manuscripta math.

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Study of multiplicities in induced representations of GL_n through a symmetric reduction

Received: 23 June 2021 / Accepted: 21 January 2022

Abstract. In this article, we develop a process to symmetrize the irreducible admissible representations of $GL_n(F)$ with F being a finite extension of $\mathbb{Q}_p$, as a consequence we obtain a more geometric understanding of the coefficient $m(\mathbf{b}, \mathbf{a})$ appearing in the decomposition of parabolic inductions, which allows us to prove a conjecture inspired by Zelevinsky.

1. Introduction

In this article we study the multiplicities in parabolic inductions of admissible representations of $\mathrm{GL}_n(F)$ with F being a finite extension of $\mathbb{Q}_p$. Zelevinsky [20] classifies the admissible irreducible representations of $\mathrm{GL}_n(F)$ in terms of multi-segments. More precisely, given a multi-segment $\mathbf{a}$, one can attach to it an irreducible representation $L_{\mathbf{a}}$, described as the unique irreducible sub-representation in some standard representation $\pi(\mathbf{a})$ constructed by parabolic induction. In other words, in the Grothendieck group of the category of admissible representations, we have

$$\pi(\mathbf{a}) = L_{\mathbf{a}} + \sum_{\mathbf{b}<\mathbf{a}} m(\mathbf{b}, \mathbf{a}) L_{\mathbf{b}}, \quad m(\mathbf{b}, \mathbf{a}) \in \mathbb{Z}_{\geq 0},$$

where "$<$" is suitable partial order imposed on the set of multi-segments. Zelevinsky [18] conjectured these coefficients can be computed through the intersection cohomology of certain nilpotent orbits constructed from the multi-segments. This conjecture was later proved by Ginzburg [5, Theorem 8.6.23] for standard modules defined by geometry and Ariki [1, 3] showed that Ginzburg's standard module conincide with Zelevinsky's.

It was first observed by Zelevinsky [19] that the above nilpotent orbits could be studied through an open embedding into some Schubert varieties. Therefore the coefficient $m(\mathbf{b}, \mathbf{a})$ can be identified to be the evaluation at $q = 1$ of some Kazhdan-Lusztig polynomial $P_{t(\mathbf{a}),t(\mathbf{b})}(q)$, where $t(\mathbf{a})$ and $t(\mathbf{b})$ live in some possibly very big symmetric group compared to the number of segments in $\mathbf{a}$ and $\mathbf{b}$.

This was further studied by A.Henderson [7], where he used a cancellation property of Kazhdan-Lusztig polynomials due to S. C. Billey and G. S. Warrington

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Mathematics Subject Classification: Primary 14L30 · 22E50

https://doi.org/10.1007/s00229-022-01375-1
Published online: 09 February 2022

to find new permutations in a symmetric group of smaller size to compute the coefficient $m(\mathbf{b}, \mathbf{a})$. Similar results were also obtained by T. Susuki [17].

Our starting point is different. We consider a pair of segments $\{\Delta, \Delta'\}$, there are four possible relations between them (cf. Definition 2.6). We say two multisegments $\mathbf{a}$ and $\mathbf{a}'$ have the same relation type if there exists an order preserving bijection between them which preserves also the relation type of segments and induces bijections between the set of beginnings and ends (cf. Definition 7.1). Note that the fact that $\mathbf{a}$ and $\mathbf{a}'$ have the same relation type naturally induces a bijection

$$\Xi : S(\mathbf{a}) := \{\mathbf{b} : \mathbf{b} < \mathbf{a}\} \to S(\mathbf{a}')$$

which is compatible with the poset structure. Then we have the following conjecture inspiring by Zelevinsky [20, 8.7].

Conjecture 1.1. *For* $\mathbf{a}$ *and* $\mathbf{a}'$ *having the same relation type, then for* $\mathbf{b} \in S(\mathbf{a})$ *with* $\mathbf{b}' = \Xi(\mathbf{b})$*, we have*

$$P_{\mathbf{a},\mathbf{b}}(q) = P_{\mathbf{a}',\mathbf{b}'}(q).$$

A simple example of multi-segments of the same relation type is

$$\mathbf{a} = \{[1, 2], [2, 3]\}, \quad \mathbf{a}' = \{[1, 3], [2, 4]\}. \tag{1}$$

Our main result of the paper is

Theorem 1.2. (Theorem 7.5) *The Conjecture 1.1 is true.*

Specializing to $q = 1$, we obtain the equality between $m(\mathbf{b}, \mathbf{a})$ and $m(\mathbf{b}', \mathbf{a}')$.

We remark that at its first glance, it seems that Conjecture 1.1 can be read directly from the Kazhdan-Lusztig description of the coefficients. But chasing the description of Zelevinsky (cf. [19]) on the coefficient $m(\mathbf{b}, \mathbf{a})$ in terms of Kazhdan-Lusztig polynomials, we find that in the Example (1), the coefficient $m(\mathbf{a}, \mathbf{b})$ corresponds to a Kazhdan-Lusztig polynomial in the symmetric group S_4, while $m(\mathbf{a}', \mathbf{b}')$ corresponds to a Kazhdan-Lusztig polynomial in S_6. So in fact we show how to relate pairs of elements in different symmetric groups with the same Kazhdan-Lusztig polynomial. These provide a nontrivial family of examples to the following old conjecture of Lusztig.

Conjecture 1.3. (G.Lusztig, M. Dyer) *Any poset isomorphism* ψ *between two Bruhat intervals* $[u, v]$ *and* $[u', v']$ *in possibly distinct Coxeter groups* W *and* W' *preserves Kazhdan-Lusztig polynomials, i.e.,*

$$\forall x, y \in [u, v], \quad P_{x,y} = P_{\psi(x),\psi(y)}.$$

For recent progress on this conjecture, see [6], [13] and [4].

In the literature, the coefficients $m(\mathbf{b}, \mathbf{a})$ are treated as some invariant depending on the cuspidal support, like the dependence on the Coxeter groups of the Kazhdan-Lusztig polynomials. However, from the point of view of the combinatorial invariance conjecture of the Kazhdan-Lusztig polynomials, one should treat all Coxeter groups as special objects in some larger (conjectured) combinatorial category. In the case of $m(\mathbf{b},\mathbf{a})$, we have a natural category, i.e., the category of multi-segments, and our goal is to treat the coefficient $m(\mathbf{b},\mathbf{a})$ as a combinatorial invariant of this category. The category of multi-segments, on the one hand admits many natural operations from representation theory, on the other hand connects to many natural geometric objects: graded nilpotent classes, Schubert varieties of different types.

We offer some other conjectures regarding the notion of "same relation type", which requires understanding deeper the combinatorial nature of the category of multi-segments and also motivates the paper from its very beginning.

Conjecture 1.4. *Let* $\{\mathbf{a},\mathbf{a}'\}$ *and* $\{\mathbf{b},\mathbf{b}'\}$ *be pairs of multi-segments of the same relation type and*

$$\mathbf{c}=\mathbf{a}\coprod\mathbf{b},\quad \mathbf{c}'=\mathbf{a}'\coprod\mathbf{b}'.$$

Assume further that $\mathbf{c}$ *and* $\mathbf{c}'$ *are of the same relation type, hence we have*

$$\Xi: S(\mathbf{c})\to S(\mathbf{c}').$$

Then

$$L_{\mathbf{a}}\times L_{\mathbf{b}}=L_{\mathbf{c}}+\sum_{\mathbf{d}<\mathbf{c}}n_{\mathbf{c},\mathbf{d}}L_{\mathbf{d}},\, n_{\mathbf{c},\mathbf{d}}\in\mathbb{Z}_{\geq 0},$$

implies

$$L_{\mathbf{a}'}\times L_{\mathbf{b}'}=L_{\mathbf{c}'}+\sum_{\mathbf{d}<\mathbf{c}}n_{\mathbf{c},\mathbf{d}}L_{\Xi(\mathbf{d})}.$$

The parabolic induction $L_{\mathbf{a}}\times L_{\mathbf{b}}$, in its special case where $L_{\mathbf{a}}$ is cuspidal, is closely related to the computation of Jacquet functor. Special case of this conjecture follows from [14] and [2].

Conjecture 1.5. *Let* $\mathbf{a}$ *and* $\mathbf{a}'$ *be multi-segments of the same relation type, then* $L_{\mathbf{a}}$ *is square irreducible if and only if* $L_{\mathbf{a}'}$ *is.*

We say that an irreducible representation π of GL_n is square irreducible if the parabolic induction of $\pi\otimes\pi$ is an irreducible representation of GL_{2n}. Otherwise we say that π is non-square. The first example of non-square irreducible representation was discovered by B. Leclerc [12] as a counter-example to a conjecture of Bernstein and Zelevinsky(where he also noticed that the counter-example was closely related to a counter-example produced by Kashiwara and Saito [9] to a conjecture by G. Lusztig). Later work [11] relates it to various geometric and representation theoretic properties.

We summarize the technique we use in the paper. In section two we setup the notations and the main object to study. We introduce the notion of partial Bernstein-Zelevinsky operator in our context, similar notion were introduced by Kashiwara [8] in quantum groups. One can show that the two are related by certain exponential maps. In section three, we review the classical theory of describing the coefficients $m(\mathbf{b},\mathbf{a})$'s in terms of geometry of graded nilpotent classes and Kazhdan-Lusztig polynomials. In section four, we introduce the notion of symmetric multisegment. For a symmetric multisegment $\mathbf{a}$ consisting of n segments, we show that $S(\mathbf{a})$ is indexed by S_n (cf. Proposition 4.7). Then we proceed to study the geometry corresponding to symmetric multisegments. Consider

$$p_\varphi : E_\varphi \to Z_\varphi (cf.\text{Definition } 4.9),$$

we observe that the projection p_φ restricting to the open part O_φ^{sym} (cf. Theorem 4.12) is a fibration with fiber $GL_{n,n-1}$. Furthermore, the stratification on O_φ^{sym} induces the stratification by Schubert varieties. This should be compared with the open embedding of Zelevinsky [19] of E_φ into certain flag varieties, where the corresponding orbit is in general only known to be open in the corresponding Schubert cell.

In section five, we deal with the general multisegment. The idea is that we want to reduce the general case to the symmetric case, as long as one is concerned about the coefficient $m(\mathbf{b}, \mathbf{a})$ (or the corresponding Kazhdan-Lusztig polynomial). The main result of this section is Proposition 5.39. Roughly speaking, we show that given the multisegment $\mathbf{a}'$, we can associate to it a symmetric multisegment $\mathbf{a}$ (in a sense, minimal), such that we have an embedding of poset

$$\psi : S(\mathbf{a}') \to S(\mathbf{a})$$

without changing the Kazhdan-Lusztig polynomials. Note that for convenience we actually reverse the procedure, we show how to start with $\mathbf{a}$ to obtain $\mathbf{a}'$, but in this section we show that we obtain all multisegment from symmetric ones (cf. Corollary 6.11). To achieve our result, we develop a geometric version of the partial Bernstein-Zelevinsky operators introduced in section two. We study a locally closed subvariety $(X_{\mathbf{a}}{}^k)_W$ in E_{φ_a} (the graded nilpotent class attached to $\mathbf{a}$) (cf.5.3), we relate it to another graded nilpotent class denoted by $Z^{k,\mathbf{a}}$ of a smaller multisegment $\mathbf{a}^{(k)}$ through certain fibration. The most technical part is to show that the map τ_W under consideration is an open immersion, which allows us to relate the Kazhdan-Lusztig polynomials of different symmetric groups (hence also the coefficient $m(\mathbf{b}, \mathbf{a})$'s for different pairs of $\{\mathbf{a}, \mathbf{b}\}$). We expect further applications of such a refined study of the graded nilpotent classes. Section 7 is just a direct application of what we have proved before. The upshot is the proof of Theorem 7.5, which is our Conjecture 1.1.

2. Zelevinsky classification of induced representations

In this section we recall the Zelevinsky classification of induced representations of $GL_n(F)$, with F a finite extension of $\mathbb{Q}_p$ (Similar classification for non-Archimedean local field of positive characteristic, but we will not need it).

Notation 2.1. *We fix a uniformizer ϖ_F of F, and an absolute value $|.|$ on F such that $|\varpi_F| = 1/q$, where q is the cardinal of its residue field. Note ν the character of $GL_n(F)$ defined by $\nu(g) = |\det g|$.*

Definition 2.2. By segment Δ, we mean a finite consecutive subset of integers

$$\Delta = \{k_1, k_1 + 1, \cdots, k_2\}, \quad k_1 \leq k_2, \quad (k_1, k_2) \in \mathbb{Z}^2.$$

We define a multisegment $\mathfrak{m}$ to be a multiset of segments,

$$\mathfrak{m} = \{\Delta_1, \cdots, \Delta_r\}.$$

We denote by $\sharp\Delta$ the cadinality of Δ. We call

$$\deg \mathfrak{m} = \sum_{i=1}^{r} \sharp\Delta_i$$

the degree of $\mathfrak{m}$.

Notation 2.3. *For convenience, most of the time we will simply write the multisegment* $\mathfrak{m}$ *as a formal sum of segments:*

$$\mathfrak{m} = \sum_{i=1}^{r} \Delta_i.$$

Following Zelevinsky,

Proposition-Definition 2.4. *[20, 3.1] To any irreducible cuspidal representation* ρ *of* $GL_n(F)$ *and a segment* $\Delta = [i, i+1, \ldots, i+\deg(\Delta)]$, *we can associate an irreducible representation* $L_{(\Delta,\rho)}$ *of* $GL_{n\deg\Delta}(F)$*: it is the unique irreducible subrepresentation of the normalized parabolic induction* $\mathrm{Ind}_P^{GL_{n\deg\Delta}(F)}(\rho\nu^i \otimes \cdots \otimes \rho\nu^{i+\deg(\Delta)})$*, where* P *is the parabolic subgroup of* $GL_{n\deg\Delta}(F)$ *consisting of upper triangular matrices with Levi* $GL_n \times \cdots \times GL_n$*. When* $\rho = 1$ *be the trivial character of* $GL_1(F)$*, we write directly* L_Δ*.*

Definition 2.5. (1) We say two segments Δ_1 and Δ_2 are linked if $\Delta_1 \cup \Delta_2$ is again a segment and different from Δ_1 and Δ_2.
(2) We define the following total order on the set of segments

$$\begin{cases} [j,k] \prec [m,n], \text{ if } k < n, \\ [j,k] \prec [m,n], \text{ if } j > m, n = k. \end{cases}$$

Definition 2.6. The relation type between segments Δ, Δ' is one of the following

- Δ cover Δ' if $\Delta \supseteq \Delta'$;
- linked but not juxtaposed if Δ does not cover Δ' and $\Delta \cup \Delta'$ is a segment but $\Delta \cap \Delta' \neq \emptyset$;
- juxtaposed if $\Delta \cup \Delta'$ is a segment but $\Delta \cap \Delta' = \emptyset$;
- unrelated if $\Delta \cap \Delta' = \emptyset$ and Δ, Δ' are not linked.

Definition 2.7. A multisegment of cuspidal representations

$$\mathbf{a} = \{(\Delta_1, \rho_1), \cdots, (\Delta_r, \rho_r)\}$$

is said to be well ordered if $\Delta_1 \succeq \Delta_2 \succeq \cdots \succeq \Delta_r$.

Definition 2.8. For any pair of representation $(\pi_1, \pi_2) \in Rep(GL_n(F)) \times Rep(GL_m(F))$, let $\pi_1 \times \pi_2$ be the normalized induction of $\pi_1 \otimes \pi_2$, which is a representation of $GL_{n+m}(F)$.

Proposition 2.9. ([20, Theorem 4.2]) *The following are equivalent:*

(1) The induced representation

$$L_{(\Delta_1,\rho)} \times L_{(\Delta_2,\rho)} \times \cdots \times L_{(\Delta_r,\rho)}$$

is irreducible.

(2) For any $1 \leq i, j \leq r$, the segments Δ_i and Δ_j are not linked with each other.

Proposition 2.10. (cf. [18, 4.6]) *Let Δ_1 and Δ_2 be two linked segments with $\Delta_1 \succeq \Delta_2$, then*

$$L_{(\Delta_1,\rho)} \times L_{(\Delta_2,\rho)}$$

contains a unique sub-representation $L_{\mathbf{a}_1}$ and a unique quotient $L_{\mathbf{a}_2}$ with

$$\mathbf{a}_1 = \{(\Delta_1, \rho),\ (\Delta_2, \rho)\}, \quad \mathbf{a}_2 = \{(\Delta_1 \cup \Delta_2, \rho),\ (\Delta_1 \cap \Delta_2, \rho)\}.$$

Definition 2.11. Let $\mathbf{a} = \{\Delta_1, \Delta_2, \ldots, \Delta_r\}$ be a multisegment such that Δ_1 and Δ_2 are linked. By an elementary operation we mean replacing the segments Δ_1 and Δ_2 by $\Delta_1 \cap \Delta_2$ and $\Delta_1 \cup \Delta_2$. In this case, we say $\mathbf{a}' = \{\Delta_1 \cap \Delta_2, \Delta_1 \cup \Delta_2, \ldots, \Delta_r\}$ is obtained from $\mathbf{a}$ via an elementary operation.

Definition 2.12. We define $\mathbf{b} \leq \mathbf{a}$ if $\mathbf{b}$ can be obtained from $\mathbf{a}$ via a sequence of elementary operations. Denote

$$S(\mathbf{a}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{a}\},$$

then $\leq$ defines a partial order on $S(\mathbf{a})$(cf. [20, 7.1]).

We recall the following classifying theorem due to Zelevinsky.

Theorem 2.13. ([20, Theorem 6.1]) *Let $\mathbf{a} = \{(\Delta_1, \rho_1), \ldots, (\Delta_r, \rho_r)\}$ be a multisegment of cuspidal representations with $\Delta_1 \succeq \Delta_2 \succeq \cdots \succeq \Delta_r$, then*

(1) The representation

$$L_{(\Delta_1,\rho_1)} \times \cdots \times L_{(\Delta_r,\rho_r)}$$

contains a unique subrepresentation, which we denote by $L_{\mathbf{a}}$.

(2) The representations $L_{\mathbf{a}'}$ and $L_{\mathbf{a}}$ are isomorphic if and only if $\mathbf{a} = \mathbf{a}'$.

(3) Any irreducible representation of $GL_n(F)$ is of the form $L_{\mathbf{a}}$.

Definition 2.14. For a cuspidal representation ρ, we call the set

$$\Pi_\rho = \{\rho\nu^s : s \in \mathbb{Z}\}$$

a Zelevinsky line. We denote by $\mathcal{O}(\rho)$ the set of multisegments supported on Π_ρ. As for $\rho = 1$, we simply write Π and $\mathcal{O}$ instead.

Notation 2.15. *From now on, for* $\mathbf{a} = \{(\rho, \Delta_1), \ldots, (\rho, \Delta_r)\}$ *being well ordered, we denote*

$$\pi(\mathbf{a}) = L_{(\rho,\Delta_1)} \times \cdots \times L_{(\rho,\Delta_r)}.$$

According to Theorem 2.13, let $\mathbf{a} = \{(\rho, \Delta_1), \ldots, (\rho, \Delta_r)\}$ be a multisegment with support contained in some Zelevinsky line Π_ρ, then we can write

$$\pi(\mathbf{a}) = \sum_{\mathbf{b}\in\mathcal{O}(\rho)} m(\mathbf{b}, \mathbf{a}) L_{\mathbf{b}} \tag{2}$$

where $m(\mathbf{b}, \mathbf{a}) \in \mathbb{Z}_{\geq 0}$. The aim of this paper is to give some new insights on these $m(\mathbf{b}, \mathbf{a})$.

Remark. it is conjectured in [20, 8.7] that the coefficient $m(\mathbf{b}, \mathbf{a})$ depends only on the combinatorial relations of **b** and **a**, and not on the specific cuspidal representation ρ. The independence of specific cuspidal representation can be shown by type theory, see for example [15, Remark 4.18]. **In other words, as far as we are concerned with the coefficient** $m(\mathbf{b}, \mathbf{a})$, **we can restrict ourselves to the special case** $\rho = 1$, **the trivial representation of** $GL_1(F)$.

Notation 2.16. *We denote by* $\mathcal{R}_n$ *the Grothendieck group of the category of finite length unipotent representations of* $GL_n(F)$ *(i.e., the irreducible constituents are of the form* $L_{\mathbf{a}}$ *for some* $a \in \mathcal{O}$*). Let*

$$\mathcal{R} = \oplus_{n\geq 1}\mathcal{R}_n.$$

As was observed by Zelevinsky, the group $\mathcal{R}$ can be endowed with a Hopf algebra structure via

Proposition 2.17. *The set* $\mathcal{R}$ *is a bi-algebra with the multiplication* μ *and co-multiplication* c *given by*

$$\mu(\pi_1 \otimes \pi_2) = \pi_1 \times \pi_2, \qquad c(\pi) = \sum_{r=0}^{n} J_{P_{r,n-r}}^{GL_n(F)}(\pi),$$

where $J_{P_{r,n-r}}^{GL_n(F)}$ *denotes the Jacquet functor from the category of smooth representations of* $GL_n(F)$ *to the category of smooth representations of* $M_{r,n-r} = GL_r(F) \times GL_{n-r}$ *regarded as the Levi subgroups of* $P_{r,n-r}$*, where* $P_{r,n-r}$ *is the unique parabolic subgroup containing the upper triangular matrices with the given Levi subgroups.*

Now Zelevinsky's classification theorem can be reformulate into the following

Corollary 2.18. *The algebra* $\mathcal{R}$ *is a polynomial ring with indeterminates* $\{L_\Delta : \Delta \subset \Pi\}$*. Moreover, as a* $\mathbb{Z}$*-module, the set* $\{L_{\mathbf{a}} : \mathbf{a} \in \mathcal{O}\}$ *form a basis for* $\mathcal{R}$*.*

In the final part of this section we show how to define some analogue of the Bernstein-Zelevinsky operator, which serve as a tool for us in the sequel and motivates the development of this paper.

Definition 2.19. We define a left partial Bernstein-Zelevinsky operator with respect to the index i to be a morphism of algebras

$$
\begin{aligned}
&{}^{i}\mathscr{D}:\mathcal{R}\to\mathcal{R},\\
&{}^{i}\mathscr{D}(L_{[j,k]}) = L_{[j,k]}+\delta_{i,j}L_{[j+1,k]} \text{ if } k>j,\\
&{}^{i}\mathscr{D}(L_{[j]}) = L_{[j]}+\delta_{[i],[j]}.
\end{aligned}
$$

Here $\delta_{[i],[j]}$ is the Kronecker symbol. Also we define a right partial Bernstein-Zelevinsky operator with respect to the index i to be a morphism of algebras

$$
\begin{aligned}
&\mathscr{D}^{i}:\mathcal{R}\to\mathcal{R}\\
&\mathscr{D}^{i}(L_{[j,k]}) = L_{[j,k]}+\delta_{i,k}L_{[j,k-1]} \text{ if } j<k,\\
&\mathscr{D}^{i}(L_{[j]}) = L_{[j]}+\delta_{[i],[j]}.
\end{aligned}
$$

Definition 2.20. We define

$$
\begin{aligned}
\mathscr{D}^{[i,j]} &= \mathscr{D}^{j}\circ\cdots\circ\mathscr{D}^{i}\\
{}^{[i,j]}\mathscr{D} &= ({}^{i}\mathscr{D})\circ\cdots\circ({}^{j}\mathscr{D})
\end{aligned}
$$

For $\mathbf{c}=\{\Delta_1,\ldots,\Delta_s\}$ with

$$\Delta_1\preceq\cdots\preceq\Delta_s,$$

we define

$$\mathscr{D}^{\mathbf{c}}=\mathscr{D}^{\Delta_1}\circ\cdots\circ\mathscr{D}^{\Delta_s}$$

and

$${}^{\mathbf{c}}\mathscr{D}=({}^{\Delta_s}\mathscr{D})\circ\cdots\circ({}^{\Delta_1}\mathscr{D}).$$

Remark. we recall that in [3, 4.5], Bernstein and Zelevinsky define an operator $\mathscr{D}$ to be an algebra homomorphism

$$\mathscr{D}:\mathcal{R}\to\mathcal{R},$$

which plays a crucial role in Zelevinsky's classification theorem.

The relation between Jacquet functor and Bernstein-Zelevinsky operator is given by

Proposition 2.21. ([20, 3.8]) *Let δ be the algebraic morphism such that $\delta(\rho)=1$ for all $\rho\in\Pi$ and $\delta(L_\Delta)=0$ for all non cuspidal representations L_Δ. Then*

$$\mathscr{D}=(1\otimes\delta)\circ c,$$

where c is the co-multiplication(cf. Proposition 2.17).

The main advantage to work with partial Bernstein-Zelevinsky operator instead of the operator defined by Bernstein and Zelevinsky is that they are much more simpler but share the following positivity properties:

Theorem 2.22. *Let* **a** *be any multisegment, then we have*

$$\mathscr{D}^i(L_{\mathbf{a}}) = \sum_{\mathbf{b}\in\mathcal{O}} n(\mathbf{b},\mathbf{a})L_{\mathbf{b}},$$

such that $n(\mathbf{b},\mathbf{a}) \geq 0$*, for all* **b**.

Remark. the same property of positivity holds for ${}^i\mathscr{D}$.

The theorem is deduced from Lemma 2.24 and Lemma 2.25 below.

Definition 2.23. For $i \in \mathbb{Z}$, let ϕ_i be the morphism of algebras defined by

$$\phi_i : \mathcal{R} \to \mathbb{Z}$$
$$\phi_i([j,k]) = \delta_{[i],[j,k]}.$$

Lemma 2.24. *For all multisegment* **a***, we have* $\phi_i(L_{\mathbf{a}}) = 1$ *if and only if* **a** *contains no other segments than* $[i]$*, otherwise it is zero.*

Proof. We prove this result by induction on the cardinality of $S(\mathbf{a})$, denoted by $|S(\mathbf{a})|$. If $|S(\mathbf{a})| = 1$, then $\mathbf{a} = \mathbf{a}_{min}$, hence $\phi_i(L_{\mathbf{a}}) = \phi_i(\pi(\mathbf{a}))$, which is nonzero if and only if **a** contains no other segments than $[i]$, and in latter case it is 1. Let **a** be a general multi-segment,

$$\pi(\mathbf{a}) = L_{\mathbf{a}} + \sum_{\mathbf{b}<\mathbf{a}} m(\mathbf{b},\mathbf{a})L_{\mathbf{b}}.$$

Now $|S(\mathbf{a})| > 1$, we know that **a** is not minimal in $S(\mathbf{a})$, hence **a** contains segments other than $[i]$, which implies $\phi_i(\pi(\mathbf{a})) = 0$.

Since $|S(\mathbf{b})| < |S(\mathbf{a})|$ for any $\mathbf{b} < \mathbf{a}$, by induction, we know that $\phi_i(L_{\mathbf{b}}) = 0$ because **b** must contain segments other than $[i]$. So we are done. □

Lemma 2.25. *We have* $\mathscr{D}^i = (1 \otimes \phi_i) \circ c$.

Proof. Since both are algebraic morphisms, we only need to check that they coincide on generators. We recall the equation from [20, Proposition 3.4],

$$c(L_{[j,k]}) = 1 \otimes L_{[j,k]} + \sum_{r=j}^{k-1} L_{[j,r]} \otimes L_{[r+1,k]} + L_{[j,k]} \otimes 1.$$

Now applying ϕ_i,

$$(1\otimes\phi_i)c(L_{[j,k]}) = L_{[j,k]} + \delta_{i,k}L_{[j,k-1]} \text{ if } (k > j)$$
$$(1\otimes\phi_i)c(L_{[j]}) = L_{[j,k]} + \delta_{i,j}.$$

Comparing this with the definition of $\mathscr{D}^i$ yields the result. □

Remark. We have the following relation between the partial Bernstein-Zelevinsky operator and the Bernstein-Zelevinsky operator. Let $e(\mathbf{a}) = \{[i_1],\ldots,[i_\alpha] : i_1 \leq \cdots \leq i_\alpha\}$ be the end of **a**, then

$$\mathscr{D}(\mathbf{a}) = \mathscr{D}^{[i_1,i_\alpha]}(\mathbf{a}).$$

Remark. In later sections we will develop a geometric version of the partial Bernstein-Zelevinsky operator, which allows us to draw more information about the Kazhdan-Lusztig polynomials $P_{\mathbf{a},\mathbf{b}}(q)$ as well as its value at $q = 1$, the latter being our $m(\mathbf{b},\mathbf{a})$.

3. Graded nilpotent classes and Kazhdan-Lusztig polynomials

A geometric interpretation of Zelevinsky's classification, which is also due to Zelevinsky, is to consider the graded nilpotent classes associated to multisegments, cf. [18], [19].

Definition 3.1. Let $\mathbf{a}$ be a multisegment, we define a function

$$\varphi_{\mathbf{a}} : \mathbb{Z} \to \mathbb{N}$$

by letting $\mathbf{a} = \sum_{i \leq j} a_{ij}[i, j]$, and

$$\varphi_{\mathbf{a}}(k) = \sum_{i \leq k \leq j} a_{ij}.$$

We call $\varphi_{\mathbf{a}}$ the weight function of $\mathbf{a}$.

Definition 3.2. Let $\varphi : \mathbb{Z} \to \mathbb{N}$ be a function with finite support. Consider the $\mathbb{Z}$-graded $\mathbb{C}$-vector space

$$V_\varphi = \oplus_k V_{\varphi,k}, \quad \dim(V_{\varphi,k}) = \varphi(k).$$

Moreover,

(1) let E_φ be the set of endomorphisms $T = (T_{\varphi,i})$ of V_φ of degree 1:

$$T_{\varphi,i} : V_{\varphi,i} \to V_{\varphi,i+1};$$

(2) let $G_\varphi = \prod_k GL(V_{\varphi,k})$ be the automorphism of V_φ.

We call E_φ the variety of representations of dimension φ.

Remark. The group G_φ acts naturally on the endomorphism space E_φ via conjugations.

Proposition 3.3. *[18, 2.3] The orbits of E_φ under G_φ are naturally parametrized by multisegments of weight function φ. Moreover, let $O_{\mathbf{a}}$ be the orbit associated to the multisegment $\mathbf{a}$, then*

$$\mathbf{a} \leq \mathbf{b} \Leftrightarrow O_{\mathbf{b}} \subseteq \overline{O}_{\mathbf{a}},$$

where $\overline{O}_{\mathbf{a}}$ denotes the orbit closure of $O_{\mathbf{a}}$ in the Zarisky topology.

Remark. Let $\mathbf{a} = \sum_{i \leq j} a_{ij}[i, j]$ such that $\varphi_{\mathbf{a}} = \varphi$, then the orbit associated consists of the operators having exactly a_{ij} Jordan cells starting from $V_{\varphi,j}$ and ending in $V_{\varphi,i}$.

Remark. A particular instance of the above Proposition is the most important for our application. Let us fix an element $T = (T_{\varphi_{\mathbf{a}},i}) \in O_{\mathbf{a}}$ and and an integer k such that $\varphi_{\mathbf{a}}(k) \neq 0$. Consider the set U of endomorphisms $T' \in E_{\varphi_{\mathbf{a}}}$ such that $T'_{\varphi_{\mathbf{a}},i} = T_{\varphi_{\mathbf{a}},i}$ for $i > k$ or $i < k$. Then the stratum containing $T' \in U$ is uniquely determined by its image in the coset

$$P_{k+1} \backslash \mathrm{Hom}(V_{\varphi_{\mathbf{a}},k}, V_{\varphi_{\mathbf{a}},k+1})/P_k,$$

where P_{k+1} is the parabolic subgroup of $GL(V_{\varphi_{\mathbf{a}},k+1})$ determined by the filtration

$$\{0 \subseteq \ker(T_{\varphi_{\mathbf{a}},k+1}) \subseteq \ker(T_{\varphi_{\mathbf{a}},k+2}T_{\varphi_{\mathbf{a}},k+1}) \subseteq \cdots \subseteq V_{\varphi_{\mathbf{a}},k+1}\}$$

and P_k is the parabolic subgroup of $GL(V_{\varphi_{\mathbf{a}},k})$ determined by the filtration

$$\{0 \subseteq \cdots \subseteq \mathrm{Im}(T_{\varphi_{\mathbf{a}},k-1}T_{\varphi_{\mathbf{a}},k-2}) \subseteq \mathrm{Im}(T_{\varphi_{\mathbf{a}},k-1}) \subseteq V_{\varphi_{\mathbf{a}},k}\}$$

Example 3.4. We consider the function $\varphi : \mathbb{Z} \to \mathbb{N}$ with

$$\varphi(0) = \varphi(1) = 2, \quad \varphi(i) = 0, \quad \forall i \neq 1, 2.$$

Then $E_\varphi = \{T : V_{\varphi,0} \to V_{\varphi,1}\}$. In this case E_φ contains 3 orbits which are determined by the rank of T:

(1) the orbit $\{T : \text{rank } T = 0\} = O_{\mathbf{a}_0}$ with $\mathbf{a}_0 = \{[0], [0], [1], [1]\}$;
(2) the orbit $\{T : \text{rank } T = 1\} = O_{\mathbf{a}_1}$ with $\mathbf{a}_1 = \{[0], [1], [0, 1]\}$;
(3) the orbit $\{T : \text{rank } T = 2\} = O_{\mathbf{a}_2}$ with $\mathbf{a}_2 = \{[0, 1], [0, 1]\}$.

Remark. The orbits $\{O_{\mathbf{b}} : \varphi_{\mathbf{b}} = \varphi\}$ give rise to a stratification of the affine space E_φ.

Definition 3.5. Let $\mathbf{a}, \mathbf{b}$ be two multisegments such that $\mathbf{b} \in S(\mathbf{a})$. Then we define the polynomial

$$P_{\mathbf{a},\mathbf{b}}(q) = \sum_i q^{(i+d_{\mathbf{b}})/2} \dim \mathcal{H}^i(\overline{O}_{\mathbf{b}})_{x_{\mathbf{a}}},$$

where

- $\mathcal{H}^i(\overline{O}_{\mathbf{b}}) := \mathcal{H}^i(IC(\overline{O}_{\mathbf{b}}))$ is the intersection complex supported on $\overline{O}_{\mathbf{b}}$ which is constant with stalk $\mathbb{C}$ on $O_{\mathbf{b}}$;
- $x_{\mathbf{a}} \in O_{\mathbf{a}}$ is an arbitrary point and $d_{\mathbf{b}} = \dim(O_{\mathbf{b}})$.

We call it the Kazhdan-Lusztig polynomial associated to $\{\mathbf{a}, \mathbf{b}\}$.

We recall the following fundamental result, which is conjectured by Zelevinsky and named by whom the p-adic analogue of Kazhdan-Lusztig conjecture.

Theorem 3.6. ([18], [5, Theorem 8.6.23], [1, 3]) *Let $\mathcal{H}^i(\overline{O}_{\mathbf{b}})_{\mathbf{a}}$ denote the stalk at a point $x \in O_{\mathbf{a}}$ of the i-th intersection cohomology sheaf of the variety $\overline{O}_{\mathbf{b}}$. Then*

$$m(\mathbf{b}, \mathbf{a}) = P_{\mathbf{a},\mathbf{b}}(1).$$

Remark. In [19, Theorem 1], Zelevinsky showed that the varieties $\overline{O}_{\mathbf{b}}$ are locally isomorphic to some Schubert varieties of type A_m, where $m = \deg(\mathbf{b})$. Hence we know that $P_{\mathbf{a},\mathbf{b}}(q)$ is a polynomial in q. More precisely, for each $\mathbf{a}$, Zelevinsky associated a permutation $w(\mathbf{a})$ in the symmetric group $S_{\deg(\mathbf{a})}$ such that we have

$$P_{\mathbf{a},\mathbf{b}}(q) = P_{w(\mathbf{a}),w(\mathbf{b})}(q).$$

Remark. In this paper, for symmetric multisegments $\mathbf{a}$ and $\mathbf{b}$ (cf. Sect. 4), we will give more concrete description about the coefficient $m(\mathbf{b}, \mathbf{a})$ in terms of the elements in S_n with n equals to the number of segments contained in $\mathbf{a}$, cf. Corollary 4.15. For the general case, we will use the reduction method from Sect. 6 to give a more concrete description of $P_{\mathbf{a},\mathbf{b}}(q)$.

4. Symmetric multisegments and the associated graded nilpotent classes

In this section we introduce the notion of symmetric multisegment, which plays an essential role in our present paper.

Definition 4.1. Let $\Delta = [i, j]$ be a segment, then we define the beginning and the end of Δ to be

$$b(\Delta) = i, \quad e(\Delta) = j.$$

Definition 4.2. We say that a multisegment $\mathbf{a} = \{\Delta_1, \ldots, \Delta_r\}$ is regular if $b(\Delta_1), \ldots, b(\Delta_r)$ are distinct and $e(\Delta_1), \ldots, e(\Delta_r)$ are distinct.

Example 4.3. The segment $\mathbf{a} = \{[1, 2], [2, 4], [4, 5]\}$ is regular.

Proposition 4.4. *Let* $\mathbf{a}$ *be a regular multisegment, then any* $\mathbf{b} \leq \mathbf{a}$ *is also regular.*

Proof. This follows from the fact that if $\mathbf{a}_1$ is obtained from $\mathbf{a}$ by elementary operation, then $b(\mathbf{a}_1) \subseteq b(\mathbf{a})$ and $e(\mathbf{a}_1) \subseteq e(\mathbf{a})$. □

Definition 4.5. Let $\mathbf{a} = \{\Delta_1, \ldots, \Delta_n\}$ be regular. We say that $\mathbf{a}$ is symmetric if

$$\max\{b(\Delta_i) : i = 1, \ldots, n\} \leq \min\{e(\Delta_i) : i = 1, \ldots, n\}.$$

Example 4.6. The multisegment $\mathbf{a} = \{[1, 4], [2, 5], [3, 6]\}$ is symmetric.

We have

Proposition 4.7. *Fix a symmetric multisegment* $\mathbf{a}_{\mathrm{Id}} = \{\Delta_1, \ldots, \Delta_n\}$ *satisfying*

$$b(\Delta_1) < \cdots < b(\Delta_n),$$
$$e(\Delta_1) < \cdots < e(\Delta_n).$$

Then for any permutation in the symmetric group S_n*, the formula*

$$\Phi(w) = \sum_{i=1}^{n} [b(\Delta_i), e(\Delta_{w(i)})]$$

defines a bijection between S_n *and* $S(\mathbf{a}_{\mathrm{Id}})$*. Moreover, the order relation on* $S(\mathbf{a}_{\mathrm{Id}})$ *induces the inverse Bruhat order, i.e.,*

$$w \leq v \Leftrightarrow \Phi(w) \geq \Phi(v).$$

Proof. We observe that the elementary transformations correspond exactly to transpositions. □

For the rest of the section, we consider a special case of symmetric multisegments, we assume that

$$\mathbf{a}_{\mathrm{Id}} = \sum_{i=1}^{n} [i, n+i-1].$$

We remind that we already constructed a bijection

$$\Phi : S_n \to S(\mathbf{a}_{\mathrm{Id}})$$

such that $\Phi(\mathrm{Id}) = \mathbf{a}_{\mathrm{Id}}$.

To ease the notation, from now on until the end of this section we will write φ as the weight function of $\mathbf{a}_{\mathrm{Id}}$ instead of $\varphi_{\mathbf{a}_{\mathrm{Id}}}$.

We consider the variety E_φ of representations of dimension φ.

Definition 4.8. Let

$$O_w = O_{\Phi(w)}, \quad \text{and} \quad O_\varphi^{\mathrm{sym}} = \coprod_{w \in S_n} O_w \subseteq E_\varphi.$$

Also, let

$$\overline{O}_w^{sym} = \overline{O}_w \cap O_\varphi^{\mathrm{sym}}.$$

Definition 4.9. Let $M_{i,j}$ be the space of $i \times j$ matrices. Let

$$\begin{array}{l} E_\varphi = M_{2,1} \times \cdots \times M_{n-1,n-2} \times M_{n,n-1} \times M_{n-1,n} \times \cdots \times M_{1,2} \\ \big\downarrow p_\varphi \\ Z_\varphi := M_{2,1} \times \cdots \times M_{n-1,n-2} \times M_{n-1,n} \times \cdots \times M_{1,2}. \end{array}$$

be the natural projection with fiber $M_{n,n-1}$.

Now we want to describe the fiber of the restriction $p_\varphi|_{O_\varphi^{\mathrm{sym}}}$.

Definition 4.10. We define $GL_{n,n-1}$ to be the subset of $M_{n,n-1}$ consisting of the matrices of rank $n-1$.

We denote by $p_n : M_{n,n} \twoheadrightarrow M_{n,n-1}$ the morphism of forgetting the last column in $M_{n,n}$.

Remark. Now by restriction to GL_n, we have the morphism

$$p_n : GL_n \twoheadrightarrow GL_{n,n-1},$$

which satisfies the property that $p_n(g_1 g_2) = g_1 p_n(g_2)$ for $g_1, g_2 \in GL_n$.

Proposition 4.11. *The morphism*

$$p_n : GL_n \twoheadrightarrow GL_{n,n-1},$$

is a fibration. Furthermore, it induces a bijection

$$\bar{p}_n : B_n \backslash GL_n / B_n \to B_n \backslash GL_{n,n-1} / B_{n-1},$$

where B_n denotes the Borel subgroup of GL_n consisting of upper triangular matrices.

Proof. Note that p_n is GL_n equivariant with GL_n acting transitively on itself and on $GL_{n,n-1}$. By Bruhat decomposition we obtain that p_n induces

$$\bar{p}_n : B_n \backslash GL_n / B_n \to B_n \backslash GL_{n,n-1} / B_{n-1},$$

which is a bijection. □

Fix a basis $\{v_{ij} | i = 1, \ldots, 2n-1, j = 1, \ldots, \varphi(i)\}$ of V_φ such that $V_{\varphi,i} = \oplus_{j=1}^{\varphi(i)} \mathbb{C} v_{ij}$.

Theorem 4.12. *The morphism*

$$p_\varphi|_{O_\varphi^{\mathrm{sym}}}$$

is smooth with fiber $GL_{n,n-1}$. Moreover, the morphism $p_\varphi|_{O_w} : O_w \to p_\varphi(O_\varphi^{\mathrm{sym}})$ is surjective with fiber $B_n p_n(w) B_{n-1}$ for any $w \in S_n$.

Proof. Note that smoothness follows from that $p_\varphi : E_\varphi \to Z_\varphi$ is smooth and that O_φ^{sym} is open in E_φ. Here the openness of O_φ^{sym} follows from that it contains the open stratum indexed by the minimal element in $S(\mathbf{a}_{\mathrm{Id}})$ (or the maximal element in S_n by Proposition 4.7). To see the rest of the properties, for $w \in S_n$, consider an element $e_w \in E_\varphi$ satisfying

$$\begin{cases} e_w(v_{ij}) = v_{i+1,j}, & \text{for } i < n-1 \\ e_w(v_{n-1,j}) = v_{n,w(j)}, & \\ e_w(v_{ij}) = v_{i+1,j-1}, & \text{for } i \geq n. \end{cases},$$

here we put $v_{i,0} = 0$ and $v_{2n,j} = 0$ for all i and j.

Example 4.13. Let $w = (1, 2)$ and $n = 3$, then by the strategy in the proof, e_w is given by the Figure 1.

We claim that $e_w \in O_w$. In fact, it suffices to observe that

$$e_w : v_{ii} \to \cdots \to v_{n-1,i} \to v_{n,w(i)} \to v_{n+1,w(i)-1} \to \cdots \to v_{n+w(i)-1,1},$$

which by Proposition 3.3, implies that the multisegment indexing e_w contains $[i, w(i) + n - 1]$ for all $i = 1, \ldots, n$, hence it must be $\Phi(w)$. Note that, by definition, we have

$$p_\varphi(e_{\mathrm{Id}}) = p_\varphi(e_w), \text{ for all } w \in S_n.$$

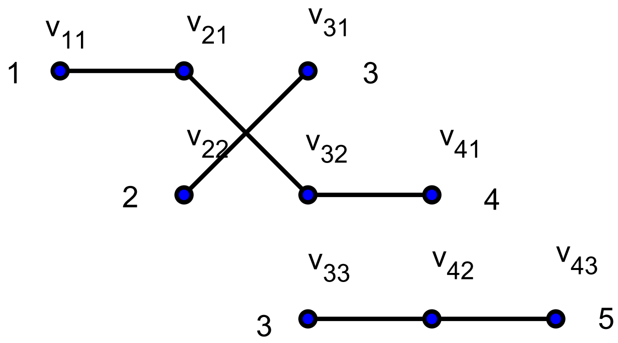

Fig. 1. Construction of e_w in case $n = 3$

Since p_φ is compatible with the action of G_φ, we get

$$p_\varphi(O_\varphi^{\mathrm{sym}}) = p_\varphi(O_w), \text{ for all } w \in S_n,$$

which implies that $p|_{O_w}$ is surjective. Now it remains to characterize its fiber. Let $T' \in p_\varphi(O_\varphi^{\mathrm{sym}})$, then $p_\varphi^{-1}(T') \simeq M_{n,n-1}$ in E_φ. Moreover, for $T = (T_1, \ldots, T_{2n-2}) \in p_\varphi^{-1}(T')$, then $T \in O_\varphi^{\mathrm{sym}}$ if and only if

$$T_{n-1} \in GL_{n,n-1}.$$

Therefore, the map $T \mapsto T_{n-1}$ induces

$$p_\varphi^{-1}(T') \cap O_\varphi^{\mathrm{sym}} \simeq GL_{n,n-1}.$$

Consider the variety $p_\varphi^{-1}(T') \cap O_w$. Note that since G_φ acts transitively on $p_\varphi(O_\varphi^{\mathrm{sym}})$, we may assume that $T' = p_\varphi(e_{\mathrm{Id}})$.

It suffices to show that the set of $f_w \in O_w$ satisfying

$$\begin{cases} f_w(v_{ij}) = v_{i+1,j}, & \text{for } i < n-1 \\ f_w(v_{ij}) = v_{i+1,j-1}, & \text{for } i \geq n. \end{cases}$$

is in bijection with $B_n p_n(w) B_{n-1}$ via $p_\varphi^{-1}(p_\varphi(e_{\mathrm{Id}})) \cap O_\varphi^{\mathrm{sym}} \simeq GL_{n,n-1}$.

Now the element $f_w \in O_w$ is completely determined by the component

$$f_{w,n-1} : V_{\varphi,n-1} \to V_{\varphi,n}.$$

We know by Proposition 3.3 and the remark following it that $f_{w,n-1}$ is injective hence of rank $n-1$. Hence we have $f_{w,n-1} \in GL_{n,n-1}$.

Now by Proposition 4.11 we get $B_n \backslash GL_{n,n-1}/B_{n-1}$ is indexed by S_n, it remains to see that $f_{w,n-1}$ is in the class indexed by $p_n(w)$.

Finally, we note that p_φ is a morphism equivariant under the action of

$$G_\varphi = GL_1 \times GL_2 \times \cdots \times GL_{n-1} \times GL_n \times \cdots \times GL_2 \times GL_1.$$

Since G_φ acts transitively on O_w, the image of O_w is $G_\varphi.(p_\varphi(e_w))$, hence is $p_\varphi(O_{\mathrm{Id}})$. Now we prove that the stabilizer of $p_\varphi(e_w)$ is $B_n \times B_{n-1}$. Let $e_{\mathrm{Id}} =$

$(e_1, \ldots, e_{n-1}, e_n, \ldots, e_{2n-2})$ with $e_i \in M_{i,i+1}$ if $i < n$ and $e_i \in M_{i,i-1}$ if $i \geq n$. We have

$$p_\varphi(e_{\mathrm{Id}}) = (e_1, \ldots, e_{n-2}, e_n, \ldots, e_{2n-2}).$$

Let $g = (g_1, \ldots, g_n, g_{n+1}, \ldots, g_{2n-1})$ such that $g.p_\varphi(e_{\mathrm{Id}}) = p_\varphi(e_{\mathrm{Id}})$. Then by definition for $i < n-1$ we know that $g_{i+1} e_i g_i^{-1} = e_i$. We prove by induction on i that $g_i \in B_i$ for $i \leq n-1$. For $i = 1$, we have nothing to prove. Now assume that $i \leq n-2$, and $g_i \in B_i$, we show that $g_{i+1} \in B_{i+1}$. Consider

$$g_{i+1} e_i g_i^{-1}(g_i(v_{ij})) = g_{i+1} e_i(v_{ij}) = g_{i+1}(v_{i+1,j}).$$

On the other hand, by induction, we know that

$$g_{i+1} e_i g_i^{-1}(g_i(v_{ij})) = e_i(g_i(v_{ij})) \in \oplus_{k \leq j} \mathbb{C} v_{i+1,k}.$$

Therefore we have $g_{i+1} \in B_{i+1}$. Actually, since e_i is injective, the equality $e_i(g_i(v_{ij})) = g_{i+1}(v_{i+1,j})$ implies that g_i is completely determined by g_{i+1}. This shows that $g_{n-1} \in B_{n-1}$ and it determines all g_i for $i < n-1$. The same method proves that $g_n \in B_n$ and it determines all g_i for $i > n$. We conclude that the fiber of the morphism $p_\varphi|_{O_w}$ is isomorphic to $B_n p_n(w) B_{n-1}$. □

Corollary 4.14. *Let $v, w \in S_n$ such that $v \leq w$, and let X_w denote the closure of $B_n w B_n$ in GL_n. Then we have*

$$\dim \mathcal{H}^i(\overline{O}_w^{sym})_v = \dim \mathcal{H}^i(X_w)_v,$$

for all $i \in \mathbb{Z}$, here the index v on the left hand side indicates that we localize at a generic point in O_v and on the right hand side means that we localize at a generic point in $B_n v B_n$.

Proof. From the above theorem and its proof, we obtain a stratified algebraic morphism

$$O_\varphi^{\mathrm{sym}} \to GL_{n,n-1} \times Z_\varphi$$

where $GL_{n,n-1}$ is identified with the fiber $p_\varphi^{-1}(p_\varphi(e_{\mathrm{Id}}))$. Here the stratification on $GL_{n,n-1}$ is the one obtained by Proposition 4.11, and the stratification on Z_φ is taken to be the trivial stratification.

We get

$$\dim \mathcal{H}^i(\overline{O}_w^{sym})_v = \dim \mathcal{H}^i(\overline{B_n p_n(w) B_{n-1}})_{B_n p_n(v) B_{n-1}}.$$

Now apply Proposition 4.11, we have

$$\dim \mathcal{H}^i(\overline{B_n p_n(w) B_{n-1}})_{B_n p_n(v) B_{n-1}} = \dim \mathcal{H}^i(X_w)_v.$$

□

Corollary 4.15. *We have for $v \leq w$ in S_n,*

$$m(\Phi(v), \Phi(w)) = P_{v,w}(1).$$

Proof. This follows from the fact that

$$\dim \sum_i \mathcal{H}^i(X(w))_v = P_{v,w}(1)$$

(cf. [10]). □

5. Lowering the degree of a multisegment

In this section we describe a procedure to decrease the degree of a multisegment **a** without affecting the coefficients $m(\mathbf{b}, \mathbf{a})$.

5.1. Notation and Combinatorics

Definition 5.1. For $\Delta = [i, j]$ a segment, we put

$$\Delta^- = [i, j-1], \quad \Delta^+ = [i, j+1].$$

Definition 5.2. Let $k \in \mathbb{Z}$ and Δ be a segment, we define

$$\Delta^{(k)} = \begin{cases} \Delta^-, & \text{if } e(\Delta) = k; \\ \Delta, & \text{otherwise .} \end{cases}$$

For a multisegment $\mathbf{a} = \{\Delta_1, \ldots, \Delta_r\}$, we define

$$\mathbf{a}^{(k)} = \{\Delta_1^{(k)}, \ldots, \Delta_r^{(k)}\}.$$

Definition 5.3. We say that the multisegment $\mathbf{b} \in S(\mathbf{a})$ satisfies **the hypothesis** $H_k(\mathbf{a})$ if the following two conditions are verified

(1) $\deg(\mathbf{b}^{(k)}) = \deg(\mathbf{a}^{(k)})$;
(2) there exists no pair of linked segments $\{\Delta, \Delta'\}$ in **b** such that $e(\Delta) = k - 1,\ e(\Delta') = k$.

Definition 5.4. Let

$$\tilde{S}(\mathbf{a})_k = \{\mathbf{c} \in S(\mathbf{a}) : \deg(\mathbf{c}^{(k)}) = \deg(\mathbf{a}^{(k)})\}.$$

Remark. Let $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$. Then

$$\sharp\{\Delta \in \mathbf{a} : e(\Delta) = k\} = \sharp\{\Delta \in \mathbf{c} : e(\Delta) = k\}.$$

Here we count segments with multiplicities.

Lemma 5.5. *Let* $k \in \mathbb{Z}$.

(1) For any $\mathbf{b} \in S(\mathbf{a})$, *we have* $\deg(\mathbf{b}^{(k)}) \geq \deg(\mathbf{a}^{(k)})$.
(2) Let $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$, *then for* $\mathbf{b} \in S(\mathbf{a})$ *such that* $\mathbf{b} > \mathbf{c}$, *we have* $\mathbf{b} \in \tilde{S}(\mathbf{a})_k$.
(3) Let $\mathbf{b} \in \tilde{S}(\mathbf{a})_k$, *then* $\mathbf{b}^{(k)} \in S(\mathbf{a}^{(k)})$. *Moreover, if we suppose that* **a** *satisfies the hypothesis* $H_k(\mathbf{a})$ *and* $\mathbf{b} \neq \mathbf{a}$, *then*

$$\mathbf{b}^{(k)} \in S(\mathbf{a}^{(k)}) - \{\mathbf{a}^{(k)}\}$$

.

(4) Suppose that **a** *does not verify the hypothesis* $H_k(\mathbf{a})$, *then there exists* $\mathbf{b} \in S(\mathbf{a})$ *satisfying the hypothesis* $H_k(\mathbf{a})$, *such that* $\mathbf{b}^{(k)} = \mathbf{a}^{(k)}$.

Proof. For (1), note that for any $\mathbf{b} \in S(\mathbf{a})$, $e(\mathbf{b}) := \{e(\Delta) : \Delta \in \mathbf{b}\}$ is a sub-multisegment of $e(\mathbf{a})$. From $\mathbf{b}$ to $\mathbf{b}^{(k)}$, we replace those segments Δ such that $e(\Delta) = k$ by Δ^-. Now (1) follows by counting the segments ending in k.

For (2), by (1), we have

$$\deg(\mathbf{a}^{(k)}) \leq \deg(\mathbf{b}^{(k)}) \leq \deg(\mathbf{c}^{(k)}).$$

The fact that $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$ implies that $\deg(\mathbf{a}^{(k)}) = \deg(\mathbf{c}^{(k)})$, hence $\deg(\mathbf{a}^{(k)}) = \deg(\mathbf{b}^{(k)})$ and $\mathbf{b} \in \tilde{S}(\mathbf{a})_k$.

As for (3), suppose that $\deg(\mathbf{b}^{(k)}) = \deg(\mathbf{a}^{(k)})$, we prove $\mathbf{b}^{(k)} < \mathbf{a}^{(k)}$. Let

$$\mathbf{a} = \mathbf{a}_0 > \cdots > \mathbf{a}_r = \mathbf{b}$$

be a maximal chain of multisegments, then by (2), we know $\deg(\mathbf{a}_j^{(k)}) = \deg(\mathbf{a}^{(k)})$, for all $j = 1, \ldots, r$. Our proof breaks into two parts.

(I) We show that

$$\deg(\mathbf{a}_j^{(k)}) = \deg(\mathbf{a}_{j+1}^{(k)}) \Rightarrow \mathbf{a}_j^{(k)} \geq \mathbf{a}_{j+1}^{(k)}.$$

Let $\mathbf{a}_{j+1}$ be obtained from $\mathbf{a}_j$ by applying the elementary operation to two linked segments Δ, Δ'.

- If none of them ends in k, then $\mathbf{a}_j^{(k)}$ contains both of them. We obtain $\mathbf{a}_{j+1}^{(k)}$ by applying the elementary operation to them. If one of them ends in k, we assume $e(\Delta') = k$.
- If Δ precedes Δ', we know that if $e(\Delta) < k - 1$, Δ is still linked to Δ'^-, and one obtains $\mathbf{a}_{j+1}^{(k)}$ by applying elementary operation to $\{\Delta, \ \Delta'^-\}$, otherwise $e(\Delta) = k - 1$, which implies $\mathbf{a}_{j+1}^{(k)} = \mathbf{a}_j^{(k)}$.
- If Δ is preceded by Δ', then the fact that

$$\deg(\mathbf{a}_{j+1}^{(k)}) = \deg(\mathbf{a}_j^{(k)})$$

implies $b(\Delta) \leq k$, hence Δ'^- is linked to Δ, and we obtain $\mathbf{a}_{j+1}^{(k)}$ from $\mathbf{a}_j^{(k)}$ by applying elementary operation to them.

Here we conclude that $\mathbf{b}^{(k)} \in S(\mathbf{a}^{(k)})$.

(II) Assuming that $\mathbf{a}$ satisfies the hypothesis $H_k(\mathbf{a})$, we show that

$$\mathbf{a}_1^{(k)} < \mathbf{a}^{(k)}.$$

Let $\mathbf{a}_1$ be obtained from $\mathbf{a}$ by performing the elementary operation to Δ, Δ'.

We do it as in (I) but put $j = 0$. Note that in (I), the only case where we can have $\mathbf{a}_1^{(k)} = \mathbf{a}^{(k)}$ is when Δ precedes Δ' and $e(\Delta') = k, \ e(\Delta) = k - 1$. But such a case can not exist since $\mathbf{a}$ verifies the hypothesis $H_k(\mathbf{a})$. Hence we are done.

Finally, for (4), we construct $\mathbf{b}$ in the following way. Suppose that $\mathbf{a}$ does not satisfy the hypothesis $H_k(\mathbf{a})$, then there exists a pair of linked segments $\{\Delta, \Delta'\}$ such that

$$e(\Delta) = k - 1, \quad e(\Delta') = k,$$

let $\mathbf{a}_1$ be the multisegment obtained by applying the elementary operation to Δ and Δ'. We have

$$\mathbf{a}_1^{(k)} = \mathbf{a}^{(k)}.$$

If again $\mathbf{a}_1$ fails the hypothesis $H_k(\mathbf{a})$, we repeat the same construction to get $\mathbf{a}_2, \ldots$, since

$$\mathbf{a} > \mathbf{a}_1 > \cdots .$$

In finite step, we get $\mathbf{b}$ satisfying the conditions and

$$\mathbf{b}^{(k)} = \mathbf{a}^{(k)}.$$

□

Remark. Actually, the multisegment constructed in (4) is unique, as we shall see later (Proposition 5.39).

Definition 5.6. We define a morphism

$$\psi_k : \tilde{S}(\mathbf{a})_k \to S(\mathbf{a}^{(k)})$$

by sending $\mathbf{c}$ to $\mathbf{c}^{(k)}$.

Proposition 5.7. *The morphism ψ_k is surjective.*

Proof. Fix $\mathbf{d} \in S(\mathbf{a}^{(k)})$, and then choose a maximal chain of multisegments,

$$\mathbf{a}^{(k)} = \mathbf{d}_0 > \cdots > \mathbf{d}_r = \mathbf{d}.$$

By induction, we can assume that there exists $\mathbf{c}_i \in \tilde{S}(\mathbf{a})_k$ such that $\mathbf{c}_i^{(k)} = \mathbf{d}_i$, for all $i < r$. Assume we obtain $\mathbf{d}$ from $\mathbf{d}_{r-1}$ by performing the elementary operation on the pair of linked segments $\{\Delta, \Delta'\}$. We assume that $\Delta \prec \Delta'$.

- If $e(\Delta) \neq k-1$ and $e(\Delta') \neq k-1$, then we observe that the pair of segments are actually contained in $\mathbf{c}_{r-1}$. Let $\mathbf{c}_r$ be the multisegment obtained by performing the elementary operation to them . We conclude that $\mathbf{c}_r^{(k)} = \mathbf{d}_r$, and $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$.
- If $e(\Delta) = k-1$, then $\Delta^+ \in \mathbf{c}_{r-1}$ and $\Delta' \in \mathbf{c}_{r-1}$ or $\Delta \in \mathbf{c}_{r-1}$. The fact that $\mathbf{d}_{r-1} = \mathbf{c}_{r-1}^{(k)}$ implies that $k \notin e(\mathbf{d}_{r-1})$, hence $e(\Delta') > k$. Hence both Δ and Δ^+ are linked to Δ'. In either case we perform the elementary operation to get $\mathbf{c}_r$ such that $\mathbf{c}_r^{(k)} = \mathbf{d}$.
- If $e(\Delta') = k-1$, then $\Delta'^+ \in \mathbf{c}_{r-1}$ and $\Delta \in \mathbf{c}_{r-1}$ or $\Delta' \in \mathbf{c}_{r-1}$. The same argument as in the second case shows that there exists $\mathbf{c}_r$ such that $\mathbf{c}_r^{(k)} = \mathbf{d}$.

□

Furthermore, the proof of Proposition 5.7 yields the following refinement.

Corollary 5.8. *Let* $\mathbf{c} \in \tilde{S}(\mathbf{a})_k, \mathbf{d} \in S(\mathbf{a}^{(k)})$ *such that*

$$\mathbf{c}^{(k)} > \mathbf{d},$$

then there exists a multisegment $\mathbf{e} \in \tilde{S}(\mathbf{a})_k$ *such that*

$$\mathbf{c} > \mathbf{e},\ \mathbf{e}^{(k)} = \mathbf{d}.$$

Proof. Note that $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$ implies $\tilde{S}(\mathbf{a})_k \supseteq \tilde{S}(\mathbf{c})_k$. Combine with the surjectivity of

$$\psi_k : \tilde{S}(\mathbf{c})_k \to S(\mathbf{c}^{(k)}),$$

we get the result. □

Definition 5.9. For a multisegment $\mathbf{a}$, and $k \in \mathbb{Z}$ we define

$$S(\mathbf{a})_k = \{\mathbf{c} \in \tilde{S}(\mathbf{a})_k : \mathbf{c} \text{ satisfies the hypothesis } H_k(\mathbf{a})\}.$$

Proposition 5.10. *The restriction*

$$\begin{aligned} \psi_k : S(\mathbf{a})_k &\to S(\mathbf{a}^{(k)}) \\ \mathbf{c} &\mapsto \mathbf{c}^{(k)} \end{aligned}$$

is also surjective. For $\mathbf{c} \in \tilde{S}(\mathbf{a})_k, \mathbf{d} \in S(\mathbf{a}^{(k)})$ *such that* $\mathbf{c}^{(k)} > \mathbf{d}$, *there exists a multisegment* $\mathbf{e} \in S(\mathbf{c})_k$ *such that* $\mathbf{e}^{(k)} = \mathbf{d}$.

Proof. For $\mathbf{d} \in S(\mathbf{a}^{(k)})$, by Proposition 5.7, we know that there exists $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$ such that $\mathbf{c}^{(k)} = \mathbf{d}$. But by (4) in Lemma 5.5, we know that there exists $\mathbf{c}' \in S(\mathbf{c})_k$ such that $\mathbf{c}'^{(k)} = \mathbf{c}^{(k)} = \mathbf{d}$. We conclude that ψ_k is surjective by the observation that if $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$, then

$$S(\mathbf{c})_k \subseteq S(\mathbf{a})_k.$$

Assume that $\mathbf{c} \in \tilde{S}(\mathbf{a})_k, \mathbf{d} \in S(\mathbf{a}^{(k)})$ satisfying $\mathbf{c}^{(k)} > \mathbf{d}$. By Corollary 5.8, we know that there exists an $\mathbf{e}' \in \tilde{S}(\mathbf{c})_k$ such that $\mathbf{e}'^{(k)} = \mathbf{d}$. By (4) in Lemma 5.5, we know that there exists $\mathbf{e} \in S(\mathbf{e}')_k$ such that $\mathbf{e}^{(k)} = \mathbf{e}'^{(k)} = \mathbf{d}$. Hence we conclude by the fact that if $\mathbf{e}' \in \tilde{S}(\mathbf{c})_k$, then

$$S(\mathbf{e}')_k \subseteq S(\mathbf{c})_k.$$

□

Remark. In the following sections, most of the time we will work with $\mathbf{a}^{(k)}$ and the hypothesis $H_k(\mathbf{a})$. But all our results will remain valid if we replace $\mathbf{a}^{(k)}$ by their left hand side versions ${}^{(k)}\mathbf{a}$ and $H_k(\mathbf{a})$ by ${}_kH(\mathbf{a})$ etc., see the end of this section.

5.2. Injectivity of ψ_k: First Step

Notation 5.11. *For any multisegment* $\mathbf{a}$*, we denote by* $\mathbf{a}_{\min}$ *the minimal element in* $S(\mathbf{a})$.

By Proposition 5.10, we know there exists $\mathbf{c} \in S(\mathbf{a})_k$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$. In this section, we show that such a $\mathbf{c}$ is unique in $S(\mathbf{a})_k$.

Notation 5.12. *Let* $\ell_{\mathbf{a},k} = \sharp\{\Delta \in \mathbf{a} : e(\Delta) = k\}$.

Definition 5.13. Let

$$\mathbf{a}_0 = \{\Delta \in (\mathbf{a}^{(k)})_{\min} : e(\Delta) = k-1\}.$$

Proposition 5.14. *Let* $\mathbf{a}_0 = \{\Delta_1 \succeq \cdots \succeq \Delta_r\}$. *Let* $\mathbf{c}$ *be a multisegment such that*

(1) If $\varphi_{\mathbf{a}}(k-1) > \varphi_{\mathbf{a}}(k)$*, then* $r = \varphi_{\mathbf{a}}(k-1) - \varphi_{\mathbf{a}}(k) + \ell_{\mathbf{a},k}$. *Let*

$$\mathbf{c} = ((\mathbf{a}^{(k)})_{\min} \setminus \mathbf{a}_0) \cup \{\Delta_1^+ \succeq \cdots \succeq \Delta_{\ell_{\mathbf{a},k}}^+ \succeq \Delta_{\ell_{\mathbf{a},k}+1} \succeq \cdots \succeq \Delta_r\}.$$

(2) If $\varphi_{\mathbf{a}}(k) - \ell_{\mathbf{a},k} < \varphi_{\mathbf{a}}(k-1) \leq \varphi_{\mathbf{a}}(k)$*, then* $r = \varphi_{\mathbf{a}}(k-1) - \varphi_{\mathbf{a}}(k) + \ell_{\mathbf{a},k}$. *Let*

$$\mathbf{c} = ((\mathbf{a}^{(k)})_{\min} \setminus \mathbf{a}_0) \cup \{\Delta_1^+ \succeq \cdots \succeq \Delta_r^+ \succ \underbrace{[k] = \cdots = [k]}_{\ell_{k,\mathbf{a}}-r}\}$$

(3) If $\varphi_{\mathbf{a}}(k-1) \leq \varphi_{\mathbf{a}}(k) - \ell_{\mathbf{a},k}$*, then* $\mathbf{a}_0 = \emptyset$ *and*

$$\mathbf{c} = \mathbf{a}^{(k)} + \ell_{\mathbf{a},k}[k].$$

Then $\mathbf{c}$ *satisfies the hypothesis* $H_k(\mathbf{c})$ *and* $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$.

Proof. We show only the case $\varphi_{\mathbf{a}}(k-1) > \varphi_{\mathbf{a}}(k)$, the proof for other cases is similar. Note that we have the following equality

$$\varphi_{\mathbf{a}}(k-1) = \varphi_{(\mathbf{a}^{(k)})_{\min}}(k-1) = r + \sharp\{\Delta \in (\mathbf{a}^{(k)})_{\min} : \Delta \supseteq [k-1,k]\}.$$

Moreover, $\varphi_{\mathbf{a}}(k-1) > \varphi_{\mathbf{a}}(k)$ implies that no segment in $(\mathbf{a}^{(k)})_{\min}$ starts at k by minimality, hence we also have

$$\varphi_{\mathbf{a}}(k) = \varphi_{(\mathbf{a}^{(k)})_{\min}}(k) + \ell_{k,\mathbf{a}} = \sharp\{\Delta \in (\mathbf{a}^{(k)})_{\min} : \Delta \supseteq [k-1,k]\} + \ell_{k,\mathbf{a}}.$$

Now comparing the two formulas gives the equality $r = \varphi_{\mathbf{a}}(k-1) - \varphi_{\mathbf{a}}(k) + \ell_{\mathbf{a},k}$. By definition we have $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$. To check that $\mathbf{c}$ satisfies the hypothesis $H_k(\mathbf{c})$, it suffices to note that $(\mathbf{a}^{(k)})_{\min} \setminus \mathbf{a}_0$ does not contain segment which ends in $k-1$. □

Lemma 5.15. *Let* $\mathbf{c} \in S(\mathbf{c})_k$ *be a multisegment such that* $\mathbf{c}^{(k)}$ *is minimal* (i.e., $S(\mathbf{c}^{(k)}) = \{\mathbf{c}^{(k)}\}$). *Then for any* $\mathbf{d} \in S(\mathbf{c})\backslash\{\mathbf{c}\}$, $\mathbf{d} \notin \tilde{S}(\mathbf{c})_k$.

Proof. Suppose that $\mathbf{d} < \mathbf{c}$ is a multisegment such that $\mathbf{d}^{(k)} = \mathbf{c}^{(k)}$. Consider the maximal chain of multisegments

$$\mathbf{c} = \mathbf{c}_0 > \cdots > \mathbf{c}_t = \mathbf{d}.$$

Our assumption implies that $\mathbf{c}_i^{(k)} = \mathbf{c}^{(k)}$ for all $i = 1, \ldots, t$ by Lemma 5.5. Hence we can assume $t = 1$ and consider $\mathbf{d} \in S(\mathbf{c})$ to be a multisegment obtained by applying the elementary operation to the pair of linked segments $\{\Delta, \Delta'\}$ with $\Delta \prec \Delta'$.

- If $e(\Delta) \neq k$, $e(\Delta') \neq k$, then the pair $\{\Delta, \Delta'\}$ also appears in $\mathbf{c}^{(k)}$, contradicting the fact that $\mathbf{c}^{(k)}$ is minimal.
- If $e(\Delta') = k$, then by the fact that $\mathbf{c} \in S(\mathbf{c})_k$, we know that $e(\Delta) < k-1$, which implies that the pair $\{\Delta, \Delta'^{-}\}$ is linked and belongs to $\mathbf{c}^{(k)}$,contradiction.
- If $e(\Delta) = k$ and $b(\Delta') < k+1$, then the pair $\{\Delta^{-}, \Delta'\}$ is still linked and belongs to $\mathbf{c}^{(k)}$, contradiction.

Hence we must have $e(\Delta) = k$ and $b(\Delta') = k+1$, this implies that $\deg(\mathbf{d}^{(k)}) > \deg(\mathbf{c}^{(k)})$ and $\mathbf{d} \notin \tilde{S}(\mathbf{c})_k$. Finally, (b) of Lemma 5.5 implies that for all $\mathbf{d} < \mathbf{c}$, we have $\mathbf{d} \notin \tilde{S}(\mathbf{c})_k$. □

Proposition 5.16. *Let $\mathbf{c} \in S(\mathbf{c})_k$ be a multisegment such that $\mathbf{c}^{(k)}$ is minimal. Then there is a unique term $L_{\mathbf{c}^{(k)}}$ of minimal degree in $\mathscr{D}^k(L_{\mathbf{c}})$ viewed as an element in $\mathcal{R}$* ($\mathscr{D}^k$ is the partial Bernstein-Zelevinsky operator)*, which appears with multiplicity one.*

Proof. Let $\mathbf{c} = \{\Delta_1, \cdots, \Delta_r\}$ such that $e(\Delta_t) = k$ if and only if $t = i, \ldots, j$ with $i \leq j$. Then

$$\begin{aligned}\mathscr{D}^k(\pi(\mathbf{c})) = \Delta_1 \times \cdots \times \Delta_{i-1} \times (\Delta_i + \Delta_i^{-}) \\ \times \cdots \times (\Delta_j + \Delta_j^{-}) \times \Delta_{j+1} \times \cdots \times \Delta_r\end{aligned}$$

with minimal degree term given by

$$\pi(\mathbf{c}^{(k)}) = \Delta_1 \times \cdots \times \Delta_{i-1} \times \Delta_i^{-} \times \cdots \times \Delta_j^{-} \times \Delta_{j+1} \times \cdots \times \Delta_r.$$

The same calculation shows that for any $\mathbf{d} \neq \mathbf{c} \in S(\mathbf{c})$, the minimal degree term in $\mathscr{D}^k(\pi(\mathbf{d}))$ is given by $\pi(\mathbf{d}^{(k)})$, whose degree is strictly greater than that of $\mathbf{c}^{(k)}$ since by Lemma 5.15 we know that $\mathbf{d} \notin \tilde{S}(\mathbf{c})_k$. Note that $\mathscr{D}^k(L_{\mathbf{d}})$ is a non-negative sum of irreducible representations (Theorem 2.22), which does not contain any representation of degree equal to that of $\mathbf{c}^{(k)}$, by comparing the minimal degree term in $\mathscr{D}^k(\pi(\mathbf{d}))$ and $\sum_{\mathbf{e} \in S(\mathbf{d})} m(\mathbf{e}, \mathbf{d}) \mathscr{D}^k(L_{\mathbf{e}})$. Finally, comparing the minimal degree term in $\mathscr{D}^k(\pi(\mathbf{c}))$ and $\sum_{\mathbf{e} \in S(\mathbf{c})} m(\mathbf{e}, \mathbf{c}) \mathscr{D}^k(L_{\mathbf{e}})$ gives the proposition. □

Proposition 5.17. *Let $\mathbf{a}$ be a multisegment. Then $S(\mathbf{a})_k$ contains a unique multisegment $\mathbf{c}$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$.*

Proof. Let $\mathbf{a} = \{\Delta_1, \cdots, \Delta_s\}$ such that $e(\Delta_t) = k$ if and only if $t = i, \ldots, j$ with $i \leq j$. Then

$$\mathscr{D}^k(\pi(\mathbf{a})) = \Delta_1 \times \cdots \times \Delta_{i-1} \times (\Delta_i + \Delta_i{}^-) \\ \times \cdots \times (\Delta_j + \Delta_j^-) \times \Delta_{j+1} \times \cdots \times \Delta_s$$

with minimal degree term given by

$$\pi(\mathbf{a}^{(k)}) = \Delta_1 \times \cdots \times \Delta_{i-1} \times \Delta_i^- \times \cdots \times \Delta_j^- \times \Delta_{j+1} \times \cdots \times \Delta_r.$$

Note that in $\pi(\mathbf{a}^{(k)})$, $m((\mathbf{a}^{(k)})_{\min}, \mathbf{a}^{(k)}) = 1$(cf. [18, Corollary 4.2]). Now compare with the terms of minimal degree in $\sum_{\mathbf{d} \in S(\mathbf{a})} m(\mathbf{d}, \mathbf{a}) \mathscr{D}^k(L_{\mathbf{d}})$ and apply the Proposition 5.16 yields the uniqueness of $\mathbf{c}$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$.

Proposition 5.18. *Let* $\mathbf{c}$ *be the multisegment constructed in Proposition 5.14. Then* $\mathbf{c} \in S(\mathbf{a})$.

Proof. Let

$$\mathbf{a}_1 = \mathbf{a}^{(k)} + \ell_{\mathbf{a},k}[k],$$

then we observe that $\mathbf{a} \in S(\mathbf{a}_1)$. Because of $\mathbf{c} \in S((\mathbf{a}^{(k)})_{\min} + \ell_{\mathbf{a},k}[k])$, we have $\mathbf{c} \in S(\mathbf{a}_1)$. Note that since $\deg((\mathbf{a}_1)^{(k)}) = \deg(\mathbf{c}^{(k)})$, the fact that $\mathbf{c} \in S(\mathbf{c})_k$ implies that $\mathbf{c} \in S(\mathbf{a}_1)_k$. Now let $\mathbf{d} \in S(\mathbf{a})_k$, then we have $\mathbf{d} \in S(\mathbf{a}_1)_k$ since $\deg(\mathbf{d}^{(k)}) = \deg(\mathbf{a}_1^{(k)}) = \deg(\mathbf{a}^{(k)})$. Assume furthermore that $\mathbf{d}^{(k)}$ is minimal, then by Proposition 5.17, we know that such a multisegment in $S(\mathbf{a}_1)_k$ is unique, which implies $\mathbf{d} = \mathbf{c}$. □

Corollary 5.19. *Let* $\mathbf{c} \in S(\mathbf{a})_k$ *such that* $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$, *then* $\mathbf{c}$ *is minimal in* $\tilde{S}(\mathbf{a})_k$.

Proof. By Proposition 5.10, we know that for any $\mathbf{d} \in \tilde{S}(\mathbf{a})_k$, there exists a multisegment $\mathbf{c}' \in S(\mathbf{a})_k$ with $\mathbf{c}'^{(k)} = (\mathbf{a}^{(k)})_{\min}$, such that $\mathbf{d} > \mathbf{c}'$. By uniqueness, we must have $\mathbf{c} = \mathbf{c}'$. □

5.3. Geometry of graded nilpotent classes: General Cases

In this section, we show geometrically that the morphism

$$\psi_k : S(\mathbf{a})_k \to S(\mathbf{a}^{(k)}) \\ \mathbf{c} \mapsto \mathbf{c}^{(k)}$$

is bijective, satisfying the properties

(1) For $\mathbf{c} \in S(\mathbf{a})_k$, we have $P_{\mathbf{c},\mathbf{a}}(q) = P_{\mathbf{c}^{(k)},\mathbf{a}^{(k)}}(q)$.
(2) The morphism ψ_k preserves the order, i.e, for $\mathbf{c}, \mathbf{d} \in S(\mathbf{a})_k$, $\mathbf{c} > \mathbf{d}$ if and only if $\mathbf{c}^{(k)} > \mathbf{d}^{(k)}$.

To achieve this, firstly we consider the sub-variety $X_{\mathbf{a}}^k = \coprod_{\mathbf{c}\in\tilde{S}(\mathbf{a})_k} O_{\mathbf{c}}$, and construct a fibration α from $X_{\mathbf{a}}^k$ to $Gr(\ell_{\mathbf{a},k}, V_{\varphi_{\mathbf{a}},k})$, the latter is the space of the $\ell_{\mathbf{a},k}$-dimensional subspaces of $V_{\varphi_{\mathbf{a}},k}$. Secondly, we construct an open immersion

$$\tau_W : (X_{\mathbf{a}}^k)_W \to Y_{\mathbf{a}^{(k)}} \times \mathrm{Hom}(V_{\varphi_{\mathbf{a}},k-1}, W),$$

where $(X_{\mathbf{a}}^k)_W$ is the fiber over W with respect to α and $Y_{\mathbf{a}^{(k)}} = \coprod_{\mathbf{c}\in S(\mathbf{a}^{(k)})} O_{\mathbf{c}}$.

Here we fix a multisegment $\mathbf{a}$ and let $\varphi = \varphi_{\mathbf{a}}$.

Definition 5.20.
- Let

$$X_{\mathbf{a}}^k = \coprod_{\mathbf{c}\in\tilde{S}(\mathbf{a})_k} O_{\mathbf{c}},$$

- Let $Y_{\mathbf{a}^{(k)}} = \coprod_{c\in S(\mathbf{a}^{(k)})} O_{\mathbf{c}}$.
- For $\mathbf{b} > \mathbf{c}$ in $\tilde{S}(\mathbf{a})_k$, we define

$$X_{\mathbf{b},\mathbf{c}}^k = \coprod_{\mathbf{b}\geq\mathbf{d}\geq\mathbf{c}} O_{\mathbf{d}}.$$

Let $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$, $T \in O_{\mathbf{c}}$, then

Lemma 5.21. *We have* $\dim(\ker(T|_{V_{\varphi,k}})) = \sharp\{\Delta \in \mathbf{a} : e(\Delta) = k\} = \ell_{\mathbf{a},k}$ *(Notation 5.12), which does not depend on the choice of* T.

Proof. The fact $T \in O_{\mathbf{c}}$ implies

$$\dim(\ker(T|_{V_{\varphi,k}})) = \sharp\{\Delta \in \mathbf{c} : e(\Delta) = k\}.$$

Then our lemma follows from the remark after Definition 5.4. □

Definition 5.22. Let

$$Gr(\ell_{\mathbf{a},k}, V_\varphi) = \{W \leq V_{\varphi,k} : \dim(W) = \ell_{\mathbf{a},k}\},$$

and for $W \in Gr(\ell_{\mathbf{a},k}, V_\varphi)$, let

$$V_\varphi/W = V_{\varphi,1} \oplus \cdots \oplus V_{\varphi,k-1} \oplus V_{\varphi,k}/W \oplus \cdots .$$

Also, we denote by

$$p_W : V_\varphi \to V_\varphi/W$$

the canonical projection.

Definition 5.23. We define

$$\tilde{Z}^k = \{(W, T) : W \in Gr(\ell_{\mathbf{a},k}, V_\varphi), T \in End(V_\varphi/W) \text{ of degree } +1\},$$

and the canonical projection

$$\begin{aligned} \pi : \tilde{Z}^k &\to Gr(\ell_{\mathbf{a},k}, V_\varphi) \\ (W, T) &\mapsto W. \end{aligned}$$

Proposition 5.24. *The morphism π is a fibration with fiber*

$$E_{\varphi_{\mathbf{a}^{(k)}}}.$$

Proof. This follows from the definition. $\square$

Definition 5.25. Assume $\mathbf{b}, \mathbf{c} \in S(\mathbf{a}^{(k)})$.

- Let

$$Z^{k,\mathbf{a}} = \{(W, T) \in \tilde{Z}^k : T \in Y_{\mathbf{a}^{(k)}}\}.$$

- Let

$$Z^{k,\mathbf{a}}_{\mathbf{b},\mathbf{c}} = \{(W, T) \in \tilde{Z}^k : T \in \coprod_{\mathbf{b} \geq \mathbf{d} \geq \mathbf{c}} O_{\mathbf{d}}\}, \ Z^{k,\mathbf{a}}_{\mathbf{b}} = \{(W, T) \in \tilde{Z}^k : T \in \coprod_{\mathbf{d} \geq \mathbf{b}} O_{\mathbf{d}}\}.$$

- Let

$$Z^{k,\mathbf{a}}(\mathbf{c}) = \{(W, T) \in Z^{k,\mathbf{a}}, T \in O_{\mathbf{c}}\}.$$

Remark. The restriction of π to $Z^{k,\mathbf{a}}$ is a fibration with fiber $Y_{\mathbf{a}^{(k)}}$.

Definition 5.26. Let $T \in X^k_{\mathbf{a}}$. We define $T^{(k)} \in End(V/\ker(T|_{V_{\varphi,k}}))$ such that

$$T^{(k)}|_{V_{\varphi,i}} = \begin{cases} T|_{V_{\varphi,i}}, \text{ for } i \neq k, k-1, \\ p_{T,k} \circ T|_{V_{\varphi,i}}, \text{ for } i = k-1 \\ T|_{V_{\varphi,i}} \circ p_{T,k}, \text{ for } i = k. \end{cases}$$

where $p_{T,k} : V_\varphi \to V_\varphi/\ker(T|_{V_{\varphi,k}})$ is the canonical projection.

This gives naturally an element $(T^{(k)}, \ker(T|_{V_{\varphi,k}}))$ in $Z^{k,\mathbf{a}}$. We construct a morphism

$$\gamma_k : X^k_{\mathbf{a}} \to Z^{k,\mathbf{a}}.$$

by

$$\gamma_k(T) = (T^{(k)}, \ker(T|_{V_{\varphi,k}})).$$

Definition 5.27. We define

$$\alpha : X^k_{\mathbf{a}} \to Gr(\ell_{\mathbf{a},k}, V_\varphi),$$

with $\alpha(T) = \ker(T|_{V_{\varphi,k}})$.

Remark. We have a commutative diagram

$$\begin{array}{ccc} X^k_{\mathbf{a}} & \xrightarrow{\gamma_k} & Z^{k,\mathbf{a}} \\ \downarrow{\scriptstyle \alpha} & \swarrow{\scriptstyle \pi} & \\ Gr(\ell_{\mathbf{a},k}, V_\varphi). & & \end{array}$$

where γ_k maps fibers to fibers.

Proposition 5.28. *The morphism* α *is a fiber bundle such that* $\alpha|_{O_{\mathbf{c}}}$ *is surjective for any* $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$.

Proof. We have to show that α is locally trivial. We fix $W \in Gr(\ell_{\mathbf{a},k}, V_\varphi)$ Note that $GL_{\varphi(k)}$ acts transitively on $Gr(\ell_{\mathbf{a},k}, V_\varphi)$. Let P_W be the stabilizer of W. Then by Serre [16, Proposition 3], we know that the principle bundle

$$GL_{\varphi(k)} \to GL_{\varphi(k)}/P_W$$

is étale-locally trivial. Here the base $GL_{\varphi(k)}/P_W$ is isomorphic to $Gr(\ell_{\mathbf{a},k}, V_\varphi)$. It is even Zariski-locally trivial because P_W is parabolic, which is special in the sense of Serre [16], 4. Now we can write

$$\begin{array}{ccc} X_{\mathbf{a}}^k & \xleftarrow{\ \delta\ } & GL_{\varphi(k)} \times_{P_W} \alpha^{-1}(W) \\ \downarrow & \swarrow & \\ Gr(\ell_{\mathbf{a},k}, V_\varphi) & & \end{array}$$

where

$$\delta([g, T]) = g.T.$$

We claim that δ is an isomorphism. In fact, for any $T \in X_{\mathbf{a}}^k$, we choose $g \in GL_{\varphi(k)}$ such that

$$g(\ker(T|_{V_{\varphi,k}})) = W.$$

This implies $g.T \in \alpha^{-1}(W)$, thus

$$\delta([g^{-1}, g.T]) = T.$$

This shows the surjectivity. By the definition of P_W, we have,

$$\delta([g, T]) = g.T \in \alpha^{-1}(W)$$

implies $g \in P_W$, which implies the injectivity. Since $GL_{\varphi(k)}$ is locally trivial over $Gr(\ell_{\mathbf{a},k}, V_\varphi)$, we obtain that

$$GL_{\varphi(k)} \times_{P_W} \alpha^{-1}(W) \to Gr(\ell_{\mathbf{a},k}, V_\varphi)$$

is locally trivial, which implies that α is locally trivial.

Finally, we want to show the surjectivity of the orbit $\alpha|_{O_{\mathbf{c}}}$. This is a consequence the fact that $GL_{\varphi(k)}$ acts transitively on $Gr(\ell_{\mathbf{a},k}, V_\varphi)$. □

Proposition 5.29. *Let* $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$. *The restriction map*

$$\gamma_k : O_{\mathbf{c}} \to Z^{k,\mathbf{a}}(\mathbf{c}^{(k)})$$

is surjective.

Proof. Let $(T_0, W) \in Z^{k,\mathbf{a}}(\mathbf{c}^{(k)})$. Consider

$$m = \sharp\{\Delta \in \mathbf{c} : e(\Delta) = k, \deg(\Delta) \geq 2\} \leq \min\{\ell_{\mathbf{a},k}, \dim(\ker(T_0|_{V_{\varphi,k-1}}))\}.$$

We choose a splitting $V_{\varphi,k} = W \oplus V_{\varphi,k}/W$ and let $T' : V_{\varphi,k-1} \to W$ be a linear morphism of rank m. Finally, we define $T \in \gamma_k^{-1}((T_0, W))$ by letting

$$\begin{aligned} T|_{V_{\varphi,k-1}} &= T' \oplus T_0|_{V_{\varphi,k-1}}, \\ T|_{V_{\varphi,k}} &= T_0|_{V_{\varphi,k}/W} \circ p_W, \\ T|_{V_{\varphi,i}} &= T|_{V_{\varphi,i}}, \text{ for } i \neq k-1, k. \end{aligned}$$

Let

$$\{\Delta \in \mathbf{c} : e(\Delta) = k, \deg(\Delta) \geq 2\} = \{\Delta_1, \ldots, \Delta_m\}, \quad b(\Delta_1) \leq \cdots \leq b(\Delta_m).$$

We denote $W_i = T_0^{[b(\Delta_i),k-1]}(V_{\varphi,b(\Delta_i)}) \cap \ker(T_0|_{V_{\varphi,k-1}})$, where $T^{[i,j]}$ is the composition map:

$$V_i \xrightarrow{T} V_{i+1} \longrightarrow \cdots \xrightarrow{T} V_j.$$

then

$$W_1 \subseteq \cdots \subseteq W_r \subseteq \ker(T_0|_{V_{\varphi,k-1}}).$$

Then we have $T \in O_{\mathbf{c}}$ if and only if

$$\dim(T'(W_i)) - \dim(T(W_{i-1})) = \dim(W_i/W_{i-1}), \quad i = 1, \ldots, m.$$

Since such T' always exists, we are done. □

Notation 5.30. *We fix $W \in Gr(\ell_{\mathbf{a},k}, V_\varphi)$, and denote*

$$(X_{\mathbf{a}}^k)_W, \quad (Z^{k,\mathbf{a}})_W$$

the fibers over W.

Proposition 5.31. *The fiber $(X_{\mathbf{a}}^k)_W$ is normal and irreducible as an algebraic variety over $\mathbb{C}$.*

Proof. Note that since $\tilde{S}(\mathbf{a})_k$ contains a unique minimal element $\mathbf{c}$, the variety $X_{\mathbf{a}}^k$ is contained and is open in the irreducible variety $\overline{O_{\mathbf{c}}}$. Now by [19, Theorem 1], we know that $X_{\mathbf{a}}^k$ is actually normal.

By Proposition 5.28, we know that α is a fibration between two varieties $X_{\mathbf{a}}^k$ and $Gr(\ell_{\mathbf{a},k}, V_\varphi)$. The fact that both are normal and irreducible implies that the fiber $(X_{\mathbf{a}}^k)_W$ is normal and irreducible. □

Remark. Note that by definition, we are allowed to identify $(Z^{k,\mathbf{a}})_W$ with $Y_{\mathbf{a}^{(k)}}$. This is what we do from now on.

Definition 5.32. We choose a splitting $V_{\varphi,k} = W \oplus V_{\varphi,k}/W$ and denote by $q_W : V_{\varphi,k} \to W$ the projection. We define a morphism τ_W

$$\tau_W(T) = ((\gamma_k)_W(T), q_W \circ T|_{V_{\varphi,k-1}}).$$

Remark. Then we have the following commutative diagram

$$\begin{array}{ccc} (X^k_{\mathbf{a}})_W & \xrightarrow{\tau_W} & (Z^{k,\mathbf{a}})_W \times \mathrm{Hom}(V_{\varphi,k-1}, W) \\ \downarrow{\scriptstyle (\gamma_k)_W} & \swarrow{\scriptstyle s} & \\ (Z^{k,\mathbf{a}})_W & & \end{array}$$

where s is the canonical projection.

Lemma 5.33. *The morphism τ_W is injective.*

Proof. Note that any $T \in (X^k_{\mathbf{a}})_W$ is determined by $(\gamma_k)_W(T)$ and $T|_{V_{\varphi,k-1}}$. Furthermore, $T|_{V_{\varphi,k-1}}$ is determined by $p_W \circ T|_{V_{\varphi,k-1}}$ and $q_W \circ T|_{V_{\varphi,k-1}}$. Since $p_W \circ T|_{V_{\varphi,k-1}}$ is a component of $(\gamma_k)_W(T)$, it is determined by $(\gamma_k)_W(T)$ and $q_W \circ T|_{V_{\varphi,k-1}}$. This gives us the injectivity. □

Lemma 5.34. *Let $\mathbf{c} \in S(\mathbf{a})_k$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$. Then the image of $O_{\mathbf{c}} \cap (X^k_{\mathbf{a}})_W$ is open in $O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)$.*

Proof. Let $\mathbf{c} \in S(\mathbf{a})_k$ such that $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$. Let $T \in (O_{\mathbf{c}})_W$. We use a case by case consideration as in Proposition 5.14:

(1) If $\varphi(k-1) \leq \varphi(k) - \ell_{\mathbf{a},k}$, the fact $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$ implies that $T^{(k)}|_{V_{\varphi,k-1}}$ is injective. As a consequence we have $\mathrm{Im}(T|_{V_{\varphi,k-1}}) \cap W = 0$. Hence for any element $T_0 \in \mathrm{Hom}(V_{\varphi,k-1}, W)$, we define $T_0 \in O_{\mathbf{c}}$, such that

$$T_0|_{V_{\varphi,k-1}} = T_0 \oplus T^{(k)}|_{V_{\varphi,k-1}},$$

which lies in the fiber over $(\gamma_k)_W^{-1}((T^{(k)}, W))$. Since by Proposition 5.29, every element in $O_{\mathbf{c}^{(k)}}$ comes from some element in $O_{\mathbf{c}}$, hence

$$\tau_W(O_{\mathbf{c}} \cap (X^k_{\mathbf{a}})_W) = O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W),$$

which is open.

(2) If $\varphi(k) - \ell_{\mathbf{a},k} < \varphi(k-1) < \varphi(k)$, the fact $\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$ implies that the morphism

$$T^{(k)}|_{V_{\varphi,k-1}}$$

has a kernel of dimension

$$\varphi(k-1) - \varphi(k) + \ell_{\mathbf{a},k}.$$

Our description of $\mathbf{c}$ in Proposition 5.14 (2) shows that in this case

$$\dim(\mathrm{Im}(T|_{V_{\varphi,k-1}}) \cap W) = \varphi(k-1) - \varphi(k) + \ell_{\mathbf{a},k}.$$

In this situation, given an element $T_0 \in \operatorname{Hom}(V_{\varphi,k-1}, W)$ we define $T' \in E_\varphi$, such that

$$T'|_{V_{\varphi,k-1}} = T_0 \oplus T^{(k)}|_{V_{\varphi,k-1}},$$
$$T'|_{V_{\varphi,k}} = T^{(k)}|_{V_{\varphi,k}/W} \circ p_W,$$
$$T'|_{V_{\varphi,i}} = T^{(k)}, \text{ for } i \neq k-1, k.$$

By construction and Proposition 3.3, we know that $T' \in O_{\mathbf{c}}$ if and only if $T'|_{V_{\varphi,k-1}}$ is injective, since no segment in $\mathbf{c}$ ends in $k-1$, as described in Proposition 5.14. This is equivalent to say

$$T_0|_{\ker(T^{(k)}|_{V_{\varphi,k-1}})}$$

is injective. This is an open condition, hence $O_{\mathbf{c}} \cap (X^k_{\mathbf{a}})_W$ is open in $O_{\mathbf{c}^{(k)}} \times \operatorname{Hom}(V_{\varphi,k-1}, W)$.

(3) If $\varphi(k-1) \geq \varphi(k)$, then by Proposition 5.14

$$\mathbf{c}^{(k)} = (\mathbf{a}^{(k)})_{\min}$$

implies

$$\operatorname{Im}(T|_{V_{\varphi,k-1}}) \supseteq W.$$

Recall the notation from Proposition 5.14

$$\mathbf{a}_0 = \{\Delta_1 \succeq \cdots \succeq \Delta_r\}.$$

with $r = \varphi(k-1) - \varphi(k) + \ell_{\mathbf{a},k}$. Then

$$\mathbf{c} = ((\mathbf{a}^{(k)})_{\min} \setminus \mathbf{a}_0) \cup \{\Delta_1^+ \succeq \cdots \succeq \Delta^+_{\ell_{\mathbf{a},k}} \succeq \Delta_{\ell_{\mathbf{a},k}+1} \succeq \cdots \succeq \Delta_r\}.$$

Let $T_0 \in \operatorname{Hom}(V_{\varphi,k-1}, W)$, we define $T' \in E_\varphi$ as in the case (2) Consider the following flag over $V_{\varphi,k-1}$,

$$\ker(T^{(k)}|_{\varphi,k-1}) = V_r \supseteq \cdots \supseteq V_1 \supseteq V_0 = 0,$$

where $V_i = \operatorname{Im}((T^{(k)})^{\Delta_i}) \cap \ker(T^{(k)}|_{\varphi,k-1})$, with $i = 1, \ldots, r$. Now by Proposition 3.3, we know that $T' \in O_{\mathbf{c}}$ if and only if

$$\dim(T_0(V_i)) - \dim(T_0(V_{i-1})) = \dim(V_i/V_{i-1}),$$

for $i = 1, \ldots, \ell_{\mathbf{a},k}$. In fact, if $V_i \neq V_{i-1}$, then

$$\dim(V_i/V_{i-1}) = \sharp\{j : \Delta_j = \Delta_i\}.$$

By construction, if $i \leq \ell_{\mathbf{a},k}$, by Proposition 3.3, the fact that $\mathbf{c}$ contains Δ_i^+ implies that if $T' \in O_{\mathbf{c}}$,

$$\dim(T_0(V_i)) - \dim(T_0(V_{i-1})) = \dim(V_i/V_{i-1}).$$

The converse holds by the same reason. Again, this is an open condition, which proves that $O_{\mathbf{c}} \cap (X^k_{\mathbf{a}})_W$ is open in $O_{\mathbf{c}^{(k)}} \times \operatorname{Hom}(V_{\varphi,k-1}, W)$. □

Proposition 5.35. *The morphism τ_W is an open immersion.*

Proof. To see that it is open immersion, we shall use Zariski's main theorem. Since all Schubert varieties are normal, we observe that

$$(Z^{k,\mathbf{a}})_W \times \mathrm{Hom}(V_{\varphi,k-1}, W)$$

are normal by theorem 1 of [19]. Also, by Proposition 5.31, we know that $(X^k_{\mathbf{a}})_W$ is irreducible and normal, hence τ_W is an open immersion. □

Proposition 5.36. *Let $\mathbf{c} \in \tilde{S}(\mathbf{a})_k$. Then $\mathbf{c} \in S(\mathbf{a})_k$ if and only if*

$$O_{\mathbf{c}} \cap (X^k_{\mathbf{a}})_W$$

is open in

$$(O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)).$$

Proof. We already showed in Lemma 5.34 that

$$O_{\mathbf{c}} \cap (X^k_{\mathbf{a}})_W$$

is a sub-variety of

$$O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W).$$

Moreover, since τ_W is open by Proposition 5.35, we have

$$(O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)) \cap (X^k_{\mathbf{a}})_W$$

is open in

$$O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W).$$

Finally, by Proposition 5.29,

$$\begin{aligned}(O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)) \cap (X^k_{\mathbf{a}})_W \\ = \coprod_{\mathbf{d}\in\tilde{S}(\mathbf{a})_k, \mathbf{d}^{(k)}=\mathbf{c}^{(k)}} O_{\mathbf{d}} \cap (X^k_{\mathbf{a}})_W.\end{aligned}$$

The variety $(O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W)) \cap (X^k_{\mathbf{a}})_W$ is irreducible because $(O_{\mathbf{c}^{(k)}} \times \mathrm{Hom}(V_{\varphi,k-1}, W))$ is irreducible, hence the stratification

$$\coprod_{\mathbf{d}\in\tilde{S}(\mathbf{a})_k, \mathbf{d}^{(k)}=\mathbf{c}^{(k)}} O_{\mathbf{d}} \cap (X^k_{\mathbf{a}})_W$$

by locally closed sub-varieties can only contain one term which is open, from the point of view of Zariski topology. Since for any element

$$\mathbf{d}' \in \{\mathbf{d} \in \tilde{S}(\mathbf{a})_k, \mathbf{d}^{(k)} = \mathbf{c}^{(k)}\},$$

by (d) of Lemma 5.5, we know that there exists $\mathbf{c}' \in S(\mathbf{a})_k$ such that $\mathbf{d}' > \mathbf{c}'$. Hence we conclude that

$$\{\mathbf{d} \in \tilde{S}(\mathbf{a})_k, \mathbf{d}^{(k)} = \mathbf{c}^{(k)}\},$$

contains a unique minimal element, which lies in $S(\mathbf{a})_k$. Now our proposition follows. □

Corollary 5.37. *Let* $\mathbf{a}$ *be a multisegment and*

$$\mathbf{c} \in S(\mathbf{a})_k,$$

then

$$P_{\mathbf{a},\mathbf{c}}(q) = P_{\mathbf{a}^{(k)},\mathbf{c}^{(k)}}(q).$$

Proof. First of all, by Proposition 5.28 and Künneth formula, we know that

$$\mathcal{H}^j(\overline{O}_{\mathbf{c}})_{\mathbf{a}} = \mathcal{H}^j(\overline{O}_{\mathbf{c}} \cap (X_{\mathbf{a}}^k)_W)_{\mathbf{a}},$$

the localization being taken at a point in $O_{\mathbf{a}} \cap (X_{\mathbf{a}}^k)_W$. Since $O_{\mathbf{c}}$ is open in $\overline{O}_{\mathbf{c}}$, by Proposition 5.35 and Proposition 5.36, we can regard $\overline{O}_{\mathbf{c}} \cap (X_{\mathbf{a}}^k)_W$ as an open sub-variety of $\overline{O}_{\mathbf{c}^k} \times Hom(V_{\varphi,k-1}, W)$, hence

$$\mathcal{H}^j(\overline{O}_{\mathbf{c}} \cap (X_{\mathbf{a}}^{(k)})_W)_{\mathbf{a}} = \mathcal{H}^j(\overline{O}_{\mathbf{c}^{(k)}} \times Hom(V_{\varphi,k-1}, W))_{\mathbf{a}^{(k)}}$$

and Künneth formula implies that the latter is equal to

$$\mathcal{H}^j(\overline{O}_{\mathbf{c}^{(k)}})_{\mathbf{a}^{(k)}}.$$

□

Corollary 5.38. *Let* $\mathbf{d} \in S(\mathbf{a})$ *such that*

$$\mathbf{d}^{(k)} = \mathbf{a}^{(k)},$$

and

$$\mathbf{c} \in S(\mathbf{a})_k,$$

then $\mathbf{c} < \mathbf{d}$, *and*

$$P_{\mathbf{a},\mathbf{c}}(q) = P_{\mathbf{d},\mathbf{c}}(q).$$

Proof. By Proposition 5.10, we know that there exists $\mathbf{c}' \in S(\mathbf{a})_k$ such that

$$\mathbf{d} > \mathbf{c}',\ \mathbf{c}'^{(k)} = \mathbf{c}^{(k)}.$$

Proposition 5.36 implies $\mathbf{c}' = \mathbf{c}$ (note that the open stratum is unique). Finally, applying the Corollary 5.37 to the pairs $\{\mathbf{a}, \mathbf{c}\}$ and $\{\mathbf{d}, \mathbf{c}\}$ yields the result. □

5.4. Some consequeces and remarks

In this section, we draw some conclusions from what we have done before, espectially the properties related to ψ_k.

Proposition 5.39. *The map*

$$\psi_k : S(\mathbf{a})_k \to S(\mathbf{a}^{(k)})$$
$$\mathbf{c} \mapsto \mathbf{c}^{(k)}$$

is bijective. Moreover,

- *for* $\mathbf{c} \in S(\mathbf{a})_k$

$$m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{(k)}, \mathbf{a}^{(k)}).$$

- *for* $\mathbf{b}, \mathbf{c} \in S(\mathbf{a})_k$*, we have* $\mathbf{b} > \mathbf{c}$ *if and only if* $\mathbf{b}^{(k)} > c^{(k)}$.

Proof. By Proposition 5.36, we know that ψ_k is injective. Surjectivity is given by Proposition 5.10.

For $\mathbf{c} \in S(\mathbf{a})_k$,

$$m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{(k)}, \mathbf{a}^{(k)})$$

is by Corollary 5.37 by putting $q = 1$, and applying Theorem 3.6.

Finally, for $\mathbf{b}, \mathbf{c} \in S(\mathbf{a})_k$, if $\mathbf{b} > \mathbf{c}$, then $\mathbf{c} \in S(\mathbf{b})_k$, and by Lemma 5.5 (3), we know that $\mathbf{b}^{(k)} > \mathbf{c}^{(k)}$. Reciprocally, if $\mathbf{b}^{(k)} > \mathbf{c}^{(k)}$, by Proposition 5.36, we know that $\overline{O}_{\mathbf{b}} \subseteq \overline{O}_{\mathbf{c}}$, hence $\mathbf{b} > \mathbf{c}$. □

Corollary 5.40. *We have*

- $$\pi(\mathbf{a}^{(k)}) = \sum_{\mathbf{c} \in S(\mathbf{a})_k} m(\mathbf{c}, \mathbf{a}) L_{\mathbf{c}^{(k)}}, \tag{3}$$

- *let* $\mathbf{b} \in S(\mathbf{a})$ *such that* $\mathbf{b}$ *satisfies the hypothesis* $H_k(\mathbf{a})$ *and* $\mathbf{b}^{(k)} = \mathbf{a}^{(k)}$*, then*

$$m(\mathbf{b}, \mathbf{a}) = 1, \; S(\mathbf{a})_k = S(\mathbf{b})_k.$$

Proof. The first part follows from the fact that ψ_k is bijective and $m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{(k)}, \mathbf{a}^{(k)})$. For the second part of the corollary, we note that $L_{\mathbf{b}^{(k)}} = L_{\mathbf{a}^{(k)}}$ appears with multiplicity one in $\pi(\mathbf{a}^{(k)})$, then equation (3) implies $m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}^{(k)}, \mathbf{a}^{(k)}) = 1$. To see that $S(\mathbf{a})_k = S(\mathbf{b})_k \subseteq S(\mathbf{b})$, note that we have $S(\mathbf{b})_k \subseteq S(\mathbf{a})_k$ together with two bijections

$$\psi_k : S(\mathbf{a})_k \to S(\mathbf{a}^{(k)}),$$
$$\psi_k : S(\mathbf{b})_k \to S(\mathbf{b}^{(k)}) = S(\mathbf{a}^{(k)}).$$

Hence comparing the cardinality gives $S(\mathbf{a})_k = S(\mathbf{b})_k$. □

Before we finish this section, let us remark that we can also work out a left handed version of everything we have done in this sections. Let us briefly introduce the main definitions, which will be of use in later sections.

Analogous to Definition 5.1, Definition 5.2 and Definition 5.3,

Definition 5.41. For $\Delta = [i, j]$ a segment, we put

$$^{-}\Delta = [i + 1, j], \quad ^{+}\Delta = [i - 1, j].$$

Definition 5.42. Let $k \in \mathbb{Z}$ and Δ be a segment, we define

$$^{(k)}\Delta = \begin{cases} ^{-}\Delta, & \text{if } b(\Delta) = k; \\ \Delta, & \text{otherwise}. \end{cases}$$

For a multisegment $\mathbf{a} = \{\Delta_1, \ldots, \Delta_r\}$, define

$$^{(k)}\mathbf{a} = \{^{(k)}\Delta_1, \ldots, {}^{(k)}\Delta_r, \}.$$

Analogously, the Definition 5.3 has its left version.

Definition 5.43. We say that the multisegment $\mathbf{b} \in S(\mathbf{a})$ satisfies **the hypothesis** $_{k}H(\mathbf{a})$ if the following two conditions are verified

(1) $\deg(^{(k)}\mathbf{b}) = \deg(^{(k)}\mathbf{a})$;
(2) there exists no pair of linked segments $\{\Delta, \Delta'\}$ in $\mathbf{b}$ such that

$$b(\Delta) = k, \ b(\Delta') = k + 1.$$

Definition 5.44. Let

$$_{k}\tilde{S}(\mathbf{a}) = \{\mathbf{c} \in S(\mathbf{a}) : \deg(^{(k)}\mathbf{c}) = \deg(^{(k)}\mathbf{a})\}.$$

Definition 5.45. We define a morphism

$$_{k}\psi : {}_{k}\tilde{S}(\mathbf{a}) \to S(^{(k)}\mathbf{a})$$

by sending $\mathbf{c}$ to $^{(k)}\mathbf{c}$.

Now all we have done in this section can be restated and proved for the left handed objects above. Since the statements and their proofs are similar, we omit the details.

6. Reduction to symmetric case

6.1. Minimal degree terms

The goal of this section is to define the set $S(\mathbf{a})_{\mathbf{d}} \subseteq S(\mathbf{a})$ and describe some of its properties.

Definition 6.1. Let $(k_1, \ldots, k_r)$ be a sequence of integers. We define

$$\mathbf{a}^{(k_1, \ldots, k_r)} = (((\mathbf{a}^{(k_1)}) \cdots)^{(k_r)}).$$

Definition 6.2. Let $\Delta = [k, \ell]$, we denote

$$\mathbf{a}^{(\Delta)} = \mathbf{a}^{(k,\ldots,\ell)}.$$

More generally, for $\mathbf{d} = \{\Delta_1 \preceq \cdots \preceq \Delta_r\}$, let

$$\mathbf{a}^{(\mathbf{d})} = (\cdots((\mathbf{a}^{(\Delta_r)})^{(\Delta_{r-1})})\cdots)^{(\Delta_1)}.$$

Definition 6.3. Let $(k_1, \ldots, k_r)$ be a sequence of integers , then we define

$$S(\mathbf{a})_{k_1,\ldots,k_r} = \{\mathbf{c} \in S(\mathbf{a}) : \mathbf{c}^{(k_1,\ldots,k_{i-1})} \in S(\mathbf{a}^{(k_1,\ldots,k_{i-1})})_{k_i}, \text{ for } i = 1, \ldots, r\},$$

with the convention

$$k_0 = -\infty, \quad a^{(-\infty)} = a, \quad c^{(-\infty)} = c$$

and

$$\psi_{k_1,\ldots,k_r} : S(\mathbf{a})_{k_1,\ldots,k_r} \to S(\mathbf{a}^{(k_1,\ldots,k_r)}),$$

sending $\mathbf{c}$ to $\mathbf{c}^{(k_1,\ldots,k_r)}$.

Definition 6.4. Let $\mathbf{d} = \{\Delta_1 \preceq \cdots \preceq \Delta_r\}$ such that $\Delta_i = [k_i, \ell_i]$. We denote

$$S(\mathbf{a})_{\mathbf{d}} := S(\mathbf{a})_{k_r,\ldots,\ell_r,k_{r-1},\ldots,k_1,\ldots,\ell_1}$$

and

$$\psi_{\mathbf{d}} := \psi_{k_r,\ldots,\ell_r,k_{r-1},\ldots,k_1,\ldots,\ell_1}.$$

Proposition 6.5. *Let $(k_1, \ldots, k_r)$ be a sequence of integers. Then the morphism*

$$\psi_{k_1,\ldots,k_r} : S(\mathbf{a})_{k_1,\ldots,k_r} \to S(\mathbf{a}^{(k_1,\ldots,k_r)}).$$

is a bijection. Moreover,

(1) For $\mathbf{c} \in S(\mathbf{a})_{k_1,\ldots,k_r}$, we have

$$P_{\mathbf{a},\mathbf{c}}(q) = P_{\mathbf{a}^{(k_1,\ldots,k_r)},\mathbf{c}^{(k_1,\ldots,k_r)}}(q).$$

(2) For $\mathbf{b}, \mathbf{c} \in S(\mathbf{a})_{k_1,\ldots,k_r}$, then $\mathbf{b} > \mathbf{c}$ if and only if $\mathbf{b}^{(k_1,\ldots,k_r)} > \mathbf{c}^{(k_1,\ldots,k_r)}$.
(3) We have

$$\pi(\mathbf{a}^{(k_1,\ldots,k_r)}) = \sum_{\mathbf{c} \in S(\mathbf{a})_{k_1,\ldots,k_r}} m(\mathbf{c}, \mathbf{a}) L_{\mathbf{c}^{(k_1,\ldots,k_r)}}.$$

(4) For $\mathbf{b} \in S(\mathbf{a})_{k_1,\ldots,k_r}$ such that $\mathbf{b}^{(k_1,\ldots,k_r)} = \mathbf{a}^{(k_1,\ldots,k_r)}$, we have

$$S(\mathbf{a})_{k_1,\ldots,k_r} = S(\mathbf{b})_{k_1,\ldots,k_r}.$$

Proof. Injectivity follows from the fact

$$\psi_{k_1,\ldots,k_r} = \psi_{k_r} \circ \psi_{k_{r-1}} \circ \cdots \circ \psi_{k_1}.$$

For surjectivity, let $\mathbf{d} \in S(\mathbf{a}^{(k_1,\ldots,k_r)})$, we construct $\mathbf{b} \in S(\mathbf{a})_{k_1,\cdots,k_r}$ inductively such that $\psi_{k_1,\ldots,k_r}(\mathbf{b}) = \mathbf{d}$. Let $\mathbf{a}_r = \mathbf{d}$, assume that we already construct $\mathbf{a}_i \in S(\mathbf{a}^{(k_1,\ldots,k_i)})_{k_{i+1}}$ satisfying that

$$\mathbf{a}_i^{(k_{i+1},\ldots,k_j)} \in S(\mathbf{a}^{(k_1,\ldots,k_j)})_{k_{j+1}}$$

for all $i < j \le r$ and $\mathbf{a}_i^{(k_{i+1},\ldots,k_r)} = \mathbf{d}$.

Note that by the bijectivity of the morphism

$$\psi_{k_i} : S(\mathbf{a}^{(k_1,\ldots,k_{i-1})})_{k_i} \to S(\mathbf{a}^{(k_1,\ldots,k_i)}),$$

there exists a unique $\mathbf{a}_{i-1} \in S(\mathbf{a}^{(k_1,\ldots,k_{i-1})})_{k_i}$, such that

$$\mathbf{a}_{i-1}^{(k_i)} = \mathbf{a}_i.$$

Finally, take $\mathbf{b} = \mathbf{a}_0 \in S(\mathbf{a})_{k_1,\ldots,k_r}$. We show (1) by induction on r. The case for $r = 1$ is by Corollary 5.37. For general r, by induction we have

$$P_{\mathbf{a},\mathbf{c}}(q) = P_{\mathbf{a}^{(k_1,\ldots,k_{r-1})},\mathbf{c}^{(k_1,\ldots,k_{r-1})}}(q).$$

By definition $\mathbf{c}^{(k_1,\ldots,k_{r-1})} \in S(\mathbf{a}^{(k_1,\ldots,k_{r-1})})_{k_r}$, apply the case $r = 1$ to the pair $\{\mathbf{c}^{(k_1,\ldots,k_{r-1})}, \mathbf{a}^{(k_1,\ldots,k_{r-1})}\}$ to obtain

$$P_{\mathbf{a}^{(k_1,\ldots,k_{r-1})},\mathbf{c}^{(k_1,\ldots,k_{r-1})}}(q) = P_{\mathbf{a}^{(k_1,\ldots,k_r)},\mathbf{c}^{(k_1,\ldots,k_r)}}(q).$$

Hence

$$P_{\mathbf{a},\mathbf{c}}(q) = P_{\mathbf{a}^{(k_1,\ldots,k_r)},\mathbf{c}^{(k_1,\ldots,k_r)}}(q).$$

Also, to show (2), it suffices to apply successively the Proposition 5.39. (3) follows from the bijectivity of $\psi_{k_1,\ldots,k_r}$ and (1). As for (4), we know by definition,

$$S(\mathbf{a})_{k_1,\ldots,k_r} \supseteq S(\mathbf{b})_{k_1,\ldots,k_r}.$$

We know that any for $\mathbf{c} \in S(\mathbf{a})_{k_1,\cdots,k_r}$, we have $\mathbf{c}^{(k_1,\ldots,k_r)} \le \mathbf{b}^{(k_1,\ldots,k_r)}$, by (2), this implies that $\mathbf{c} \le \mathbf{b}$. Hence we are done. □

Similarly, we have

Definition 6.6. Let $(k_1, \ldots, k_r)$ be a sequence of integers, then we define

$${}_{k_r,\ldots,k_1}S(\mathbf{a}) = \{\mathbf{c} \in S(\mathbf{a}) : {}^{(k_i,\ldots,k_1)}\mathbf{c} \in {}_{k_{i+1}}S({}^{(k_i,\ldots,k_1)}\mathbf{a}), \text{ for } i = 1, \ldots, r\}.$$

and

$${}_{k_r,\ldots,k_1}\psi : {}_{k_r,\ldots,k_1}S(\mathbf{a}) \to S({}^{(k_r,\ldots,k_1)}\mathbf{a}),$$

sending $\mathbf{c}$ to ${}^{(k_r,\ldots,k_1)}\mathbf{c}$.

Notation 6.7. *Let* $\mathbf{d} = \{\Delta_1, \ldots, \Delta_r\}$ *such that* $\Delta_i = [k_i, \ell_i]$ *with* $k_1 \leq \cdots \leq k_r$ *We denote*

$$_{\mathbf{d}}S(\mathbf{a}) :=_{k_r,\ldots,\ell_r,k_{r-1},\ldots,k_1,\ldots,\ell_1} S(\mathbf{a}),$$

and

$$_{\mathbf{d}}\psi :=_{k_r,\ldots,\ell_r,k_{r-1},\ldots,k_1,\ldots,\ell_1} \psi.$$

Remark. Let k_1, k_2 be two integers. In general, we do not have

$$_{k_2}(S(\mathbf{a})_{k_1}) = (_{k_2}S(\mathbf{a}))_{k_1}.$$

For example, let $k_1 = k_2 = 1$, $\mathbf{a} = \{[1], [2]\}$, then

$$_{k_2}(S(\mathbf{a})_{k_1}) = \{\mathbf{a}\}, \;(_{k_2}S(\mathbf{a}))_{k_1} = \{[1, 2]\}.$$

Notation 6.8. *We write for multisegments* $\mathbf{d}_1, \mathbf{d}_2, \mathbf{a}$*,*

$$_{\mathbf{d}_2}S(\mathbf{a})_{\mathbf{d}_1} := (_{\mathbf{d}_2}S(\mathbf{a}))_{\mathbf{d}_1}, \;S(\mathbf{a})_{\mathbf{d}_1,\mathbf{d}_2} := (S(\mathbf{a})_{\mathbf{d}_1})_{\mathbf{d}_2}.$$

and

$$_{\mathbf{d}_2}\psi_{\mathbf{d}_1} := (_{\mathbf{d}_2}\psi)_{\mathbf{d}_1}, \;\psi_{\mathbf{d}_1,\mathbf{d}_2} := (\psi_{\mathbf{d}_1})_{\mathbf{d}_2}$$

For $\mathbf{b} \in S(\mathbf{a})$*,*

$$^{(\mathbf{d}_2)}\mathbf{b}^{(\mathbf{d}_1)} := (^{\mathbf{d}_2}\mathbf{b})^{(\mathbf{d}_1)}, \;\mathbf{b}^{(\mathbf{d}_1,\mathbf{d}_2)} := (\mathbf{b}^{(\mathbf{d}_1)})^{(\mathbf{d}_2)}.$$

6.2. Main result: symmetrization of multisegments

Now we return to the main question, i.e., the calculation of the coefficient $m(\mathbf{c}, \mathbf{a})$ for $\mathbf{c} \in S(\mathbf{a})$. Before we go into the details, we describe our strategies:

(i) Find a symmetric multisegment, denoted by $\mathbf{a}^{\mathrm{sym}}$, such that $L_{\mathbf{a}}$ is the unique term of minimal degree in the image $L_{\mathbf{a}^{\mathrm{sym}}}$ under some partial Bernstein-Zelevinsky operator of considered as element in $\mathcal{R}$.
(ii) For $\mathbf{c} \in S(\mathbf{a})$, find $\mathbf{c}^{\mathrm{sym}} \in S(\mathbf{a}^{\mathrm{sym}})$, such that we have $m(\mathbf{c}, \mathbf{a}) = m(\mathbf{c}^{\mathrm{sym}}, \mathbf{a}^{\mathrm{sym}})$.

Proposition 6.9. *Let* $\mathbf{a}$ *be any multisegment, then there exists a regular multisegment* $\mathbf{b}$*, and two multisegments* $\mathbf{c}_i, i = 1, 2$ *such that*

$$\mathbf{b} \in {}_{\mathbf{c}_2}S(\mathbf{b})_{\mathbf{c}_1}, \;\mathbf{a} = {}^{(\mathbf{c}_2)}\mathbf{b}^{(\mathbf{c}_1)}$$

Proof. Let $\mathbf{a} = \{\Delta_1, \ldots, \Delta_r\}$ be such that

$$\Delta_1 \preceq \cdots \preceq \Delta_r.$$

Assume that there is a j such that $e(\Delta_j)$ appears in $e(\mathbf{a})$ with multiplicity greater than 1. Furthermore, assume that Δ_j is the the smallest segment satisfying this property. Then

$$e(\Delta_1) \leq \cdots < e(\Delta_j) = \cdots = e(\Delta_i) < e(\Delta_{i+1}) \leq \cdots.$$

Let $\Delta^1 = [e(\Delta_i) + 1, \ell]$, where ℓ is the maximal integer such that for any m such that $e(\Delta_i) \leq m \leq \ell - 1$, there is a segment in $\mathbf{a}$ which ends in m. Let $\mathbf{a}_1$ be the multisegment obtained by replacing Δ_i by Δ_i^+, and all $\Delta \in \mathbf{a}$ such that $e(\Delta) \in (e(\Delta_i), \ell]$ by Δ^+. Now we continue the previous construction with $\mathbf{a}_1$ to get $\mathbf{a}_2$, and proceed in the same way until we obtain a multisegment $\mathbf{a}_{r_1}$ such that $e(\mathbf{a}_{r_1})$ contains no segment with multiplicity greater than 1. Let

$$c_1 = \{\Delta^1, \Delta^2, \ldots, \Delta^{r_1}\}.$$

Note that by construction, we have

$$\Delta^1 \prec \Delta^2 \prec \cdots \prec \Delta^{r_1}.$$

We show that $\mathbf{a}_{r_1} \in S(\mathbf{a}_{r_1})_{\mathbf{c}_1}$. Note that

$$\mathbf{a}_i = \mathbf{a}_{r_1}^{(\Delta^{r_1}, \ldots, \Delta^{i+1})},$$

by induction on r_1, we can assume that $\mathbf{a}_1 \in S(\mathbf{a}_{r_1})_{\Delta^{r_1}, \ldots, \Delta^2}$ and show that $\mathbf{a} \in S(\mathbf{a}_1)_{\Delta^1}$. We observe that in $\mathbf{a}_1$, by construction, with the notations above, $\Delta_j, \ldots, \Delta_{i-1}$ are the only segments in $\mathbf{a}_1$ that end in $e(\Delta_i)$, and Δ_i^+ is the only segment in $\mathbf{a}_1$ that ends in $e(\Delta_i) + 1$. Hence we conclude that $\mathbf{a}_1 \in S(\mathbf{a}_1)_{e(\Delta_i)+1}$. For $e(\Delta_i) + 1 < m \leq \ell$, we know that $\mathbf{a}_1^{(e(\Delta_1)+1, \ldots, m-1)}$ does not contain a segment which ends in $m-1$, hence $\mathbf{a}_1^{(e(\Delta_1)+1, \ldots, m-1)} \in S(\mathbf{a}_1^{(e(\Delta_1)+1, \ldots, m-1)})_m$. We are done by putting $m = \ell$.

Now same construction can be applied to show that there exists a multisegment $\mathbf{a}_{r_2}$ such that $b(\mathbf{a}_{r_2})$ contains no segment with multiplicity greater than 1, and

$$c_2 = \{{}^1\Delta, \ldots, {}^{r_2}\Delta\},$$

such that

$$\mathbf{a}_{r_2} \in {}_{\mathbf{c}_2}S(\mathbf{a}_2),\ \mathbf{a}_{r_1} = {}^{(\mathbf{c}_2)}\mathbf{a}_{r_2}$$

as minimal degree component.

Note that in this way we have constructed a regular multisegment $\mathbf{b} = \mathbf{a}_{r_2}$,

$$\mathbf{b} \in {}_{\mathbf{c}_2}S(\mathbf{b})_{\mathbf{c}_1},\ \mathbf{a} = {}^{(\mathbf{c}_2)}\mathbf{b}^{(\mathbf{c}_1)}$$

□

To finish our strategy (i), we are reduced to consider the case of regular multi-segments.

Proposition 6.10. *Let* $\mathbf{b}$ *be a regular multisegment, then there exists a symmetric multisegment* $\mathbf{b}^{\mathrm{sym}}$*, and a multisegment* $\mathbf{c}$ *such that*

$$\mathbf{b}^{\mathrm{sym}} \in {}_{\mathbf{c}}S(\mathbf{b}^{\mathrm{sym}}),\ \mathbf{b} = {}^{(\mathbf{c})}(\mathbf{b}^{\mathrm{sym}}).$$

Proof. In general $\mathbf{b}$ is not symmetric, i.e, we do not have $\min\{e(\Delta) : \Delta \in \mathbf{b}\} \geq \max\{b(\Delta) : \Delta \in \mathbf{b}\}$. Let

$$\mathbf{b} = \{\Delta_1, \ldots, \Delta_r\}, \quad b(\Delta_1) > \cdots > b(\Delta_r).$$

so that

$$b(\Delta_1) = \max\{b(\Delta_i) : i = 1, \ldots, r\}.$$

If $\mathbf{b}$ is not symmetric, let $\Delta^1 = [\ell, b(\Delta_1) - 1]$ with ℓ maximal satisfying that for any m such that $\ell - 1 \leq m \leq b(\Delta_1)$, there is a segment in $\mathbf{b}$ starting in m. We construct $\mathbf{b}_1$ by replacing every segment Δ in $\mathbf{b}$ ending in Δ^1 by ${}^+\Delta$. Repeat this construction with $\mathbf{b}_1$ to get $\mathbf{b}_2, \ldots$, until we get $\mathbf{b}^{\mathrm{sym}} = \mathbf{b}_s$, which is symmetric. Let $\mathbf{c} = \{\Delta^1, \ldots, \Delta^s\}$, then as before, we have

$$\mathbf{b}^{\mathrm{sym}} \in {}_{\mathbf{c}}S(\mathbf{b}^{\mathrm{sym}}),\ \mathbf{b} = {}^{(\mathbf{c})}(\mathbf{b}^{\mathrm{sym}}).$$

□

As a corollary, we know that

Corollary 6.11. *For any multisegment* $\mathbf{a}$*, we can find a symmetric multisegment* $\mathbf{a}^{\mathrm{sym}}$ *and three multisegments* $\mathbf{c}_i$*,* $i = 1, 2, 3$*, such that*

$$\mathbf{a}^{\mathrm{sym}} \in {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1},\ \mathbf{a} = {}^{(\mathbf{c}_2,\mathbf{c}_3)}\mathbf{a}^{\mathrm{sym}(\mathbf{c}_1)}.$$

Now applying Proposition 6.5

Proposition 6.12. *The morphism*

$${}_{\mathbf{c}_2,\mathbf{c}_3}\psi_{\mathbf{c}_1} : {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1} \to S(\mathbf{a})$$

is bijective, and for $\mathbf{b} \in S(\mathbf{a})$*, there exists a unique* $\mathbf{b}^{\mathrm{sym}} \in S(\mathbf{a}^{\mathrm{sym}})$ *such that*

$$P_{\mathbf{a},\mathbf{b}}(q) = P_{\mathbf{a}^{\mathrm{sym}},\mathbf{b}^{\mathrm{sym}}}(q).$$

6.3. Examples

In this section we shall give some examples to illustrate the idea of reduction to symmetric case.

We first take $\mathbf{a} = \{[1], [2], [2], [3]\}$ to show how to reduce a general multisegment to a regular multisegment. The procedure is showed in the following picture.

Here we have $\mathbf{a}_2 = \{[0, 1], [1, 3], [2], [3, 4]\}$, such that

$$\mathbf{a}_2 \in {}_{[0,1]}S(\mathbf{a}_2)_{[3,4]},\ \mathbf{a} = {}^{([0,1])}\mathbf{a}_2^{([3,4])}$$

Next, we reduce the regular multisegment $\mathbf{a}_2$ to a multisegment $\mathbf{a}^{\mathrm{sym}}$, as is showed in the following picture.

Here,we have

$$\mathbf{a}^{\mathrm{sym}} = \{[0, 3], [1, 5], [2, 4], [3, 6]\} = \Phi(w)$$

where $w = \sigma_2 \in S_4$.

Now we take $\mathbf{b} = \{[1, 2], [2, 3]\}$, we want to find $\mathbf{b}^{\text{sym}} \in S(\mathbf{a}^{\text{sym}})$ such that $m(\mathbf{b}, \mathbf{a}) = m(\mathbf{b}^{\text{sym}}, \mathbf{a}^{\text{sym}})$. Actually, following the procedure in Fig. 2 above, we have the situation shown in Fig. 4.

Here we get $\mathbf{b}_2 = \{[0, 3], [1, 4]\}$. Following the procedure in the Fig. 3, we obtain the situation shown in Fig. 5.

Hence we get

$$\mathbf{b}^{\text{sym}} = \{[0, 5], [1, 3], [2, 6], [3, 4]\} = \Phi(v)$$

with $v = (13)(24) \in S_4$. From [20, 11.3], we know that $m(\mathbf{b}, \mathbf{a}) = 2$, hence we get $m(\mathbf{b}^{\text{sym}}, \mathbf{a}^{\text{sym}}) = 2$.

Remark. We showed in Sect. 2 that

$$m(\mathbf{b}^{\text{sym}}, \mathbf{a}^{\text{sym}}) = P_{w,v}(1),$$

where $P_{v,w}(q)$ is the Kazhdan-Lusztig polynomial associated to v, w. One knows that $P_{v,w}(q) = 1 + q$, hence $P_{v,w}(1) = 2$.

As we have seen, to each multisegment, we have (at least) two different ways to attach a Kazhdan-Lusztig polynomial:

(1) To use the Zelevinsky construction as described in section 4.2.
(2) To first construct an associated symmetric multisegment, and then attach the corresponding Kazhdan-Lusztig polynomial.

Remark. In general, for $\mathbf{a} > \mathbf{b}$, (1) gives a polynomial $P^Z_{\mathbf{a},\mathbf{b}}$ which is a Kazhdan-Lusztig polynomial for the symmetric group $S_{\deg(\mathbf{a})}$. (2) gives a polynomial $P^S_{\mathbf{a},\mathbf{b}}$, which is a Kazhdan-Lusztig polynomial for a symmetric group S_n with $n \le \deg(\mathbf{a})$. It may happen that $n = \deg(\mathbf{a})$. By Corollary 5.37, we always have $P^Z_{\mathbf{a},\mathbf{b}} = P^S_{\mathbf{a},\mathbf{b}}$.

Example 6.13. Consider $\mathbf{a} = \{1, 2, 2, 3\}$, $\mathbf{b} = \{[1, 2], [3, 4]\}$, then by [18] section 3.4, we know that $P^Z_{\mathbf{a},\mathbf{b}} = 1 + q$. The symmetrization of $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a}^{\text{sym}} = \Psi((2, 3)), \quad \mathbf{b}^{\text{sym}} = \Psi((1, 3)(2, 4)).$$

Hence $P^S_{\mathbf{a},\mathbf{b}} = P_{(2,3),(1,3)(2,4)} = 1 + q$, which is the Kazhdan-Lusztig polynomial for the pair $((2, 3), (1, 3)(2, 4))$ in S_4

7. Application

Definition 7.1. Two multisegments

$$\mathbf{a} = \{\Delta_1, \ldots, \Delta_r\} \quad \text{and} \quad \mathbf{a}' = \{\Delta'_1, \ldots, \Delta'_{r'}\}$$

have the same relation type if

- $r = r'$;

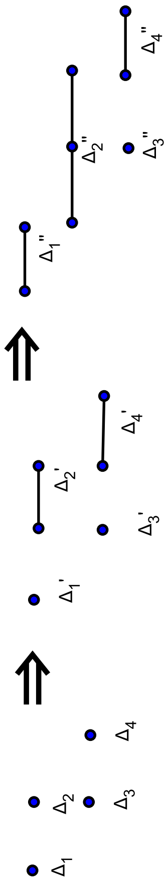

Fig. 2. Reduction to a regular multisegment of **a**

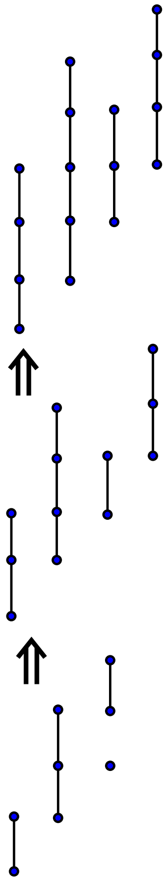

Fig. 3. Reduction to a symmetric multisegment of $\mathbf{a}_2$

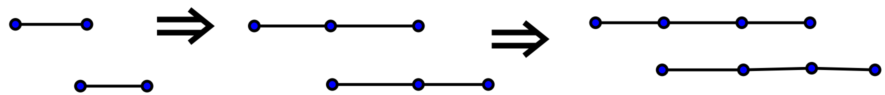

Fig. 4. Reduction to a regular multisegment of **b** which is compatible with the reduction of **a** in fig. 2

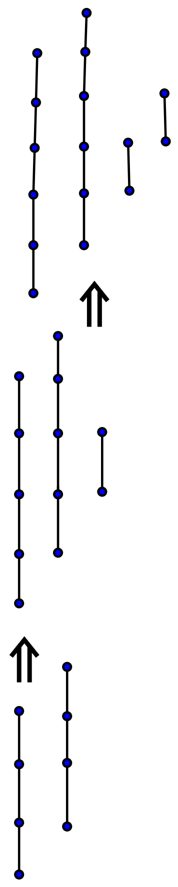

Fig. 5. Reduction to a symmetric multisegment of $\mathbf{b}_2$ which is compatible with the reduction of $\mathbf{a}_2$ in Fig. 3

- there exists a bijection

$$\xi : \mathbf{a} \to \mathbf{a}'$$

of multisets which preserves the partial order $\preceq$ and relation type of segments and induces bijection of multisets

$$e(\xi) : e(\mathbf{a}) \to e(\mathbf{a}'), \quad b(\xi) : b(\mathbf{a}) \to b(\mathbf{a}').$$

satisfying

$$e(\xi)(e(\Delta)) = e(\xi(\Delta)), \quad b(\xi)(b(\Delta)) = b(\xi(\Delta)).$$

Lemma 7.2. *Let* $\mathbf{a}$ *and* $\mathbf{a}'$ *be of the same relation type induced by* ξ_1. *Let* $\{\Delta_1 \preceq \Delta_2\}$ *be linked in* $\mathbf{a}$. *Denote by* $\mathbf{a}_1$ ($\mathbf{a}_1'$, resp.) *the multisegment obtained by applying the elementary operation to* $\{\Delta_1, \Delta_2\}$ ($\{\xi(\Delta_1), \xi(\Delta_2)\}$, resp.). *Then* $\mathbf{a}_1$ *and* $\mathbf{a}_1'$ *also have the same relation type.*

Proof. We define a bijection

$$\xi_1 : \mathbf{a}_1 \to \mathbf{a}_1'$$

by

$$\xi_1(\Delta_1 \cup \Delta_2) = \xi(\Delta_1) \cup \xi(\Delta_2), \quad \xi_1(\Delta_1 \cap \Delta_2) = \xi(\Delta_1) \cap \xi(\Delta_2)$$

and

$$\xi_1(\Delta) = \xi(\Delta), \quad \text{for all } \Delta \in \mathbf{a} \setminus \{\Delta_1, \Delta_2\}.$$

It induces a bijection between the end multisets $e(\mathbf{a}_1)$ and $e(\mathbf{a}_1')$ as well as the beginning multisets $b(\mathbf{a}_1)$ and $b(\mathbf{a}_1')$. Also the morphism ξ preserves the partial order follows from the fact that for $x, y \in e(\mathbf{a})$ such that $x \leq y$, then $e(\xi_1)(x) = e(\xi)(x) \leq e(\xi_1)(y) = e(\xi)(y)$ (The same fact holds for $b(\xi_1)$). Finally, it remains to show that ξ_1 respects the relation type. Let $\Delta \preceq \Delta'$ be two segments in $\mathbf{a}_1$, if non of them is contained in $\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}$, then $\xi_1(\Delta) = \xi(\Delta)$ and $\xi_1(\Delta') = \xi(\Delta')$ and they are in the same relation type as $\{\Delta, \Delta'\}$ by assumption. For simplicity, we only discuss the case where $\Delta = \Delta_1 \cup \Delta_2$ but Δ' is not contained in $\{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\}$, other cases are similar.

- If Δ' cover Δ, then Δ cover Δ_1 and Δ_2, hence $\xi_1(\Delta) = \xi(\Delta)$ cover $\xi(\Delta_1)$ and $\xi(\Delta_2)$, which implies $\xi_1(\Delta')$ covers $\xi_1(\Delta)$.
- If Δ' is linked to Δ but not juxtaposed, then either Δ' covers Δ_2 and linked to Δ_1, or Δ' is linked to Δ_2 but not juxtaposed. In both cases we have $\xi(\Delta')$ is linked to $\xi(\Delta_1) \cup \xi(\Delta_2)$ and not juxtaposed.
- If Δ' is juxtaposed to Δ, then Δ' is juxtaposed to Δ_2 since $\Delta_2 \succeq \Delta_1$. Therefore $\xi(\Delta')$ is juxtaposed to $\xi(\Delta_2)$ which implies $\xi_1(\Delta')$ is juxtaposed to the segment $\xi_1(\Delta)$.
- If Δ' is unrelated to $\Delta_1 \cup \Delta_2$, then it is unrelated to both Δ_1 and Δ_2 with $\Delta_2 \preceq \Delta'$, this implies that $\xi(\Delta')$ is unrelated to $\xi(\Delta_1) \cup \xi(\Delta_2)$. □

Remark. As every element $\mathbf{b} \in S(\mathbf{a})$ is obtained from $\mathbf{a}$ by a sequence of elementary operations, we can define a morphism of poset

$$\Xi : S(\mathbf{a}) \longrightarrow S(\mathbf{a}').$$

Lemma 7.3. *The application Ξ is well defined and bijective.*

Proof. We give a new definition of Ξ in the following way. For $\mathbf{b} \in S(\mathbf{a})$, we define

$$\Xi(\mathbf{b}) = \{[b(\xi)(b(\Delta)), e(\xi)(e(\Delta))] : \Delta \in \mathbf{b}\}$$

such a definition is independent of the choice of elementary operations. It remains to see that it coincides with the one using elementary operation. In fact, let $\mathbf{a}_1$ be a multisegment obtained by applying the elementary operation to the pair of segments $\{\Delta_1 \preceq \Delta_2\}$, then by our original definition of Ξ, it sends $\mathbf{a}_1$ to $\mathbf{a}'_1$ as in the previous lemma. Now by the new definition, we have $\Xi(\mathbf{a}_1)$ given by

$$\{\xi(\Delta) : \Delta \in \mathbf{a} \setminus \{\Delta_1, \Delta_2\}\} \cup \{[b(\xi)(b(\Delta_1)), b(\xi)(b(\Delta_2))], [b(\xi)(b(\Delta_2)), b(\xi)(b(\Delta_1))]\}.$$

By our definition of ξ, we get

$$[b(\xi)(b(\Delta_1)), b(\xi)(b(\Delta_2))] = \xi(\Delta_1) \cup \xi(\Delta_2),$$

and

$$[b(\xi)(b(\Delta_2)), b(\xi)(b(\Delta_1))] = \xi(\Delta_1) \cap \xi(\Delta_2).$$

Hence we conclude that Ξ is well defined. Note that by our definition, since ξ is invertible, we can use the same procedure to construct Ξ^{-1}. Now we have

$$\Xi\Xi^{-1} = \mathrm{Id}, \quad \Xi^{-1}\Xi = \mathrm{Id}$$

by our definition above using $b(\xi)$ and $e(\xi)$. This shows that Ξ is bijective. □

Let $\mathbf{a}$ and $\mathbf{a}'$ be two multisegments of the same relation type. Note that by Corollary 6.11, we have a symmetric multisegment $\mathbf{a}^{\mathrm{sym}}$ and three multisegments $\mathbf{c}_i$, $i = 1, 2, 3$ such that

$$\mathbf{a}^{\mathrm{sym}} \in {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1}, \ \mathbf{a} = {}^{(\mathbf{c}_2,\mathbf{c}_3)}\mathbf{a}^{\mathrm{sym}(\mathbf{c}_1)}.$$

In the same way, we have

$$\mathbf{a}'^{\mathrm{sym}} \in {}_{\mathbf{c}'_2,\mathbf{c}'_3}S(\mathbf{a}'^{\mathrm{sym}})_{\mathbf{c}'_1}, \ \mathbf{a}' = {}^{(\mathbf{c}'_2,\mathbf{c}'_3)}\mathbf{a}'^{\mathrm{sym}(\mathbf{c}'_1)}.$$

Lemma 7.4. *The two multisegments $\mathbf{a}^{\mathrm{sym}}$ and $\mathbf{a}'^{\mathrm{sym}}$ have the same relation type. Let $\Xi^{\mathrm{sym}} : S(\mathbf{a}^{\mathrm{sym}}) \to S(\mathbf{a}'^{\mathrm{sym}})$ be the bijection constructed in Lemma 7.3, then we have the following commutative diagram*

$$\begin{array}{ccc} {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1} & \xrightarrow{\Xi^{\mathrm{sym}}} & {}_{\mathbf{c}'_2,\mathbf{c}'_3}S(\mathbf{a}'^{\mathrm{sym}})_{\mathbf{c}'_1} \\ \downarrow {}_{\mathbf{c}_2,\mathbf{c}_3}\psi_{\mathbf{c}_1} & & \downarrow {}_{\mathbf{c}'_2,\mathbf{c}'_3}\psi_{\mathbf{c}'_1} \\ S(\mathbf{a}) & \xrightarrow{\Xi} & S(\mathbf{a}'). \end{array}$$

Proof. Note that by construction we know that the number of segments in $\mathbf{a}^{\mathrm{sym}}$ is the same as that of $\mathbf{a}$ (cf. Proposition 6.9 and Proposition 6.10). Let $\mathbf{a}^{\mathrm{sym}} = \{\Delta_1 \preceq \cdots \preceq \Delta_r\}$, then $\mathbf{a} = \{^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_1^{(\mathbf{c}_1)} \preceq \cdots \preceq {}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_r^{(\mathbf{c}_1)}\}$. Also let $\mathbf{a}'^{\mathrm{sym}} = \{\Delta_1' \preceq \cdots \preceq \Delta_r'\}$. We define

$$\xi^{\mathrm{sym}} : \mathbf{a}^{\mathrm{sym}} \to \mathbf{a}'^{\mathrm{sym}}$$
$$\Delta_i \mapsto \Delta_i'.$$

This automatically induces bijections

$$e(\xi^{\mathrm{sym}}) : e(\mathbf{a}^{\mathrm{sym}}) \to e(\mathbf{a}'^{\mathrm{sym}}), \quad b(\xi^{\mathrm{sym}}) : b(\mathbf{a}^{\mathrm{sym}}) \to b(\mathbf{a}'^{\mathrm{sym}}),$$

since all 4 are sets (instead of multisets). Note that we have

$$\xi({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}) = {}^{(\mathbf{c}_2',\mathbf{c}_3')}\Delta_i'^{(\mathbf{c}_1')}$$

It remains to show that ξ^{sym} preserves the relation type. Let $i \leq j$. Then Δ_i and Δ_j are linked if and only if one of the following happens

- ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}$ and ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_j^{(\mathbf{c}_1)}$ are linked (juxtaposed or not);
- ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}$ and ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_j^{(\mathbf{c}_1)}$ are unrelated.

Δ_j covers Δ_i if and only if ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_j^{(\mathbf{c}_1)}$ covers ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}$. Since ξ preserves relation types, this shows that ξ^{sym} also preserves relation types. Hence we conclude that $\mathbf{a}^{\mathrm{sym}}$ and $\mathbf{a}'^{\mathrm{sym}}$ have same relation type. To see that the map Ξ^{sym} sends ${}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1}$ to ${}_{\mathbf{c}_2',\mathbf{c}_3'}S(\mathbf{a}'^{\mathrm{sym}})_{\mathbf{c}_1'}$, consider $\mathbf{b} \in S(\mathbf{a})$ and its pre-image $\mathbf{b}^{\mathrm{sym}} \in {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1}$ under ${}_{\mathbf{c}_2,\mathbf{c}_3}\psi_{\mathbf{c}_1}$, whose existence and uniqueness is guaranteed by Proposition 6.12.

We introduce a length function $\ell : S(\mathbf{a}) \to \mathbb{Z}_{\geq 0}$: $\ell(a) = 0$ and in general

$$\ell(\mathbf{b}) = \min\{\ell : \mathbf{a} = \mathbf{a}_0 > \cdots > \mathbf{a}_\ell = \mathbf{b} \text{ is a maximal chain}\}.$$

- First of all, we assume that $\ell(\mathbf{b}) = 1$, i.e. $\mathbf{b}$ can be obtained from $\mathbf{a}$ by applying the elementary operation to the pair $\{{}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_i^{(\mathbf{c}_1)}, {}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta_j^{(\mathbf{c}_1)}\}(i < j)$. Let $\tilde{\mathbf{b}}$ be the element in $S(\mathbf{a}^{\mathrm{sym}})$ obtained by applying the elementary operation to the pair of segments $\{\Delta_i, \Delta_j\}$ in $\mathbf{a}^{\mathrm{sym}}$. Then we have

$$\mathbf{b} = {}^{(\mathbf{c}_2,\mathbf{c}_3)}\tilde{\mathbf{b}}^{(\mathbf{c}_1)}.$$

Let $\tilde{\mathbf{b}}' = \Xi^{\mathrm{sym}}(\tilde{\mathbf{b}})$. By construction, we have

$$\mathbf{b}' = \Xi(\mathbf{b}) = {}^{(\mathbf{c}_2',\mathbf{c}_3')}\tilde{\mathbf{b}}'^{(\mathbf{c}_1')}.$$

Now consider

$$\tilde{\mathbf{b}}_0 = \tilde{\mathbf{b}} > \cdots > \tilde{\mathbf{b}}_n = \mathbf{b}^{\mathrm{sym}}$$

be a maximal chain of multisegments and let $\tilde{\mathbf{b}}_i' = \Xi^{\mathrm{sym}}(\tilde{\mathbf{b}}_i)$, then

$$\tilde{\mathbf{b}}_0' > \cdots > \tilde{\mathbf{b}}_n' = \Xi^{\mathrm{sym}}(\mathbf{b}^{\mathrm{sym}}).$$

Let

$$\tilde{\mathbf{b}}_i = \{\Delta_{i,1} \preceq \cdots \preceq \Delta_{i,r_i}\}, \quad \tilde{\mathbf{b}}'_i = \{\Delta'_{i,1} \preceq \cdots \preceq \Delta'_{i,r_i}\}.$$

We prove by induction that

$$\mathbf{b}' = {}^{(\mathbf{c}'_2,\mathbf{c}'_3)}\tilde{\mathbf{b}}_i^{\prime(\mathbf{c}'_1)}.$$

We already showed the case where $i = 0$. Assume that we have

$$\mathbf{b}' = {}^{(\mathbf{c}'_2,\mathbf{c}'_3)}\tilde{\mathbf{b}}_j^{\prime(\mathbf{c}'_1)}$$

for $j < i$. Suppose that $\tilde{\mathbf{b}}_i$ is obtained from $\tilde{\mathbf{b}}_{i-1}$ by applying the elementary operation to the pair of segments $\{\Delta_{i-1,\alpha_{i-1}} \preceq \Delta_{i-1,\beta_{i-1}}\}$. We deduce from the fact $\tilde{\mathbf{b}}_i \geq \mathbf{b}^{\mathrm{sym}}$ that we are in one of the following situations

- ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\alpha_{i-1}} = \emptyset$ or ${}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\beta_{i-1}} = \emptyset$;
- $b({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\beta_{i-1}}) = b({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\alpha_{i-1}})$;
- $e({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\beta_{i-1}}) = e({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,\alpha_{i-1}})$.

According the our assumption that $\tilde{\mathbf{b}}'_i = \Xi^{\mathrm{sym}}(\tilde{\mathbf{b}}'_i)$, we have

$$\xi({}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{(\mathbf{c}_1)}_{i-1,j}) = {}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{\prime(\mathbf{c}_1)}_{i-1,j},$$

therefore the pair $\{{}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{\prime(\mathbf{c}_1)}_{i-1,\alpha_{i-1}}, {}^{(\mathbf{c}_2,\mathbf{c}_3)}\Delta^{\prime(\mathbf{c}_1)}_{i-1,\beta_{i-1}}\}$ also satisfies one of the listed properties above. This implies that $\tilde{\mathbf{b}}'_i$ is sent to $\mathbf{b}'$ by ${}_{\mathbf{c}'_2,\mathbf{c}'_3}\psi_{\mathbf{c}'_1}$. Note that by Proposition 5.36, which implies that $\mathbf{b}'^{\mathrm{sym}}$ is the minimal element that are mapped to $\mathbf{b}'$ in $S(\mathbf{a}'^{\mathrm{sym}})$, we know that

$$\tilde{\mathbf{b}}'_n (= \Xi^{\mathrm{sym}}(\mathbf{b}^{\mathrm{sym}})) \geq \mathbf{b}'^{\mathrm{sym}}.$$

Conversely, we have

$$\Xi^{\mathrm{sym}\,-1}(\mathbf{b}'^{\mathrm{sym}}) \geq \mathbf{b}^{\mathrm{sym}}.$$

Combine the two inequalities to get

$$\Xi^{\mathrm{sym}}(\mathbf{b}^{\mathrm{sym}}) = \mathbf{b}'^{\mathrm{sym}}.$$

- The general case where $\ell(\mathbf{b}) > 1$, we can choose a maximal chain of multisegments

$$\mathbf{a} = \mathbf{a}_0 > \cdots > \mathbf{a}_{\ell(\mathbf{b})} = \mathbf{b}.$$

Let $\mathbf{a}'_i = \Xi(\mathbf{a}_i)$, by assumption, we can assume that for $i < \ell(\mathbf{b})$, we have

$$\Xi^{\mathrm{sym}}(\mathbf{a}_i^{\mathrm{sym}}) = \mathbf{a}_i^{\prime\,\mathrm{sym}}.$$

By considering the set $S(\mathbf{a}_{\ell(\mathbf{b})-1})$, we can reduce to the case where $\ell(\mathbf{b}) = 1$. Hence we are done. □

Theorem 7.5. *For* $\mathbf{a}$ *and* $\mathbf{a}'$ *be the multisegments of the same relation type. Then for* $\mathbf{b} \in S(\mathbf{a})$ *with* $\mathbf{b}' = \Xi(\mathbf{b})$, *we have*

$$P_{\mathbf{a},\mathbf{b}}(q) = P_{\mathbf{a}',\mathbf{b}'}(q).$$

Proof. First of all, we consider the case where $\mathbf{a}$ and $\mathbf{a}'$ are symmetric multisegments. Let $\mathbf{a} = \Phi(w)$ by fixing a map

$$\Phi : S_n \to S(\mathbf{a}_{\mathrm{Id}}).$$

Now since $\mathbf{a}$ and $\mathbf{a}'$ have the same relation type, we know that there is a map $\Phi' : S_n \to S(\mathbf{a}'_{\mathrm{Id}})$ such that

$$\mathbf{a}' = \Phi'(w).$$

Finally, let $\mathbf{a} = \{\Delta_1, \dots, \Delta_n\}$ and $\mathbf{a}' = \{\Delta'_1, \dots, \Delta'_n\}$ such that

$$b(\Delta_1) < \cdots < b(\Delta_n), \quad \Delta'_i = \xi(\Delta_i).$$

Without loss of generality, we assume that $b(\Delta_1) = b(\Delta'_1)$. We can assume that $b(\Delta_i) = b(\Delta_{i-1}) + 1$. In fact, if $b(\Delta_i) > b(\Delta_{i-1}) + 1$, then by replacing Δ_i by ${}^{+}\Delta_i$, we get a new symmetric multisegment $\mathbf{a}_1$ which has the same relation type as $\mathbf{a}$. Moreover, let $\mathbf{b} \in S(\mathbf{a})$ and $\mathbf{b}_1$ be the image of $\mathbf{b}$ in $S(\mathbf{a}_1)$ under the map $\Xi : S(\mathbf{a}) \to S(\mathbf{a}_1)$ from Lemma 7.3. Then

$$P_{\mathbf{a},\mathbf{b}}(q) = P_{\mathbf{a}_1,\mathbf{b}_1}(q)$$

by Corollary 5.37. It suffices to prove the theorem for $\mathbf{a}_1$ and $\mathbf{a}'$. From now on, let $b(\Delta_i) = b(\Delta_{i-1}) + 1$ and $b(\Delta_i) = b(\Delta'_i)$ for all i. The same argument (as the reduction from the pair $(\mathbf{a}, \mathbf{a}')$ to the pair $(\mathbf{a}_1, \mathbf{a}')$) shows that we can furthermore assume that

$$e(\Delta_{w^{-1}(i)}) = e(\Delta_{w^{-1}(i-1)}) + 1, \quad e(\Delta'_{w^{-1}(i)}) = e(\Delta'_{w^{-1}(i-1)}) + 1.$$

Now if $e(\Delta_{w^{-1}(1)}) < e(\Delta'_{w^{-1}(1)})$, then consider the truncation functor $\mathbf{a}' \mapsto \mathbf{a}'^{(e(\Delta_{w^{-1}(1)})+1, \dots, e(\Delta'_{w^{-1}(1)}))}$, the latter is a symmetric multisegment having the same relation type as $\mathbf{a}'$, and by Proposition 6.5 (1)

$$P_{\mathbf{a}',\mathbf{b}'}(q) = P_{\mathbf{a}'^{(e(\Delta_{w^{-1}(1)})+1,\dots,e(\Delta'_{w^{-1}(1)}))},\mathbf{b}'^{(e(\Delta_{w^{-1}(1)})+1,\dots,e(\Delta'_{w^{-1}(1)}))}}(q).$$

Repeating the same procedure, in a finite number of steps, we find $\mathbf{c}$, such that

$$\mathbf{a} = \mathbf{a}'^{(\mathbf{c})}$$

and again by Proposition 6.5,

$$P_{\mathbf{a},\mathbf{b}}(q) = P_{\mathbf{a}',\mathbf{b}'}(q).$$

For general case, applying Corollary 6.11, there exist symmetric multisegment $\mathbf{a}^{\mathrm{sym}}$ and three multisegments $\mathbf{c}_i$, $i = 1, 2, 3$ such that

$$\mathbf{a}^{\mathrm{sym}} \in {}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1}, \; \mathbf{a} = {}^{(\mathbf{c}_2,\mathbf{c}_3)}\mathbf{a}^{\mathrm{sym}(\mathbf{c}_1)}.$$

Similarly for $\mathbf{a}'$,

$$\mathbf{a}'^{\,\mathrm{sym}} \in {}_{\mathbf{c}_2',\mathbf{c}_3'}S(\mathbf{a}'^{\,\mathrm{sym}})_{\mathbf{c}_1'},\ \mathbf{a}' = {}^{(\mathbf{c}_2',\mathbf{c}_3')}\mathbf{a}'^{\,\mathrm{sym}(\mathbf{c}_1')}.$$

By Proposition 6.12,

$$P_{\mathbf{a},\mathbf{b}}(q) = P_{\mathbf{a}^{\mathrm{sym}},\mathbf{b}^{\mathrm{sym}}}(q), \quad P_{\mathbf{a}',\mathbf{b}'}(q) = P_{\mathbf{a}'^{\,\mathrm{sym}},\mathbf{b}'^{\,\mathrm{sym}}}(q)$$

where $\mathbf{b}^{\mathrm{sym}}$ (resp. $\mathbf{b}'^{\,\mathrm{sym}}$) is the pre-image of $\mathbf{b}$ (resp. $\mathbf{b}'$) in ${}_{\mathbf{c}_2,\mathbf{c}_3}S(\mathbf{a}^{\mathrm{sym}})_{\mathbf{c}_1}$ (resp. ${}_{\mathbf{c}_2',\mathbf{c}_3'}S(\mathbf{a}'^{\,\mathrm{sym}})_{\mathbf{c}_1'}$). Now combining with Lemma 7.4 and the symmetric case we have just proved, we have

$$P_{\mathbf{a}^{\mathrm{sym}},\mathbf{b}^{\mathrm{sym}}}(q) = P_{\mathbf{a}'^{\,\mathrm{sym}},\mathbf{b}'^{\,\mathrm{sym}}}(q),$$

which implies $P_{\mathbf{a},\mathbf{b}}(q) = P_{\mathbf{a}',\mathbf{b}'}(q)$. □

Corollary 7.6. *Let* $\mathbf{a}_{\mathrm{Id}}$ *be a symmetric multisegment associated to the identity in* S_n *and*

$$\Phi : S_n \to S(\mathbf{a}_{\mathrm{Id}}).$$

Then

$$m(\Phi(v), \Phi(w)) = P_{w,v}(1).$$

Proof. The special case where

$$\mathbf{a}_{\mathrm{Id}} = \sum_{i=1}^{n} [i, i+n-1]$$

is already treated in Corollary 4.15. The general case can be deduced from the theorem above. □

Acknowledgements This paper is part of my thesis at University Paris 13, which is funded by the program DIM of the region Ile de France. I would like to thank my advisor Pascal Boyer for his keen interest in this work and his continuing support and countless advice. It is rewritten during my first year of posdoc at Universität Bonn, I thank their hospitality. In addition, I would like to thank Alberto Mínguez et Vincent Sécherre, Yichao Tian for their helpful discussions on the subject. Finally, I would like to thank an anonymous referee for a careful reading and many suggestions for a previous version of the paper.

Declarations

Data availability statements data in this paper include 5 figures, all of which are generated by the software Geogebra.

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