WILLMORE SPHERES IN THE 3-SPHERE REVISITED

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Abstract. Bryant [1] classified all Willmore spheres in 3-space to be given by minimal surfaces in $\mathbb{R}^3$ with embedded planar ends. This note provides new explicit formulas for genus 0 minimal surfaces in $\mathbb{R}^3$ with $2k+1$ embedded planar ends for all $k \geq 4$. Peng and Xiao claimed these examples to exist in [6], but in the same paper they also claimed the existence of a minimal surface with 7 embedded planar ends, which was falsified by Bryant [2].

1. Surfaces

Let $\phi\colon \hat{\Sigma} \to S^3$ be a compact, smooth, conformally parametrised, and immersed surface such that for a suitable chosen point $p \in \phi(\hat{\Sigma}) \subset S^3$, with its stereographic projection $\pi_p\colon S^3 \setminus \{p\} \to \mathbb{R}^3$, the composition

$$f = \pi_p \circ \phi\colon \Sigma = \hat{\Sigma} \setminus \phi^{-1}\{p\} \to \mathbb{R}^3$$

is a minimal surface in $\mathbb{R}^3$. We call such surfaces f minimal surfaces with embedded planar ends. It was shown by Bryant [1] that all Willmore spheres in the 3-sphere are of this type. Conversely, every ϕ as above is a Willmore surface. By definition, (immersed) Willmore surfaces $\phi\colon \hat{\Sigma} \to S^3$ are the critical points for the Willmore functional

$$\mathcal{W}(\phi) = \int_{\hat{\Sigma}} (H^2 - K + 1)dA, \tag{1}$$

where 1 is the sectional curvature of the round 3-sphere, H is the mean curvature, K is the Gaussian curvature, and dA is the area form of the induced metric of ϕ. The Willmore energy of a Willmore sphere is quantized by $4\pi(n-1)$, with n being the number of ends of $f = \pi_p \circ \phi$.

Remark. *Peng and Xiao claim the existence of a minimal surface with 7 embedded planar ends in* [6] *and remark that the existence for $n = 2k+1 \geq 9$ follows by a long but straight forward computation. Though the $n = 7$ case was falsified by Bryant* [2]*, we show that the surfaces predicted in* [6] *do exist for $n \geq 9$ by giving a simple and explicit parametrization.*

Let X be a Riemann surface, and $g\colon X \to \mathbb{K}^r$ be a smooth map, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $r \in \mathbb{N}$. We denote by

$$dg = \partial g + \bar{\partial} g$$

the decomposition of the differential dg of g into its complex linear part ∂g and its complex antilinear part $\bar{\partial} g$. For a minimal surface $f\colon \Sigma \to \mathbb{R}^3$ there exists a holomorphic line bundle $S \to \Sigma$ with $S^2 = K_\Sigma$ and two holomorphic sections $s_1, s_2 \in H^0(\Sigma, S)$ such that

$$\partial f = (s_1^2 + s_2^2, is_1^2 - is_2^2, -2is_1s_2). \tag{2}$$

This is called the Weierstrass representation and the two spinors (s_1, s_2) are the Weierstrass data of f. In the case of a minimal surface $f\colon \Sigma \to \mathbb{R}^3$ with embedded planar ends, the Weierstrass data are meromorphic spinors on $\hat{\Sigma}$ with first order poles at $\phi^{-1}(p)$, see for example [5] and the references therein. Moreover, ∂f has no residues at the embedded planar ends.

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1.1. **Existing examples in the literature.** Peng and Xiao [6] considered the following ansatz

$$s_1 = \frac{z(z^k-c)}{(z^k-1)(z^k-\lambda)}\sqrt{dz} \quad \text{and} \quad s_2 = \frac{(z^k-a)(z^k-b)}{z(z^k-1)(z^k-\lambda)}\sqrt{dz}$$

for the Weierstrass data of a minimal surface of the $(2k+1)$-punctured sphere

$$\Sigma = \mathbb{C}P^1 \setminus \{z \in \mathbb{C} \mid z(z^k-1)(z^k-\lambda) = 0\}.$$

The parameters $a, b, c, \lambda \in \mathbb{C}$ are pairwise distinct and s_1, s_2 satisfy the conditions

$$\mathrm{res}_p s_1^2 = \mathrm{res}_p s_1 s_2 = \mathrm{res}_p s_2^2 = 0 \tag{3}$$

at every point $p \in \{z \in \mathbb{C} \mid z(z^k-1)(z^k-\lambda) = 0\}$. Peng and Xiao [6] claim that a solution (a, b, c, λ) always exist implying via (2) the existence of immersed minimal surfaces with $2k+1$ ends.

Appropriate solutions of (3) for $k \in \{4, 5, 6\}$ and explicit formulas for the corresponding minimal surfaces f can be computed easily. For example, for $k = 4$,

$$a = 10 - 4\sqrt{7} + \sqrt{\tfrac{635}{3} - 80\sqrt{7}}, \quad b = 10 + 4\sqrt{7} + \sqrt{\tfrac{635}{3} + 80\sqrt{7}}, \quad c = -3 - 4\sqrt{\tfrac{3}{5}}, \quad \lambda = -31 - 8\sqrt{15}$$

solve Equation (3). We obtain a minimal surface $f = \Re(F)$ with 9 embedded planar ends by taking the real part of a primitive

$$F\colon \mathbb{C}P^1 \setminus \{z \in \mathbb{C} \mid z(z^k-1)(z^k-\lambda) = 0\} \longrightarrow \mathbb{C}^3$$

$$F(z) = \frac{1}{45z(-1+z^4)(31+8\sqrt{15}+z^4)} \begin{pmatrix} 6\sqrt{-1}z^2(-15 - 4\sqrt{15} + 5z^4)) \\ 5(31 + 8\sqrt{15}) - 3z^4(153 + 40\sqrt{15} + 15z^4) \\ \sqrt{-1}(5(31 + 8\sqrt{15}) - 3z^4(147 + 40\sqrt{15} + 15z^4)) \end{pmatrix}$$

of (2).

2. Curves

The following description of genus 0 minimal surfaces with embedded planar ends is due to Bryant [2], see also [3]. Consider a genus 0 minimal surface $f\colon \mathbb{C}P^1 \setminus \{p_1, .., p_n\} \to \mathbb{R}^3$ with n embedded planar ends. As ∂f has no residues at the ends $p_1, .., p_n$, the surface f is the real part of a meromorphic map $F\colon \mathbb{C}P^1 \to \mathbb{C}^3$ with simple poles at $p_1, .., p_n$. Since f is conformally parametrised, F is a null curve, i.e., with respect to the standard symmetric inner product $(.,.)$ on $\mathbb{C}^3$ we have $(\partial F, \partial F) = 0$. Consider $\mathbb{C}^5$ with the inner product

$$\langle .,. \rangle = -e_0^* \otimes e_4^* - e_4^* \otimes e_0^* + e_1^* \otimes e_1^* + e_2^* \otimes e_2^* + e_3^* \otimes e_3^*,$$

the 3-quadric

$$\mathcal{Q}^3 = P\{v \in \mathbb{C}^5 \setminus \{0\} \mid \langle v, v \rangle = 0\}$$

and the holomorphic embedding

$$\Psi\colon \mathbb{C}^3 \to \mathcal{Q}^3; \quad (z_1, z_2, z_3) \mapsto [\tfrac{1}{2}(z_1^2 + z_2^2 + z_3^2), z_1, z_2, z_3, 1].$$

For a minimal sphere $f = \Re(F)$ with n embedded planar ends, $\Psi \circ F\colon \mathbb{C}P^1 \to \mathcal{Q}^3$ is an unbranched rational curve of degree n. Moreover, $\Psi \circ F$ is again a null curve, i.e., for every local holomorphic lift $\hat{\Psi}$ of $\Psi \circ F$ the condition $\langle \hat{\Psi}, \hat{\Psi} \rangle = 0 = \langle \partial\Psi, \partial\Psi \rangle$ holds. Conversely, every nondegenerate unbranched null curve gives rise to a minimal surface with embedded planar ends by reversing the above construction, where a projective curve is called nondegenerate if it is not completely contained in any hyperplane.

Let $V = \mathbb{C}^4$ be equipped with the 2-form $\Omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Consider the 5-dimensional space $W = \{\eta \in \Lambda^2 V \mid \Omega(\eta) = 0\}$ equipped with the nondegenerate symmetric inner product $(\alpha, \beta) \mapsto \frac{1}{2}(\Omega \wedge \Omega)(\alpha \wedge \beta)$ and the corresponding 3-quadric $\mathcal{Q}_\Omega$ of null lines in PW. Identifying $(W, \frac{1}{2}\Omega \wedge \Omega) \cong (\mathbb{C}^5, \langle .,. \rangle)$ yields $\mathcal{Q}_\Omega \cong \mathcal{Q}^3$. A holomorphic curve $\psi\colon \hat{\Sigma} \to \mathbb{C}P^3$ is a contact curve if $\Omega(\hat{\psi} \wedge \partial\hat{\psi}) = 0$ holds for every local holomorphic lift $\hat{\psi}$ of ψ. Then, with respect to a local holomorphic coordinate z, the map $z \mapsto \hat{\psi} \wedge \frac{\partial\hat{\psi}}{\partial z}$ gives a local lift of a well-defined holomorphic curve

$$\psi_2\colon \hat{\Sigma} \to \mathcal{Q}_\Omega \subset PW,$$

the second associated curve of ψ. It is a null curve with respect to $\frac{1}{2}\Omega \wedge \Omega$. A curve into a projective space is called nondegenerate if it is not contained in any hyperplane. The Klein correspondence (see [3]) states that every nondegenerate null curve is given by a nondegenerate contact curve in $\mathbb{C}P^3$. For $\hat{\Sigma} = \mathbb{C}P^1$ and ψ of degree d its second associated curve ψ^2 is unbranched if and only if the total branch order of ψ is $d-3$. This is a direct consequence of the Plücker relations applied to the duality between ψ and its third associated curve, see [3]. Hence, to construct genus 0 minimal surfaces with $2k+1$ embedded planar ends, we have to construct rational contact curves of degree $2k$ with total branch order $2k-3$.

2.1. **Rational contact curves of degree $2k$ with total branch order $2k-3$.** For $k \in \mathbb{N}\setminus\{3\}$ consider the map $\psi\colon \mathbb{C}P^1 \to \mathbb{C}P^3$ defined via the lift

$$\hat{\psi}(z) = \begin{pmatrix} z^3 \\ \frac{1}{6}(-6+13k-9k^2+2k^3+\frac{12(-3+2k)}{-3+k}z^k+3k(-3+2k)z^k-6z^{2k}) \\ \frac{1}{2}(-3+2k)z(-2+k+2z^k) \\ z^2(-1+k+z^k) \end{pmatrix}. \tag{4}$$

This is a nondegenerate rational curve of degree $2k$ if $k \in \mathbb{N}^{>3}$. It can be directly verified that it is a contact curve, i.e., $\Omega(\hat{\psi} \wedge \frac{\partial \hat{\psi}}{\partial z}) = 0$. For $k \in \mathbb{N}^{>3}$ its branch points are at the k-th roots of unity and $z = \infty$. The branch order at the roots of unity is 1, and at $z = \infty$ the branch order is $k-3$. Hence, the total branch order is $2k-3$. The construction fails for $k \leq 3$: for $k = 1$, the curve is of degree 2 and branched at $z = 0$ and $z = 1$. Thus, its second associate curve cannot be unbranched. For $k = 2$, the degree of the curve is $2 \neq 4$. For $k = 3$ and $k = 0$, the formula (4) gives a point in $\mathbb{C}P^3$. For $k = 4$ we obtain the first valid example

$$\hat{\psi}(z) = (z^3, 5+20z^4-z^8, 5z+5z^5, 3z^2+z^6).$$

Together with the examples of unbranched null curves of even degree $d \geq 4$ and the nonexistence results of Bryant in [2], this shows:

Theorem. *There exist a Willmore sphere $\phi_n\colon \mathbb{C}P^1 \to S^3$ with Willmore energy $4\pi(n-1)$ if and only if $n \in \mathbb{N}\setminus\{2,3,5,7\}$.*

For completeness, we state the formula for the genus 0 minimal surfaces $f = \Re(F)$ with $(2k+1)$ embedded planar ends corresponding to the contact curve (4). Note that these surfaces are only determined up to Goursat transformations [4], see also [2]. We leave it as an exercise for the interested reader to verify that we have recovered the surfaces of Peng and Xiao [6] up to a Goursat transformation and reparametrization. The meromorphic map $F\colon \mathbb{C}P^1 \to \mathbb{C}^3$ is given by

$$F(z) = \begin{pmatrix} \frac{-12\sqrt{-1}(-3+k)(-3+2k)z^2(2-3k+k^2-2z^k)}{4z((-3+k)(-1+k)^2(-3+2k)-6(-1+k)(-3+2k)z^k-3(-3+k)z^{2k})} \\ \frac{(-1+k)(-12(3-2k)^2(-2+k)z^k-12(-3+k)(-3+2k)z^{2k}+(-3+k)((6-7k+2k^2)^2-12z^4))}{4z((-3+k)(-1+k)^2(-3+2k)-6(-1+k)(-3+2k)z^k-3(-3+k)z^{2k})} \\ \frac{\sqrt{-1}(-1+k)(12(3-2k)^2(-2+k)z^k+12(-3+k)(-3+2k)z^{2k}+(-3+k)(-(6-7k+2k^2)^2-12z^4))}{4z((-3+k)(-1+k)^2(-3+2k)-6(-1+k)(-3+2k)z^k-3(-3+k)z^{2k})} \end{pmatrix}.$$

References

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