinguished grouplike elements of H and H^*, respectively. Then $(D(H), \mathcal{R})$ has a ribbon element if and only if there are $a \in G(H)$ and $\beta \in G(H^*)$ such that*

(i) $a^2 = g$ *and* $\beta^2 = \alpha$*;*
(ii) $S^2(h) = a(\beta \rightharpoonup h \leftharpoonup \beta^{-1})a^{-1}, \ h \in H.$

3. Constructing ribbon factorizable Hopf algebras by central extensions

This section is dedicated to giving a way to construct ribbon factorizable Hopf algebras. Before proceeding, let us introduce some notations. Let (H, R) be a quasitriangular Hopf algebra and let $\pi : H \to K$ be a surjective Hopf map. Then $(K, (\pi \otimes \pi)(R))$ is a quasitriangular Hopf algebra. In particular, if I is a Hopf ideal of H, then $(H/I, \overline{R})$ is also a quasitriangular Hopf algebra, where $\overline{R}$ is induced by the quotient map. For simplicity, we denote K^+ as $\operatorname{Ker} \epsilon$ for a Hopf algebra K, where ϵ denote the counit of K.

To state the main result in this section, we need to recall a lemma given by [16, Lemma 4.17]. Assume $G = \langle x_i | \ x_i^{n_i} = 1, x_i x_j = x_j x_i, \ 1 \le i, j \le m \rangle$ as groups. Suppose η is bicharacter on G defined by $\eta(x_i, x_j) = t_j^{m_{ij}}$, where t_j is a primitive n_jth root of unity. Denote the matrix $(m_{ij})_{1 \le i,j \le m}$ as M.

Lemma 3.1. *The matrix $[\eta(g,h)]_{g,h \in G}$ is non-degenerate if and only if the following equation has a unique solution in $\mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_m}$*

$$(i_1, ..., i_m)M = (0, ..., 0), \ (i_1, ..., i_m) \in \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_m}.$$

Proof. Let us denote the dual group of G as $\widehat{G}$. Define $\gamma : G \to \widehat{G}$ by $\gamma(g)(h) = \eta(g, h)$ for $g, h \in G$. Directly, we observe that $[\eta(g,h)]_{g,h \in G}$ is non-degenerate if and only if γ is injective. By definition, γ is injective if and only if the following equation has a unique solution

$$(i_1, ..., i_m)M = (0, ..., 0), \ (i_1, ..., i_m) \in \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_m}.$$

□

Let $m \in \mathbb{N}$ and let H be a Hopf algebra. Assume $G \subseteq G(H)$ is a subgroup such that $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m} = \langle g_1, \cdots, g_m | \ g_i^{n_i} = 1, \ g_i g_j = g_j g_i, \ 1 \le i, j \le m \rangle$ as groups. Suppose that $\overline{G} \subseteq G(H^*)$ is a subgroup such that $\overline{G} = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m} = \langle \chi_1, \cdots, \chi_m | \ \chi_i^{n_i} = 1, \ \chi_i \chi_j = \chi_j \chi_i, \ 1 \le i, j \le m \rangle$ as groups. Let t_j be a primitive n_jth root of 1 and let $\chi_i(g_j)\chi_j(g_i) = t_j^{m_{ij}}$ for $1 \le i, j \le m$. And we also denote the matrix $(m_{ij})_{\le i,j \le m}$ as M. Then we have

Theorem 3.2. *Keeping the above notation, assume H is generated by $\{a_i | \ 1 \le i \le n\}$ as an algebra. If the following conditions hold*

(i) $\chi_i \rightharpoonup a_j \leftharpoonup \chi_i^{-1} = g_i^{-1} a_j g_i$ *for* $1 \leq i \leq m,\ 1 \leq j \leq n$;
(ii) $(i_1, ..., i_m)M = (0, ..., 0),\ (i_1, ..., i_m) \in \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_m}$ *has a unique solution;*

then $\langle \chi_i g_i |\ 1 \leq i \leq m\rangle \subseteq Z(D(H))$ *and* $(D(H)/I, \overline{\mathcal{R}})$ *is factorizable Hopf algebra, where* $I = D(H)\langle \chi_i g_i |\ 1 \leq i \leq m\rangle^+$ *and* $\mathcal{R}$ *is the standard universal* $\mathcal{R}$*-matrix of* $D(H)$*.*

Proof. By definition of $D(H)$, we know $\chi_i a_j \chi_i^{-1} = \chi_i \rightharpoonup a_j \leftharpoonup \chi_i^{-1}$. Combing this fact with (i), we deduce that $\chi_i g_i \in Z(D(H))$. For convenience, we denote $D(H)/I$ as K. Since K is quotient Hopf algebra of $D(H)$, we can view the braided tensor category $(\mathrm{Rep}(K), \otimes, \Bbbk, \overline{c})$ as a braided tensor subcategory of $(\mathrm{Rep}(D(H)), \otimes, \Bbbk, c)$, where $\overline{c}$ (resp. c) is given by $\tau \circ \overline{\mathcal{R}}$ (resp. $\tau \circ \mathcal{R}$). Let $\mathcal{D} = \mathrm{Rep}(D(H))$ and let $\mathcal{E} = \mathrm{Rep}(K)$. Then we will apply Theorem 2.3 to complete the proof.

Since $G \subseteq [G(D(H)) \cap Z(D(H))]$, we know $f^{-1}_{\mathcal{R}_{21}\mathcal{R}}(g) \in G(D(H)^*)$ for $g \in G$, here $f_{\mathcal{R}_{21}\mathcal{R}}$ is defined by (2.1) for $(D(H), \mathcal{R})$. Let $g \in G$ and let $\Bbbk_g$ be the one-dimensional representation of $D(H)$ determined by $s.1 = f^{-1}_{\mathcal{R}_{21}\mathcal{R}}(g)(s)1$ for $s \in D(H)$. Let $\mathcal{F}$ be the full subcategory of $\mathcal{D}$ with objects

$$\mathrm{Ob}(\mathcal{F}) := \{\text{finite direct sums of } \{\Bbbk_g,\ g \in G\}\}.$$

By definition of $\mathcal{F}$, it is a topologizing braided tensor subcategory of $\mathcal{D}$. Let $V \in \mathcal{D}$. By definition of $C_{\mathcal{D}}(\mathcal{F})$, we know $V \in C_{\mathcal{D}}(\mathcal{F})$ if and only if $g.v = v$ for all $g \in G,\ v \in V$. i.e. $I.V = 0$. This implies that $\mathcal{E} = C_{\mathcal{D}}(\mathcal{F})$. By Theorem 2.3, we know that $\mathcal{F}$ is non-degenerate if and only if $\mathcal{E}$ is non-degenerate. Since Lemma 3.1 and $(f^{-1}_{\mathcal{R}_{21}\mathcal{R}}(\chi_i g_i))(\chi_j g_j) = \chi_i(g_j)\chi_j(g_i) = t_j^{m_{ij}}$ for $1 \leq i, j \leq m$, we know $\mathcal{F}$ is non-degenerate. Hence $\mathcal{E}$ is non-degenerate. □

Remark 3.3. Combing this theorem with previous Theorem 2.8 given by Radford, we actually obtain a method for constructing ribbon factorizable Hopf algebras. Using this approach, it's not difficult to recover many interesting ribbon factorizable Hopf algebras, including small quantum groups $u_q(\mathfrak{g})$. In practice, it's often straightforward to find grouplike elements $\chi_i g_i (1 \leq i \leq m)$ satisfying the conditions of above theorem, making this method convenient to use. In the next section, we will explore this advantage further.

4. Ribbon factorizable Hopf algebras E_q and related conclusions

4.1. **A family of ribbon factorizable Hopf algebras E_q.** In this subsection, we will apply Theorem 3.2 to construct a family of ribbon factorizable Hopf algebras denoted by E_q. Subsequently, we will provide an explicit description of E_q, including its universal $\mathcal{R}$-matrix and its unique ribbon element.

Recall the n-rank Taft algebra $\overline{\mathscr{A}}_q(n)$ defined by (2.3)-(2.6). Let $\chi_i : \overline{\mathscr{A}}_q(n) \longrightarrow \Bbbk$ be the algebra map determined by $\chi_i(x_j) = 0,\ \chi_i(g_j) = \theta(i,j)q^{\delta_{i,j}}$, where $1 \leq j \leq n$.

Proposition 4.1. *If* l *is odd, then the Hopf algebra* $D(\overline{\mathscr{A}}_q(n))/I$ *is a factorizable Hopf algebra with dimension* l^{3n}*, where* $I = D(\overline{\mathscr{A}}_q(n))\langle \chi_i g_i |\ 1 \leq i \leq n\rangle^+$*.*

Proof. We only need to demonstrate that the Hopf algebra $\overline{\mathscr{A}}_q(n)$ satisfies the conditions of Theorem 3.2. By definition of $\overline{\mathscr{A}}_q(n)$, it is generated by $\{g_i, x_i| \ 1 \leq i \leq n\}$. Directly, we know $g_i x_j g_i^{-1} = \chi_i^{-1} \rightharpoonup x_j \leftharpoonup \chi_i = \chi_i(g_j)x_j$. Hence the condition (i) of Theorem 3.2 holds. To verify the condition (ii) of Theorem 3.2, we will show that the matrix $[\chi_i(g_j)\chi_j(g_i)]_{1\leq i,j\leq n}$ is non-degenerate. Let $s_{ij} = \chi_{n-i}(g_j)\chi_j(g_{n-i})$ for $1 \leq i, j \leq n$. By definition, $[\chi_i(g_j)\chi_j(g_i)]_{1\leq i,j\leq n}$ is non-degenerate if and only if $[s_{ij}]_{1\leq i,j\leq n}$ is non-degenerate. Let $1 \leq i, j \leq n$. Define

$$m_{ij} = \begin{cases} 2 & i+j \leq n+1 \\ -2 & i+j > n+1 \end{cases}.$$

Obviously, the following equation has a unique solution in $\mathbb{Z}_l^n$

$$(i_1, ..., i_m)M = (0, ..., 0), \ (i_1, ..., i_m) \in \mathbb{Z}_l^n,$$

where $M = (m_{ij})_{1\leq i,j\leq n}$. By Lemma 3.1, $[s_{ij}]_{1\leq i,j\leq n}$ is non-degenerate and hence the condition (ii) holds. Consider the natural exact sequence $\langle \chi_i g_i| \ 1 \leq i \leq n\rangle \hookrightarrow D(\overline{\mathscr{A}}_q(n)) \to D(\overline{\mathscr{A}}_q(n))/I$, and hence we know the dimension of $D(\overline{\mathscr{A}}_q(n))/I$ is l^{3n}. □

Remark 4.2. Denote the above Hopf algebra $D(\overline{\mathscr{A}}_q(n))/I$ as $E(n, q)$. If $n = 1$, then $E(1, q)$ recovers the small quantum group $u_q(sl_2)$ by definition. Since our goal, we will only focus on discussing $E(3, q)$ in the following content. In fact, most of following conclusions and proofs can also be generalized to $E(n, q)$ just by adjusting the number n. For simplicity, we just denote $E(3, q)$ as E_q in following content.

To describe the Hopf algebra E_q explicitly, we need to understand more about the dual Hopf algebra $\overline{\mathscr{A}}_q(3)^*$. For this purpose, we require an alternative description of $\overline{\mathscr{A}}_q(3)^*$ distinct from the approach in [13]. Let $\beta_1, ...\beta_3, \gamma_1, ..., \gamma_3 \in G(\overline{\mathscr{A}}_q(3)^*)$ be given by

$$\beta_i(g_j) = \theta(j, i)q^{\delta_{j,i}}, \beta_i(x_j) = 0, \ 1 \leq i, j \leq 3,$$
$$\gamma_i(g_j) = q^{2\delta_{i,j}}, \gamma_i(x_j) = 0, \ 1 \leq i, j \leq 3,$$

and define $X_1, ..., X_3 \in \overline{\mathscr{A}}_q(3)^*$ by

$$X_i(gx_1^{j_1}x_2^{j_2}x_3^{j_3}) = \delta_{j_i,1}\prod_{k\neq i}\delta_{j_k,0}, \ 0 \leq i, j_i \leq l-1.$$

Let $1 \leq i, j, k \leq l$ and define $e_{g_1^i g_2^j g_3^k} \in \overline{\mathscr{A}}_q(3)^*$ by

$$e_{g_1^i g_2^j g_3^k} := \frac{1}{|G|}\sum_{1\leq u,v,w\leq l} q^{-2iu}q^{-2jv}q^{-2kw}\gamma_1^u\gamma_2^v\gamma_3^w.$$

Let $1 \leq i, j, k \leq l$ and define $\lambda(i, j, k) = q^{-i(j+k)-jk}q^{\frac{-i(i-1)}{2}+\frac{-j(j-1)}{2}+\frac{-k(k-1)}{2}}$. For convenience, the above notations will be freely used in the following content.

Lemma 4.3. *Keeping the above notations, we have*

(i) $\gamma_1 = \chi_1\chi_3, \ \gamma_2 = \chi_1^{-1}\chi_2, \ \gamma_3 = \chi_2^{-1}\chi_3$;
(ii) $\beta_1 = \chi_3, \ \beta_2 = \chi_1^{-1}, \ \beta_3 = \chi_2^{-1}$;

(iii) *The set* $\{\lambda(u,v,w)^{-1}e_{g_1^{i+u}g_2^{j+v}g_3^{k+w}}X_1^uX_2^vX_3^w|\ 0\le i,j,k,u,v,w\le l-1\}$ *is the dual basis of* $\{g_1^ig_2^jg_3^kx_1^ux_2^vx_3^w|\ 0\le i,j,k,u,v,w\le l-1\}$*;*

Proof. By definition, we have $\gamma_1(g_i)=\chi_1\chi_3(g_i)=q^{2\delta_{i,1}}$ for $1\le i\le 3$. Thus $\gamma_1=\chi_1\chi_3$. Similarly, the equations in (i)-(ii) hold. To verify (iii), note that the coproduct of $\overline{\mathscr{A}}_q(3)$ preserves the degree, hence $(X_1^uX_2^vX_3^w)(gx_1^{u'}x_2^{v'}x_3^{w'})=0$ if $(u',v',w')\neq(u,v,w)$. Next, we claim that $X_i^u(gx_i^u)=q^{-\frac{u(u-1)}{2}}$. Let $b_u=X_i^u(gx_i^u)$ for $0\le u\le l-1$. By definition, we have $\langle X_i^u,gx_i^u\rangle=\langle X_i\otimes X_i^{u-1},\Delta(gx_i)\Delta(x_i^{u-1})\rangle$. Since we have shown $(X_i^v)(gx_i^{v'})=0$ if $v\neq v'$, it's not difficult to see

$$\langle X_i^u,gx_i^u\rangle=\langle X_i\otimes X_i^{u-1},gg_1^{u-1}x_i\otimes gx_i^{u-1}\rangle=q^{-1}b_{u-1}.$$

Thus $b_u=q^{-1}b_{u-1}$ for $1\le u\le l-1$. Due to $b_0=1$, we obtain $X_i^u(gx_i^u)=q^{-\frac{u(u-1)}{2}}$ which is our claim. To complete the proof, we only need to show:

$$(e_{g_1^ig_2^jg_3^k}X_1^uX_2^vX_3^w)(g_1^{i'}g_2^{j'}g_3^{k'}x_1^{u'}x_2^{v'}x_3^{w'})=\lambda(u,v,w)\delta_{i,i'+u}\delta_{j,j'+v}\delta_{k,k'+w}\delta_{u,u'}\delta_{v,v'}\delta_{w,w'}.$$

Assume $1\le i,j,k,u,v,w\le l$ and $r+s+t\ge 1$. By definition, we have

$$e_{g_1^ig_2^jg_3^k}(g_1^ug_2^vg_3^w)=\delta_{i,u}\delta_{j,v}\delta_{k,w},\ e_{g_1^ig_2^jg_3^k}(gx_1^rx_2^sx_3^t)=0. \tag{4.1}$$

By the previous claim, we have $X_i^u(gx_i^u)=q^{-\frac{u(u-1)}{2}}$. Hence we get

$$\begin{aligned}\langle X_1^uX_2^vX_3^w,gx_1^ux_2^vx_3^w\rangle&=\langle X_1^u\otimes X_2^v\otimes X_3^w,\Delta^{(2)}(gx_1^ux_2^vx_3^w)\rangle\\&=\langle X_1^u\otimes X_2^v\otimes X_3^w,gx_1^ug_2^vg_3^w\otimes gx_2^vg_3^w\otimes gx_3^w\rangle\\&=q^{-u(v+w)-vw}\langle X_1^u\otimes X_2^v\otimes X_3^w,gg_2^vg_3^wx_1^u\otimes gg_3^wx_2^v\otimes gx_3^w\rangle\\&=\lambda(u,v,w).\end{aligned}$$

Combining this fact with equation (4.1), we obtain

$$(e_{g_1^ig_2^jg_3^k}X_1^uX_2^vX_3^w)(g_1^{i'}g_2^{j'}g_3^{k'}x_1^ux_2^vx_3^w)=\lambda(u,v,w)\delta_{i,i'+u}\delta_{j,j'+v}\delta_{k,k'+w}.$$

Using the fact that the coproduct of $\overline{\mathscr{A}}_q(3)$ preserves the degree again, we obtain what we want. □

As a result, we obtain

Corollary 4.4. *The* $\overline{\mathscr{A}}_q(3)^*$ *is generated by* $\{\chi_i,X_i|\ 1\le i\le 3\}$ *as an algebra, with the relations*

$$\chi_i\chi_j=\chi_j\chi_i,\ \chi_i^l=1,\ \chi_iX_j=\theta(i,j)q^{\delta_{i,j}}X_j\chi_i,\ X_iX_j=\theta(i,j)X_jX_i,\ X_i^l=0.$$

The coproduct, counit and antipode are given by

$$\begin{aligned}\Delta(\chi_i)=\chi_i\otimes\chi_i,\ \Delta(X_i)&=1\otimes X_i+X_i\otimes\beta_i^{-1},\\ \epsilon(\chi_i)&=1,\ \epsilon(\chi_i)=0,\\ S(\chi_i)&=\chi_i^{-1},S(X_i)=-X_i\beta_i,\end{aligned}$$

Proof. Directly, we know that $G(\overline{\mathscr{A}}_q(n)^*) = \langle \chi_i |\ 1 \leq i \leq 3\rangle$. By (iii) of Lemma 4.3, we obtain that $\{\chi_i, X_i |\ 1 \leq i \leq 3\}$ generates $\overline{\mathscr{A}}_q(n)^*$ as an algebra. To complete the proof, we only need to show following non-trivial equalities

$$\chi_i X_j = \theta(i,j) q^{\delta_{i,j}} X_j \chi_i,\ X_i X_j = \theta(i,j) X_j X_i,\ \Delta(X_i) = 1 \otimes X_i + X_i \otimes \beta_i^{-1}.$$

Firstly, we show $\chi_i X_j = \theta(i,j) q^{\delta_{i,j}} X_j \chi_i$. Since the degree reason, we only need to prove $\langle \chi_i X_j, g x_k\rangle = \theta(i,j) q^{\delta_{i,j}} \langle X_j \chi_i, g x_k\rangle$ for $1 \leq k \leq 3$. Directly, we have

$$\langle \chi_i X_j, g x_k\rangle = \theta(i,j) q^{\delta_{i,j}} \langle X_j \chi_i, g x_k\rangle = \chi_i(g g_j)\delta_{j,k}.$$

Hence we get $\chi_i X_j = \theta(i,j) q^{\delta_{i,j}} X_j \chi_i$. Similarly, we have

$$\langle X_i X_j, g x_k x_l\rangle = \langle \theta(i,j) X_j X, g x_k x_l\rangle = \theta(i,j)\delta_{i,k}\delta_{j,l}.$$

So we get $X_i X_j = \theta(i,j) X_j X_i$. Since the degree reason and following equalities

$$\langle \Delta(X_i), g \otimes h x_i\rangle = \langle \epsilon \otimes X_i, g \otimes h x_i\rangle = 1,\ \langle \Delta(X_i), h x_i \otimes g\rangle = \langle X_i \otimes \beta_i^{-1}, h x_i \otimes g\rangle = \beta_i^{-1}(g),$$

we obtain $\Delta(X_i) = 1 \otimes X_i + X_i \otimes \beta_i^{-1}$. □

Now we can describe the Hopf algebra E_q explicitly.

Proposition 4.5. *The Hopf algebra E_q is generated by $\{g_i, x_i, X_i |\ 1 \leq i \leq 3\}$ as an algebra with relations*

$$g_i g_j = g_j g_i,\ g_i^l = 1,\ g_i x_j = \beta_j(g_i) x_j g_i,\ g_i X_j = \beta_j(g_i)^{-1} X_j g_i,$$
$$x_i x_j = \theta(i,j) x_j x_i,\ X_i X_j = \theta(i,j) X_j X_i,\ x_i^l = 0; X_i^l = 0$$
$$x_i X_j - \beta_j(g_i)^{-1} x_i X_j = \delta_{i,j}(1 - h_i),\ \textit{where } (h_1, h_2, h_3) = (g_1 g_3, g_1^{-1} g_2, g_2^{-1} g_3),$$

where $g_i \in G(E_q)$. The coproduct, counit and antipode are given by

$$\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i,\ \Delta(X_i) = 1 \otimes X_i + X_i \otimes h_i g_i^{-1},$$
$$\epsilon(x_i) = \epsilon(X_i) = 0,\ S(x_i) = -g_i^{-1} x_i,\ S(X_i) = -X_i h_i^{-1} g_i.$$

Proof. Corollary 4.4 tells us that the union of $\{g_i |\ 1 \leq i \leq 3\}$ and $\{x_i, X_i |\ 1 \leq i \leq 3\}$ generates E_q as an algebra. Next, we show $x_i X_j - \beta_j(g_i)^{-1} X_j x_i = \delta_{i,j}(1 - h_i)$. Since $x_i X_j = [(x_i)_{(1)} \rightharpoonup X_j \leftharpoonup S^{-1}(x_i)_{(3)}](x_i)_{(2)}$ and

$$\Delta^{(2)}(x_i) = x_i \otimes 1 \otimes 1 + g_i \otimes x_i \otimes 1 + g_i \otimes g_i \otimes x_i,$$

we have

$$x_i X_j = x_i \rightharpoonup X_j + (g_i \rightharpoonup X_j) x_i + (g_i \rightharpoonup X_j \leftharpoonup S^{-1}(x_i)) g_i. \tag{4.2}$$

By definition, we get

$$x_i \rightharpoonup X_j = \delta_{i,j} 1,\ g_i \rightharpoonup X_j = \beta_j(g_i)^{-1} X_j,\ g_i \rightharpoonup X_j \leftharpoonup S^{-1}(x_i) = -\delta_{i,j}\beta_i^{-1}. \tag{4.3}$$

Due to Lemma 4.3, we have $(\beta_1, \beta_2, \beta_3) = (\chi_3, \chi_1^{-1}, \chi_2^{-1})$. By definition of E_q, we get $\chi_i = g_i^{-1}$ for $1 \leq i \leq 3$. This implies $\beta_i^{-1} = h_i g_i^{-1}$. Combining this with equations (4.2)-(4.3), we know $x_i X_j - \beta_j(g_i)^{-1} X_j x_i = \delta_{i,j}(1 - h_i)$.

Since Corollary 4.4, we get $\Delta(X_i) = 1 \otimes X_i + X_i \otimes \beta_i^{-1}$. Note that $\beta_i^{-1} = h_i g_i^{-1}$, hence the equation $\Delta(X_i) = 1 \otimes X_i + X_i \otimes h_i g_i^{-1}$ holds. Using Corollary 4.4 again, we obtain other equations. □

Moreover, we can provide the universal $\mathcal{R}$-matrix of E_q and the unique ribbon element as follows.

Proposition 4.6. *Keeping the notations in Proposition 4.5, then the induced universal $\mathcal{R}$-matrix of E_q is given by*

$$\overline{R} = \sum_{0\le i,j,k,u,v,w\le 2} \lambda(u,v,w)^{-1} g_1^i g_2^j g_3^k x_1^u x_2^v x_3^w \otimes e_{i+u,j+v,k+w}(X_1^u X_2^v X_3^w). \tag{4.4}$$

The unique ribbon element for $(E_q, \overline{R})$ is given by $\theta = u g_3^{-1}$, where $u = \sum_i S(\overline{R}^i)\overline{R}_i$.

Proof. By Lemma 4.3, the set $\{\lambda(u,v,w)^{-1} e_{g_1^{i+u} g_2^{j+v} g_3^{k+w}} X_1^u X_2^v X_3^w |\ 0 \le i,j,k,u,v,w \le l-1\}$ is the dual basis of $\{g_1^i g_2^j g_3^k x_1^u x_2^v x_3^w |\ 0 \le i,j,k,u,v,w \le l-1\}$. Since the definition of $\overline{R}$, we know it is given by the equation (4.4).

Denote H as $\overline{\mathscr{A}}_q(3)$ for simplicity. To prove that $(E_q, \overline{R})$ has ribbon element, we only need to show $(D(H), \mathcal{R})$ has a ribbon element. Let $\Lambda = (\sum_{g\in G(H)} g)(x_1^{l-1} \dots x_3^{l-1})$. By definition, we know Λ is a left integral of H. This implies α (the distinguished grouplike element of H^*) is determined by $\alpha(g_i) = q^{-3+2i}$ and $\alpha(x_i) = 0$ for $1 \le i \le 3$. Similarly, we know that $\Lambda' = (\sum_{\chi\in\widehat{G}} \chi)(X_1^{l-1} \dots X_3^{l-1})$ is a left integral of H^* by using Corollary 4.4. Hence we have $g_0 = g_1^{-1} \dots g_3^{-1}$ (the distinguished grouplike element of H). Since l is odd, we can assume $l = 2m-1$. Let $a = g_0^m, \beta = \alpha^m$. By definition, we know that a, β satisfy the conditions of Theorem 2.8. Thus $(D(H), \mathcal{R})$ has a ribbon element. Now we can assume $v \in E_q$ is a ribbon element. Let $g = vu^{-1}$. Then we have $g \in G(E_q)$ and $S^2(h) = g^{-1}hg$ for $h \in E_q$. By definition, we also have $S^2(h) = g_3 h g_3$ for $h \in E_q$. To complete the proof, we only need to show that $G(E_q) \cap Z(E_q) = \{1\}$. Note that the set $\{\beta_i |\ 1 \le i \le 3\}$ generate $\widehat{G}$ as a group, hence it's not difficult to see that $G(E_q) \cap Z(E_q) = \{1\}$. □

Remark 4.7. For a general n-rank Taft algebra $\overline{\mathscr{A}}_q(n)$, we can use similar discussions as above to conclude that $E(n,q)$ has a unique ribbon element, i.e., $E(n,q)$ is a ribbon factorizable Hopf algebra for all $n \in \mathbb{N}^+$.

4.2. A prime decomposition of $\mathrm{Rep}(E_q)$.

In this subsection, we will mainly show the following result

Proposition 4.8. *We have $\mathrm{Rep}(E_q) \simeq \mathrm{Rep}(u_q(sl_2))^{\boxtimes 3}$ as ribbon categories, thereby providing a prime decomposition for $\mathrm{Rep}(E_q)$.*

To provide a proof, we rely on following result that combines [6, Lemma 2.1] and [6, Lemma 2.5] by P. Etingof et al.

Proposition 4.9. *Suppose the order of q' is odd and it is coprime to the determinant of the Cartan matrix of $\mathfrak{g}$. Then $u_{q'}(\mathfrak{g})$ has only two universal $\mathcal{R}$-matrices.*

Suppose the order of q' is odd and it is coprime to the determinant of the Cartan matrix of $\mathfrak{g}$. It's known that $u_{q'}(\mathfrak{g})$ is a factorizable Hopf algebra (see [35, XI.6.3] for example). Further, we have

Corollary 4.10. *Suppose the order of q' is odd and it is coprime to the determinant of the Cartan matrix of $\mathfrak{g}$. Then any universal $\mathcal{R}$-matrix of $u_{q'}(\mathfrak{g})$ yields a factorizable Hopf algebra.*

Proof. Since $u_{q'}(\mathfrak{g})$ is a factorizable Hopf algebra, we can assume R is a factorizable universal $\mathcal{R}$-matrix of $u_{q'}(\mathfrak{g})$. Let $R' = \tau(R^{-1})$, where τ is the flip map. By definition, R' is also a universal $\mathcal{R}$-matrix. Since R is a factorizable universal $\mathcal{R}$-matrix, we know $R' \neq R$. By Lemma 2.6, R' is also a factorizable universal $\mathcal{R}$-matrix. Since Proposition 4.9, we know the set $\{R, R'\}$ gives all the universal $\mathcal{R}$-matrices. Hence we get what we want. □

For the Hopf algebra E_q, let $1 \leq i \leq 3$ and let I_i be the Hopf ideal generated by the following set

$$\{x_j, X_j, 1 - h_j |\ 1 \leq j \neq i \leq 3\}.$$

Denote the quotient Hopf algebra E_q/I_i as H_i. Then H_i is a quasitriangular Hopf algebra with an induced universal $\mathcal{R}$-matrix denoted by $\overline{R_i}$. Hence $(\mathrm{Rep}(H_i), \otimes, \Bbbk, c_i)$ is a braided tensor category, where c_i is given by $\tau \circ \overline{R_i}$. And it can be viewed as a braided tensor subcategory of $(\mathrm{Rep}(E_q), \otimes, \Bbbk, c)$ naturally, where c is given by $\tau \circ \overline{R}$ (see Proposition 4.6 for definition of $\overline{R}$).

Lemma 4.11. *Let $1 \leq i \neq j \leq 3$. Then any object of $\mathrm{Rep}(H_i)$ belongs to the Muger centralizer of $\mathrm{Rep}(H_j)$.*

Proof. Assume V_i is an object of $\mathrm{Rep}(H_i)$ and V_j is an object of $\mathrm{Rep}(H_j)$. By definition, we only need to prove that $\overline{R_{21}R}(v_i \otimes v_j) = v_i \otimes v_j$ for $v_i \in V_i, v_j \in V_j$. To begin, we'll demonstrate this for the case $i = 1, j = 2$. For simplicity, we define $g^{(i,j,k)} := g_1^i g_2^j g_3^k$. By definition, we have $x_j.V_1 = X_j.V_1 = \{0\}$ for $j \neq 1$ and $x_k.V_2 = X_k.V_2 = \{0\}$ for $k \neq 2$. Using this fact and $\lambda(0,0,0) = 1$, we know that $\overline{R} = \sum_{0 \leq i,j,k \leq 2} g^{(i,j,k)} \otimes e_{i,j,k}$ when they act on $V_1 \otimes V_2$. By a similar discussion and the relations $g_i x_j = \beta_j(g_i) x_j g_i$, $g_i X_j = \beta_j(g_i)^{-1} X_j g_i$, we know that $\overline{R_{21}R} = \sum_{0 \leq i,i',j,j',k,k' \leq 2} [e_{i',j',k'} g^{(i,j,k)} \otimes g^{i',j',k'} e_{i,j,k}]$ when they act on $V_1 \otimes V_2$. By definition, we have

$$e_{g_1^i g_2^j g_3^k} = \frac{1}{|G|} \sum_{1 \leq u,v,w \leq l} q^{-2iu} q^{-2jv} q^{-2kw} \gamma_1^u \gamma_2^v \gamma_3^w.$$

Using (i) of Lemma 4.3 and $\chi_k g_k = 1$ for $1 \leq k \leq 3$, we get

$$e_{g_1^i g_2^j g_3^k} = \frac{1}{|G|} \sum_{1 \leq u,v,w \leq l} q^{-2iu} q^{-2jv} q^{-2kw} h_1^{-u} h_2^{-v} h_3^{-w}.$$

Since $(h_j - 1).V_1 = \{0\}$ for $j \neq 1$, we know $e_{i,j,k} = \delta_{j,0}\delta_{k,0} e_{i,0,0}$ when they act on V_1. Similarly, since $(h_k - 1).V_2 = \{0\}$ for $j \neq 2$, we know that $e_{i,j,k} = \delta_{i,0}\delta_{k,0} e_{0,j,0}$ when they act on V_2. Thus the following equation holds when both sides act on $V_1 \otimes V_2$

$$\overline{R_{21}R} = \frac{1}{l^2} \sum_{i,j',u,v} q^{-2iu} g_1^{-2u} g_2^{j'} \otimes q^{-2j'v} g_1^{i} g_2^{-2v}.$$

By definition, we know $g_2 = g_1$ when they act on V_1. Similarly, we obtain $g_2 = g_1^{-1}$ when they act on V_2. Hence the following equation holds when both sides act on $V_1 \otimes V_2$

$$\overline{R_{21}R} = \frac{1}{l^2} \sum_{i,j',u,v} q^{-2iu} g_1^{-2u+j'} \otimes q^{-2j'v} g_2^{-2v-i}.$$

Let $1 \leq j \leq 2$. Since $g_j^l = 1$, we know g_j is diagonalizable when it acts on V_j. Thus we can assume that $v_j \in V_j$ such that $g_j.v_j = q^{m_j}$. Then we get

$$(\overline{R_{21}R}).(v_1 \otimes v_2) = (\frac{1}{l^2} \sum_{i,j',u,v} q^{-2iu-2m_1u+m_1j'} \otimes q^{-2j'v-2m_2v-m_2i})(v_1 \otimes v_2).$$

Directly, we have $(\frac{1}{l^2} \sum_{i,j',u,v} q^{-2iu-2m_1u+m_1j'} \otimes q^{-2j'v-2m_2v-m_2i}) = 1$. Hence we get $(\overline{R_{21}R}).(v_1 \otimes v_2) = (v_1 \otimes v_2)$, i.e. we have shown that any object of $\mathrm{Rep}(H_1)$ belongs to the Muger centralizer of $\mathrm{Rep}(H_2)$. Similarly, we can show $\overline{R_{21}R}(v_i \otimes v_j) = v_i \otimes v_j$ for other cases $i \neq j$.

□

Now we can give following proof.

Proof of Proposition 4.8. Let $1 \leq i \leq 3$. By definition, we know $H_i \cong u_q(sl_2)$ as Hopf algebras. Since Corollary 4.10, we know $\mathrm{Rep}(H_i)$ is a non-degenerate braided tensor subcategory. Moreover, it is a topologizing subcategory by definition. By Theorem 2.5, we can get $\mathrm{Rep}(E_q) = \mathrm{Rep}(H_1) \boxtimes \mathrm{Rep}(H_2) \boxtimes \mathrm{Rep}(H_3)$. Since $H_i \cong u_q(sl_2)$ and it's known that $\mathrm{Rep}(u_q(sl_2))$ with any braiding structure is a prime modular tensor category (see [6, Lemma 2.5] for example), we know the above decomposition for $\mathrm{Rep}(E_q)$ is a prime decomposition. Hence, we get what we want. □

Recall that a normalized twist for a Hopf algebra H is an invertible element $J \in H \otimes H$ which satisfies $(\epsilon \otimes \mathrm{Id})(J) = (\mathrm{Id} \otimes \epsilon)(J) = 1$ and

$$(\Delta \otimes \mathrm{Id})(J)(J \otimes 1) = (\mathrm{Id} \otimes \Delta)(J)(1 \otimes J).$$

Then there is a twisted Hopf algebra H^J whose coproduct Δ^J is given by $\Delta^J(h) = J^{-1}\Delta(h)J$ for $h \in H$ and its algebra is same with H. By [30, Theorem 2.2], two Hopf algebras H and H' are gauge equivalent (their representation categories are equivalent as tensor categories) if and only if there exists a twist J of H such that $H' \cong H^J$ as Hopf algebras. Then we have

Corollary 4.12. *There is a twist J of $u_q(sl_2^{\oplus 3})$ such that $E_q \cong u_q(sl_2^{\oplus 3})^J$ as Hopf algebras.*

Proof. By Proposition 4.8 and above discussion, we get what we want. □

4.3. **Compare E_q with some known ribbon factorizable Hopf algebras.** We call a Hopf algebra *trivial* if it is a group algebra or its dual. This subsection is devoted to proving following Theorem 4.14. To do this, we introduce following proposition. Denote the center of H as $Z(H)$.

Proposition 4.13. *The Hopf algebra E_q is a simple Hopf algebra.*

Proof. Suppose $K \neq \Bbbk.1$ is normal sub-Hopf algebra of E_q. To complete the proof, we only need to show $\{g_i, x_i, X_i |\ 1 \leq i \leq n\} \subseteq K$.

Since K is finite dimensional pointed Hopf algebra over $\Bbbk$ of characteristic 0, we can find $1 \neq g \in G(K)$. Let $1 \leq j \leq 3$. Denote the dual group of $G(\overline{\mathscr{A}}_q(3))$ as $\hat{G}$. Directly, we have $\mathrm{ad}_{x_j}(g) = (1 - \chi_j(g))gx_j$ for $1 \leq j \leq 3$, where the "ad" means the adjoint action. Since $g \neq 1$ and $\{\chi_j |\ 1 \leq j \leq 3\}$ generates $\hat{G}$ as group, there is some $1 \leq j_0 \leq 3$ such that $\chi_{j_0}(g) \neq 1$. This implies that $gx_{j_0} \in K$ by normality of K. Then we know $x_{j_0} \in K$ and $g_{j_0} \in K$. For $1 \leq j \leq 3$, we get $\mathrm{ad}_{x_j}(g_{j_0}) = (1 - \chi_{j_0}(g_j))x_j g_{j_0}$ by definition. Since $\chi_{j_0}(g_j) \neq 1$, we get $x_j \in K$ for $1 \leq j \leq 3$. Using $\Delta(x_j) \in K \otimes K$, we obtain $g_j \in K$ for $1 \leq j \leq 3$. As a result, we have $G(E_q) \subseteq K$. Let $k_j = h_j g_j^{-1}$. By definition, we get $\mathrm{ad}_{X_j}(k_j) = (1 - \beta_j(k_j))X_j$. Directly, we have $\beta_j(k_j) = q$. Hence we know $X_j \in K$ for $1 \leq j \leq 3$. Since we have shown that $\{g_i, x_i, X_i |\ 1 \leq i \leq n\} \subseteq K$, we know $K = H$. □

Now, we can give following comparison.

Theorem 4.14. *If the order of q is not a square number, then the ribbon factorizable Hopf algebra E_q is not isomorphic to any tensor products of trivial Hopf algebras, Drinfeld doubles, and small quantum groups as Hopf algebras.*

Proof. Denote the order of q as l. Since the assumption and the dimension of E_q is l^9, we know it is not isomorphic to any Drinfeld double. By Proposition 4.13, we only need to show that E_q is not isomorphic to any small quantum group of a simple Lie algebra. Let $\mathfrak{g}$ be a complex simple Lie algebra and let q be a root of unity with odd order not divisible by 3 if $\mathfrak{g} = G_2$. Note that $u_q(\mathfrak{g})$ has no non-trivial Hopf quotient (see [6] for example), i.e. the only quotient Hopf algebras are $\Bbbk$ or itself. But we have known that E_q has non-trivial quotient Hopf algebras, such as the quotient Hopf algebra H_1 in Lemma 4.11. Thus $E_q \ncong u_q(\mathfrak{g})$ as Hopf algebras. □

Remark 4.15. Since Corollary 4.12 and above theorem, we conclude that there exists some twist J for $u_q(sl_2^{\oplus 3})$ such that the twisted Hopf algbra is simple. This implies that the notion of simplicity of a Hopf algebra H is not determined by its tensor category of representations, even if H is assumed to be a ribbon factorizable Hopf algebra. Hence, we have reinforced a part of the conclusion in [10]. Furthermore, the following table illustrates that there is no relation between simplicity of H and primality of

Rep(H), thereby providing answers to the questions posed in the introduction.

H	H simple Hopf algebra?	Rep(H) prime?
$D(\mathbb{Z}_2)$	No	Yes
$D(\mathbb{Z}_2\times\mathbb{Z}_2)$	No	No
$\mathbb{Z}_3$	Yes	Yes
E_q	Yes	No

Lastly, let us recall the ribbon factorizable Hopf algebras given in [28, Example 5.18] and compare them with E_q. Let q' be a primitive $2n$-th root of unity, where $n \geq 1$ is an odd integer. The Hopf algebra $\mathrm{Drin}_{K^*}(\mathfrak{B}_{q'}, \mathfrak{B}^*_{q'})$ is generated by $\{x_i, y_i, k_i | \ 1 \leq i \leq 2\}$ as an algebra, subject to relations, for $1 \leq i \neq j \leq 2$,

$$k_ik_j = k_jk_i,\ k_i^{2n} = 1,\ k_ix_i = x_ik_i,\ k_iy_i = y_ik_i,\ k_ix_j = q'x_jk_i,\ k_iy_j = q'^{-1}y_jk_i,$$
$$x_iy_j + y_jx_i = \delta_{i,j}(1 - k_i),\ x_i^2 = y_i^2 = 0,\ (x_1x_2 - x_2x_1)^{2n} = (y_2y_1 - y_1y_2)^{2n} = 0,$$

The coproduct, counit and antipode are given by $\Delta(k_i) = k_i \otimes k_i$ and

$$\Delta(x_1) = x_1 \otimes 1 + k_2^n \otimes x_1,\ \Delta(x_2) = x_2 \otimes 1 + k_1^nk_2 \otimes x_2,$$
$$\Delta(y_1) = y_1 \otimes 1 + k_1k_2^n \otimes y_1,\ \Delta(y_2) = y_2 \otimes 1 + k_1^n \otimes y_2,$$
$$\epsilon(k_i) = 1,\ \epsilon(x_i) = \epsilon(y_i) = 0,$$
$$S(k_i) = k_i^{-1}, S(x_1) = -k_2^{-n}x_1, S(x_2) = -k_1^{-n}k_2^{-1}x_2,$$
$$S(y_1) = -k_1^{-1}k_2^{-n}y_1,\ S(y_2) = -k_1^{-n}y_2.$$

The dimension of $\mathrm{Drin}_{K^*}(\mathfrak{B}_{q'}, \mathfrak{B}^*_{q'})$ is $256n^4$ ([28, Example 5.18]). Due to dimension reason, we know that Rep(E_q) is not equivalent to representation category of $\mathrm{Drin}_{K^*}(\mathfrak{B}_{q'}, \mathfrak{B}^*_{q'})$ as tensor categories.

5. Declarations

5.1. **Conflict of interest.** There is no conflicts of interest to declare.

5.2. **Data availability statement.** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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