



Tensor algebras over the Steenrod algebra [☆]

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ABSTRACT

It is known that unstable Steenrod module structure on the polynomial algebra $\mathbb{F}_2[t_0, \dots, t_{N-1}] \cong H^*((\mathbb{R}P^\infty)^N; \mathbb{F}_2)$ obtained by forgetting the multiplication is isomorphic to that arising from a twisted action of Sq^1 . We show that the same theorem holds for tensor algebras. As in the abelian case, the result is applied to produce a decomposition of the tensor algebra into “weight spaces”.

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1. Introduction

Let $V := \langle t_0, \dots, t_{N-1} \rangle$ be a graded vector space over $\mathbb{F}_2$ with generators in degree 1. The axioms yield a unique unstable Steenrod algebra structure on the polynomial algebra $\mathbb{F}_2[V] \cong H^*((\mathbb{R}P^\infty)^N; \mathbb{F}_2)$ determined by $Sq^1[t_j] = t_j^2$. It was shown in [4] that the unstable Steenrod module structure on $\mathbb{F}_2[V]$ obtained by forgetting the multiplication is isomorphic, as unstable Steenrod modules, to $\mathbb{F}_2[v_0, \dots, v_{N-1}]$ with the twisted action $Sq^1(v_j) = v_{j-1}^2$ with $Sq^1(v_0) = v_{N-1}^2$. We consider the non-abelian case and show that the same result holds for the tensor algebra $T_{\mathbb{F}_2}[t_0, \dots, t_{N-1}]$. In [4] this isomorphism was applied to obtain a decomposition, as unstable Steenrod modules, of $\mathbb{F}_2[V]$ into “weight spaces”. A corresponding decomposition is shown to hold in the non-abelian case. As in the abelian case, the weight decomposition yields a second grading in which the subspace of weight 0 forms an algebra over which the other weight spaces are modules.

In the final section we develop a method for computing the Hilbert series of the weight 0 subspace and use it to compute the answer for $N \leq 5$.

For further information concerning topics discussed in this paper see [1], [6], [9] and [10].

2. Twisted Steenrod action

Let $\phi(a) = a^2$ denote the Frobenius automorphism of $\mathbb{F}_{2^N}$, the generator of the Galois group $\text{Gal}(\mathbb{F}_{2^N}/\mathbb{F}_2)$. Extend ϕ to an action on the tensor algebra $T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}]$ by setting its restriction to $T_{\mathbb{F}_2}[t_0, \dots, t_{N-1}]$

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to be the identity. By the primitive normal basis theorem (Davenport [2]) it is possible to choose an element $\omega \in \mathbb{F}_{2^N}$ so that ω generates the cyclic multiplicative group of units in $\mathbb{F}_{2^N}$ and $\{\omega, \phi(\omega), \dots, \phi^{N-1}(\omega)\}$ forms a basis for $\mathbb{F}_{2^N}$ over $\mathbb{F}_2$. Let

$$p(x) = a_0 + a_1x + \dots + a_{N-1}x^{N-1} + x^N$$

be the minimum polynomial of ω . Let

$$T = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & a_{N-1} \end{pmatrix}$$

be the $N \times N$ matrix over $\mathbb{F}_2$ representing multiplication by ω in the basis $\{1, \omega, \dots, \omega^{N-1}\}$. Since ω is a generator of the $\mathbb{F}_{2^N}^*$, the multiplicative group of units, we see that T has order $2^N - 1$ in $\text{GL}_N(\mathbb{F}_2)$. Regard T as a linear transformation in the basis $\{t_0, \dots, t_{N-1}\}$ on the linear subspace $V := \langle t_0, \dots, t_{N-1} \rangle$ of $T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}]$ and extend T multiplicatively to a selfmap of $T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}]$. Since the characteristic polynomial of T is $p(x)$, ω becomes an eigenvalue of T over $\mathbb{F}_{2^N}$. Let $v_0 \in V$ be a nonzero eigenvector for the eigenvalue ω . Set $v_j := \phi^j v_0$ for $j = 0, 1, \dots, N-1$. The entries of T lie in $\mathbb{F}_2$ so T commutes with ϕ . Therefore $Tv_j = \omega^{2^j} v_j$. Thus $\{v_0, v_1, \dots, v_{N-1}\}$ is a basis for V consisting of eigenvectors of T . Extending multiplicatively the map taking this basis of V to the other gives an isomorphism

$$B : T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}] \cong T_{\mathbb{F}_{2^N}}[v_0, \dots, v_{N-1}].$$

Grade $T_{\mathbb{F}_2}[t_0, \dots, t_{N-1}]$ by tensor length. It is generated by elements in degree 1 so it has a unique structure as an unstable algebra over the Steenrod algebra $\mathcal{A}$ determined by $\text{Sq}(t_j) = t_j + t_j^2$ and the Cartan formula. Extend the action to $T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}]$ by requiring that the action of each element of $\mathcal{A}$ be $\mathbb{F}_{2^N}$ -linear. Define an $\mathcal{A}$ -algebra action on $T_{\mathbb{F}_{2^N}}[v_0, \dots, v_{N-1}]$ by means of the isomorphism B . In this way $T_{\mathbb{F}_{2^N}}[v_0, \dots, v_{N-1}] \cong T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}]$ becomes an $\mathcal{A}$ -algebra which is unstable as an $\mathcal{A}$ -module. The action on $T_{\mathbb{F}_{2^N}}[v_0, \dots, v_{N-1}]$ of each element of $\mathcal{A}$ commutes both with ϕ and with multiplication by ω which are thus each $\mathcal{A}$ -module homomorphisms. Also observe that if $v \in V$ then $\text{Sq}^1 \phi v = v^2$. In particular $\text{Sq}^1 v_j = v_{j-1}^2$ for $j = 0, \dots, N-1$ where we conventionally set $v_j = v_{j'}$ when $j \equiv j' \pmod{N}$. Thus $T_{\mathbb{F}_2}[v_0, \dots, v_{N-1}]$ is a sub- $\mathcal{A}$ algebra of $T_{\mathbb{F}_{2^N}}[v_0, \dots, v_{N-1}]$.

Theorem 2.1. $T_{\mathbb{F}_2}[v_0, \dots, v_{N-1}] \cong T_{\mathbb{F}_2}[t_0, \dots, t_{N-1}]$ as $\mathcal{A}$ -modules.

Proof. Define $\lambda : T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}] \rightarrow T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}]$ by

$$\lambda(x) := \text{Tr}(\omega x) := \omega x + \phi(\omega x) + \dots + \phi^{N-1}(\omega x).$$

Then λ is an $\mathcal{A}$ -module homomorphism. Since $\phi(\lambda x) = \lambda x$, λx is invariant under the action of $\text{Gal}(\mathbb{F}_{2^N}/\mathbb{F}_2)$ on $T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}]$ and so $\text{Im } \lambda \subset T_{\mathbb{F}_2}[t_0, \dots, t_{N-1}]$. Define θ to be the composite

$$T_{\mathbb{F}_2}[v_0, \dots, v_{N-1}] \hookrightarrow T_{\mathbb{F}_{2^N}}[v_0, \dots, v_{N-1}] \xrightarrow[\cong]{B^{-1}} T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}] \xrightarrow{\lambda} T_{\mathbb{F}_2}[t_0, \dots, t_{N-1}].$$

Since $\{\omega, \phi(\omega), \dots, \phi^{N-1}(\omega)\}$ forms a basis for $\mathbb{F}_{2^N}$ over $\mathbb{F}_2$, θ is a monomorphism. However the Hilbert series of the two sides are identical so this implies θ is an isomorphism. The inverse to θ can be described explicitly as follows. For $x \in T_{\mathbb{F}_2}[t_0, \dots, t_{N-1}]$, write

$$B(x) = \omega q_1 + \phi(\omega)q_2 + \dots \phi^{s-1}(\omega)q_s$$

with $q_j \in T_{\mathbb{F}_2}[v_0, \dots, v_{N-1}]$, and set $\psi(x) := q_1$. The fact that x is invariant under the action of $\text{Gal}(\mathbb{F}_{2^N}/\mathbb{F}_2)$ on $T_{\mathbb{F}_{2^N}}[t_0, \dots, t_{N-1}]$ shows that $q_j = \phi^{j-1}(q_1)$ so $B(x) = \text{Tr}(\omega q_1)$. Thus $\psi = \theta^{-1}$. $\square$

Set $M^N := T_{\mathbb{F}_2}[v_0, v_1, \dots, v_{N-1}]$ with action $\text{Sq}^1(v_j) = v_{j-1}^2$ as above. (We write M^N for the tensor algebra with this “twisted” $\mathcal{A}$ -action and $T(V)$ for the tensor algebra with the standard action. In the notation M_n^N the superscript refers to the number of variables and the subscript refers to tensor length.)

Let R be a ring. Define the “raw weight” $\hat{w}(m)$ of a monomial $m \in T_R^N := T_R[v_0, \dots, v_{N-1}]$ by $\hat{w}(v_j) = 2^j$ and $\hat{w}(yz) = \hat{w}(y) + \hat{w}(z)$. Let $T_R^N(j)$ be the subspace of $T_R[v_0, \dots, v_{N-1}]$ having the monomials of raw weight j as basis.

For monomials $m \in M^N$ define the weight $w(m) \in \mathbb{Z}/(2^N - 1)$ to be the reduction of $\hat{w}(m)$ modulo $2^N - 1$. Weight is well defined under our convention that $v_j = v_{j'}$ when $j \equiv j' \pmod{N}$ since $2^N \equiv 1 \pmod{2^N - 1}$. Let $M^N(j)$ be the subspace of M^N having the monomials of weight j as basis. Observe that Sq^1 , and thus all Steenrod operations, preserve weight. Thus $M^N(j)$ is a graded (by tensor length) $\mathcal{A}$ -module (under the twisted $\mathcal{A}$ -action) for each j and $M^N(0)$ is a graded $\mathcal{A}$ -algebra. The vector space decomposition

$$M^N \cong \bigoplus_{j \in \mathbb{Z}/(2^N - 1)} M^N(j)$$

is thus a decomposition of M^N as $\mathcal{A}$ -modules and exhibits an augmented graded $M^N(0)$ -algebra structure on M^N .

The coproduct on M^N respects the grading by weight. Letting $\pi_j : M \twoheadrightarrow M^N(j)$ be the projection, the composite

$$\tilde{\psi} : M^N(0) \hookrightarrow M^N \xrightarrow{\psi} M^N \otimes M^N \xrightarrow{\pi_0 \otimes \pi_0} M^N(0) \otimes M^N(0)$$

defines a coalgebra structure on $M^N(0)$.

The map sending v_j to v_{j+1} induces $\mathcal{A}$ -module isomorphisms $\alpha : M^N(k) \xrightarrow{\cong} M^N(2k)$ for all k . Note that the orbits of $\mathbb{Z}/(2^N - 1)$ under multiplication by 2 do not have the same size. For example, for $N = 4$ the orbits are: $M(0)$; $M(1) \cong M(2) \cong M(4) \cong M(8)$; $M(3) \cong M(6) \cong M(12) \cong M(9)$; $M(5) \cong M(10)$; $M(7) \cong M(14) \cong M(13) \cong M(11)$.

Summarizing we have

Theorem 2.2.

- (1) $M := M^N := T_{\mathbb{F}_2}[v_0, v_1, \dots, v_{N-1}]$ is a $\mathbb{Z} \times \mathbb{Z}/(2^N - 1)$ bigraded Hopf algebra over the Steenrod algebra under the “tensor length” $\times$ “weight” bigrading (using the “twisted” Steenrod action). The Steenrod operation Sq^i has bidegree $(i, 0)$.
- (2) $M(0)$ forms a graded Hopf algebra over the Steenrod algebra (with coproduct $\tilde{\psi}$) and a graded subalgebra of M . For each $j \in \mathbb{Z}/(2^N - 1)$, $M(j)$ forms a Steenrod submodule of M and becomes a two-sided $M(0)$ bimodule under left and right multiplication in M .
- (3) $v_j \mapsto v_{j+1}$ describes an $(\mathcal{A} \times M(0))$ -module isomorphism $M(j) \rightarrow M(2j)$. $\square$

As in the abelian case, the weight modules are not, in general, indecomposable as unstable Steenrod modules. Finer decompositions can be obtained by applying Theorem 2.4 below, the non-abelian analogue of a theorem in [4].

Let $N' = lN$ for some integer l .

Define an algebra homomorphism $f_R^{N, N'} : T_R^N \rightarrow T_R^{N'}$ by defining it on the algebra generators by

$$f_R^{N,N'}(v_j) := \sum_{\{n \in \mathbb{Z}/N' \mid n \equiv j \pmod{N}\}} v_n = \sum_{k=0}^{l-1} v_{j+kN}$$

Note that on the left, j is an equivalence class in $\mathbb{Z}/(2^N - 1)$ while in the second expression the subscript is in $\mathbb{Z}/(2^{N'} - 1)$. However that sum runs through all elements in $\mathbb{Z}/(2^{N'} - 1)$ which are congruent to j modulo N , so the result is independent of the choice of representative of v_j .

Let $f_R^{N,N'}(j)$ be the composite

$$T_R^N(j) \hookrightarrow T_R^N \xrightarrow{f_R^{N,N'}} T_R^{N'} \longrightarrow T_R^{N'}(j).$$

Define an algebra homomorphism $\gamma_R^{N,N'} : T_R^{N'} \rightarrow T_R^N$ on generators by $\gamma_R^{N,N'}(v_j) := v_j$ and let $\gamma_R^{N,N'}(j)$ denote the composite

$$T_R^{N'}(j) \hookrightarrow T_R^{N'} \xrightarrow{\gamma_R^{N,N'}} T_R^N \longrightarrow T_R^N(j).$$

In the domain of the definition $\gamma_R^{N,N'}(v_j) := v_j$, j denotes an equivalence class in $\mathbb{Z}/N'$ while on the right it denotes the reduction of that class to $\mathbb{Z}/N$.

In the special case $R = \mathbb{F}_2$ note that $f_{\mathbb{F}_2}^{N,N'}$ commutes with Sq^1 on generators and thus is an $\mathcal{A}$ -algebra homomorphism. Similarly $\gamma_{\mathbb{F}_2}^{N,N'}$ is an $\mathcal{A}$ -algebra homomorphism.

Example 2.3. $N = 2$; $N' = 4$.

Consider $X = v_0 v_1 v_0 \in M^2(1)$

$$f(X) = (v_0 + v_2)(v_1 + v_3)(v_0 + v_2).$$

$$\pi(f(X)) = v_2 v_3 v_2 \in M^4(1)$$

$$\gamma(1) \circ f(1)(X) = \gamma(\pi(f(X))) = \gamma(v_2 v_3 v_2) = X$$

Theorem 2.4. Let $j \in \{1, \dots, 2^N\}$. The composite $\gamma_{\mathbb{F}_2}^{N,N'}(j) \circ f_{\mathbb{F}_2}^{N,N'}(j)$ is the identity on $M^N(j)$.

Proof. For any monomial $m \in T_R^N(j)$ it is immediate from the definitions that $\gamma_{\mathbb{Z}}^{N,N'}(j) \circ f_{\mathbb{Z}}^{N,N'}(j)(m) = \lambda_m m$ for some scalar $\lambda_m \in \mathbb{Z}$ and that $\gamma_{\mathbb{F}_2}^{N,N'}(j) \circ f_{\mathbb{F}_2}^{N,N'}(j)(m) = \bar{\lambda}_m m$ where $\bar{\lambda}_m$ is the reduction of λ_m modulo 2.

$$\begin{array}{ccccc}
 T_{\mathbb{Z}}^N(j) & \xrightarrow{\gamma_{\mathbb{Z}}^{N,N'} \circ f_{\mathbb{Z}}^{N,N'}(j)} & T_{\mathbb{Z}}^N(j) & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & M^N(j) & \xrightarrow{\gamma_{\mathbb{F}_2}^{N,N'} \circ f_{\mathbb{F}_2}^{N,N'}(j)} & M^N(j) & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 \mathbb{Z}[v_0, \dots, v_{N-1}](j) & \xrightarrow{\gamma_{\mathbb{Z}}^{N,N'} \circ f_{\mathbb{Z}}^{N,N'}(j)} & \mathbb{Z}[v_0, \dots, v_{N-1}](j) & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 \mathbb{F}_2[v_0, \dots, v_{N-1}](j) & \xrightarrow{\gamma_{\mathbb{F}_2}^{N,N'} \circ f_{\mathbb{F}_2}^{N,N'}(j)} & \mathbb{F}_2[v_0, \dots, v_{N-1}](j) & &
 \end{array}$$

Since the front-bottom is the identity by the abelian case considered by Campbell-Selick [4], λ_m is odd. $\square$

Define the Dynkin-Specht-Weber operator $\beta_k : V^{\otimes k} \rightarrow V^{\otimes k}$ by

$$\beta_k(x_1, x_2, \dots, x_k) := [\dots [[x_1, x_2], x_3], \dots, x_k]$$

extended linearly. It satisfies the Dynkin-Specht-Weber relation $\beta_k^2 = k\beta_k$. The image of β_k is written as $L_k(V)$. The module $L_k(V)$ has been studied extensively in the classical literature. Its dimension was computed by Witt [5], p. 169, using the Möbius Inversion Formula. Our decomposition of M^N into weight spaces yields a corresponding decomposition of $L_N(V)$ as follows.

Observe that $\beta_N : M^N \rightarrow M^N$ preserves weight. In particular, $M^N(0)$ is closed under the bracket operation and so becomes a graded (by tensor length) Lie sub-algebra of M^N over the Steenrod algebra. Set $L^N(j) := \beta_N(M^N(j))$. Then $L^N := \bigoplus_j L^N(j)$ becomes an $\mathcal{A}$ -module under the twisted Steenrod action.

Let γ_N denote the linear subspace of $V^{\otimes N}$ generated by the basis $\{v_{\sigma(0)}, v_{\sigma(1)}, \dots, v_{\sigma(N-1)}\}$ where σ runs through all elements of the symmetric/permutation group $S(\{0, \dots, N-1\})$. The module $\text{Lie}(N)$, which has been well studied, can be defined as $\text{Lie}(N) := \gamma_N \cap L_N(V)$. The basis elements of γ_N have weight $1 + 2 + 4 \dots 2^{N-1} = 2^N - 1$. That is, $\gamma_N \subset M^N(0)$ and so $\text{Lie}(N) \subset M^N(0)$.

3. Hilbert series

In this section we provide a method for computing the Hilbert series of $M^N(0)$. The method works for arbitrary N although the amount of work required to obtain the answer grows with N . As examples, we do the computation for $N \leq 5$.

Molien's Theorem [7] says that for an n -dim representation of a finite group G on the vector space $V = \langle x_1, \dots, x_n \rangle$, the Hilbert series of the ring of invariants $\mathbb{C}[x_1, \dots, x_n]^G$ of the induced action of G on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ is given by

$$\frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - tg)}$$

where we have abused notation by writing $g : V \rightarrow V$ for the linear transformation corresponding to the action of g .

More generally, if $\chi : G \rightarrow \mathbb{C}^*$ is an irreducible character and

$$\mathbb{C}[x_1, \dots, x_n]^\chi := \{f : \mathbb{C}[x_1, \dots, x_n] \mid g \cdot f = \chi(g)f\}$$

then the Hilbert series of $\mathbb{C}[x_1, \dots, x_n]^\chi$ is given by

$$\frac{\chi(1)}{|G|} \sum_{g \in G} \frac{\bar{\chi}(g)}{\det(1 - tg)}$$

For example, in the polynomial case for $n = 2$ we have $G = \mathbb{Z}/3$ and the Hilbert series of $M_{\text{AB}}^2(j) := \mathbb{F}_2[v_0, v_1](j)$, and that of the weight j polynomial in the polynomial algebra $\mathbb{F}_2[v_0, v_1]$, is given by

$$\frac{1}{3} \left(\frac{1}{(1-t)^2} + \frac{\bar{\omega}^j}{(1-\omega t)(1-\omega^2 t)} + \frac{\bar{\omega}^{2j}}{(1-\omega t)(1-\omega^2 t)} \right)$$

where $\omega = e^{2\pi i/3}$.

For $j = 0$ this becomes $\frac{1+t^2+t^4}{(1-t^3)^2}$.

In more detail, for $j = 0$ the Formal Character is $\frac{1+st+s^2t^2}{(1-s^3)(1-t^3)}$ (from which the Hilbert series can be obtained by setting $s = t$).

The formal character for the nonabelian case can be obtained from that in the abelian case by replacing $s^k t^m$ by $\binom{m+k}{m} s^k t^m$.

The nonabelian version of Moli en's Theorem (Dicks-Formanek [3]) for the action of a finite group G on the vector space $V = \langle x_1, \dots, x_n \rangle$ says the Hilbert Series of the ring $T[x_1, \dots, x_n]^G$ of invariants of the tensor algebra is given by

$$\frac{1}{|G|} \sum_{g \in G} \frac{1}{1 - \text{Tr } tg}$$

A subalgebra of a tensor algebra is always a tensor algebra, in contrast to the abelian case where a subalgebra of a polynomial algebra need not be a polynomial algebra. The Hilbert series of $T(V)$ is related to that of V by $\chi(T(V)) = 1/(1 - \chi(V))$, so one determines the other.

Since Moli en and Dicks-Formanek refer to rings of invariants over $\mathbb{C}$, to apply them to compute the Hilbert series of $M^N(0)$ over $\mathbb{F}_2$ we consider an action of $G = \{1, t, t^2, \dots, t^{2^N-2}\}$, the cyclic group of order $2^N - 1$, on $\mathbb{C}[v_0, \dots, v_{N-1}]$ such that the Hilbert series of $\mathbb{C}[v_0, \dots, v_{N-1}]^G$ over $\mathbb{C}$ equals that of $M^N(0)$ over $\mathbb{F}_2$.

Define an action of G on $\mathbb{C}[v_0, \dots, v_{N-1}]$ by extending multiplicatively the action defined on the polynomial generators by $t \cdot v_j := \zeta^{2^j} v_j$ where $\zeta := e^{2\pi i}/(2^N - 1)$. This action decomposes $\mathbb{C}[v_0, \dots, v_{N-1}]$ into weight spaces in the same manner as the decomposition of $M^N(0)$, resulting in the same Hilbert series. We apply Dicks-Formanek this action to obtain Hilbert series of $M^N(0)$ over $\mathbb{F}_2$.

$N = 2$

For $M^2(0)$ we get $(\frac{1}{1-2t} + \frac{2}{1+t})/3$. Writing this as $1/(1 - \chi(Y))$ gives $M^2(0) = T_{\mathbb{F}_2}(Y)$ where the Hilbert series of Y is $2t^2/(1 - t)$.

$N = 3$

Set $\zeta := e^{2\pi i/7}$

$A := \zeta + \zeta^2 + \zeta^4$

$B := \zeta^3 + \zeta^5 + \zeta^6$

Then the Hilbert series of $M^3(0)$ is given by

$$\chi(M^3(0)) = \frac{1/(1 - 3t) + 3/(1 - At) + 3/(1 - Bt)}{7}$$

$A + B = -1$

$A^2 = A + 2B = -A - 2$

$A = (-1 + \sqrt{7}i)/2$ $B = (-1 - \sqrt{7}i)/2$

$$\begin{aligned} \chi(M^3(0)) &= \frac{1/(1 - 3t) + 3/(1 - At) + 3/(1 + t + At)}{7} \\ &= \frac{7 - 14t - 9t^2 - At^2 - A^2t^2}{7(1 - 3t)(1 - At)(1 + t + At)} = \frac{1 - 2t - t^2}{1 - 2t - t^2 - 6t^3} \end{aligned}$$

$M^3(0) = T_{\mathbb{F}_2}(Y)$ where the Hilbert series of Y is

$$t^2 \frac{2 + A(1 - 3t) + A^2(1 - 3t)}{1 - 2t - t^2} = t^2 \frac{2 - 2(1 - 3t)}{1 - 2t - t^2} = \frac{6t^3}{1 - 2t - t^2}.$$

$N = 4$

Set $\zeta := e^{2\pi i/15}$

$A := \zeta + \zeta^2 + \zeta^4 + \zeta^8$

$$\begin{aligned} B &:= \zeta^3 + \zeta^6 + \zeta^{12} + \zeta^9 \\ C &:= \zeta^7 + \zeta^{14} + \zeta^{13} + \zeta^{11} \\ D &:= \zeta^5 + \zeta^{10} \end{aligned}$$

Then the Hilbert series of $M^4(0)$ is given by

$$\chi(M^4(0)) = \frac{1/(1-4t) + 4/(1-At) + 4/(1-Bt) + 4/(1-Ct) + 2/(1-2Dt)}{15}$$

$$\begin{aligned} B &= e^{2\pi i/5} + e^{4\pi i/5} + e^{8\pi i/5} + e^{6\pi i/5} = -1 \\ D &= e^{2\pi i/3} + e^{4\pi i/3} = -1 \\ A + B + C + D &= -1 \\ C &= \bar{A} \\ A^2 &= A + 2B + 2D = A - 4 \\ A &= (1 + \sqrt{15}i)/2 \\ C &= (1 - \sqrt{15}i)/2 \end{aligned}$$

Since

$$\frac{1}{1-At} + \frac{1}{1-Ct} = \frac{1}{1-At} + \frac{1}{1-\bar{A}t} = \frac{1-\bar{A}t+1-At}{1-At-\bar{A}t+A\bar{A}t^2} = \frac{2-t}{1-t+4t^2}$$

setting $Q(t) = 1/(1-4t) + 4(2-t)/(1-t+4t^2) + 4/(1+t) + 2/(1+2t)$ gives

$$\chi(M^4(0)) = \frac{Q(t)}{15} = \frac{1-2t-5t^2-2t^3-8t^4}{1-2t-5t^2-2t^3-32t^4-32t^5}$$

$M^4(0) = T_{\mathbb{F}_2}(Y)$ where the Hilbert series of Y is

$$\frac{8t^4(3+4t)}{1-2t-5t^2-2t^3-8t^4}.$$

$N = 5$

$$\begin{aligned} \text{Set } \zeta &:= e^{2\pi i/31} \\ A &:= \zeta + \zeta^2 + \zeta^4 + \zeta^8 + \zeta^{16} \\ B &:= \zeta^3 + \zeta^6 + \zeta^{12} + \zeta^{24} + \zeta^{17} \\ C &:= \zeta^5 + \zeta^{10} + \zeta^{20} + \zeta^9 + \zeta^{18} \\ D &:= \zeta^7 + \zeta^{14} + \zeta^{28} + \zeta^{25} + \zeta^{19} \\ E &:= \zeta^{11} + \zeta^{22} + \zeta^{13} + \zeta^{26} + \zeta^{21} \\ F &:= \zeta^{15} + \zeta^{30} + \zeta^{29} + \zeta^{27} + \zeta^{23} \end{aligned}$$

The Galois group of $\mathbb{Q}[\zeta]$ over $\mathbb{Q}$ is $\{\zeta \mapsto \zeta^k \mid k = 1, \dots, 30\}$.

Therefore A, B, C, D, E, F are Galois conjugates, sharing the same minimum polynomial.

Let $s_1, \dots, s_6$ be elementary symmetric polynomials in A, B, C, D, E, F and let $\psi_1, \dots, \psi_6$ be the power functions. We can evaluate ψ_k as follows.

$$\begin{aligned} \text{In } \mathbb{Z}[z]/(z^{31}-1) \text{ set} \\ a(z) &:= z + z^2 + z^4 + z^8 + z^{16}; \end{aligned}$$

...

$$f(z) := z^{15} + z^{30} + z^{29} + z^{27} + z^{23}.$$

Thus $A = a[\zeta]; \dots; F = f[\zeta]$.

Expanding $\psi_k(a, \dots, f)$ gives $p + 5qz + 5qz^2 + 5qz^3 + 5qz^4 + 5qz^5 + 5qz^6$ for a total of $6(5^k)$ terms, where p equals 6 times the coefficient of 1 in the expansion of $a(z)^5 \pmod{(z^{31}-1)}$. Then $\psi_k(A, \dots, F) = p - q$ where $p + 30q = 6(5^k)$.

For $k \leq 4$ we have $p = 0$ so $\psi_k = -5^{k-1}$. For $k = 5$ we find that $p = 720$ giving $q = 601$ and $\psi_5 = 119$. For $k = 6$ we find that $p = 3600$ giving $q = 3005$ and $\psi_6 = 595$.

Power functions can be written in the elementary symmetric polynomials by means of Newton's formula $ks_k = \sum_{j=1}^k (-1)^{j-1} \psi_j s_{k-j}$. (See [8], Page 23.)

Solving inductively:

$$s_1 = -1$$

$$s_2 = (1 - (-5))/2 = 3$$

$$s_3 = (-3 + (-5) - 25)/3 = -11$$

$$s_4 = (11 - (-5)3 + 25 + 125)/4 = 44$$

$$s_5 = (-44 - (-5)(-11) + (-25)3 + (-125)(-1) + 119)/5 = -36$$

$$s_6 = (36 - (-5)44 + (-25)(-11) - (-125)3 + 119(-1) - 595)/6 = 32$$

so the minimum polynomial is

$$q(x) = x^6 + x^5 + 3x^4 + 11x^3 + 44x^2 + 36x + 32.$$

Thus, setting

$$Q(t) = 1/(1-5t) + 5/(1-At) + 5/(1-Bt) + 5/(1-Ct) + 5/(1-Dt) + 5/(1-Et) + 5/(1-Ft),$$

the Hilbert series of $M^5(0)$ is given by

$$\begin{aligned} \chi(M^5(0)) &= \frac{Q(t)}{31} = \frac{1}{31(1-5t)} + \frac{5(6-5s_1t+4s_2t^2-3s_3t^3+2s_4t^2-s_5t^5)}{31(1-s_1t+s_2t^2-s_3t^3+s_4t^4-s_5t^5+s_6t^6)} \\ &= \frac{1}{31(1-5t)} + \frac{5(6+5t+12t^2+33t^3+88t^4+36t^5)}{31(1+t+3t^2+11t^3+44t^4+36t^5+32t^6)} \\ &= \frac{1-4t-2t^2-4t^3-11t^4-64t^5-28t^6}{(1-5t)(1+t+3t^2+11t^3+44t^4+36t^5+32t^6)} \end{aligned}$$

$M^5(0) = T_{\mathbb{F}_2}(Y)$ where the Hilbert series of Y is

$$\frac{40t^5(3+3t+4t^2)}{1-4t-2t^2-4t^3-11t^4-64t^5-28t^6}.$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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