Calabi-Yau in Mirror Symmetry

Calabi-Yau 50 : Legacy and Future

September 10, 1976 + 50.

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Supported by RGC grants of HK Government

Yau Theorem

(Y, J, ω_0) compact Kähler manifold w/ $c_1(Y)_{\mathbb{R}} = 0$,

$\Longrightarrow \exists!$ Kähler metric g w/ $[\omega] = [\omega_0]$ s.t. $Ric(g) = 0$.

Up to a finite cover, Y is a product of :

1° flat torus, e.g. $\mathbb{C}^n / \mathbb{Z}^{2n}$ $\quad Hol(Y^n, \nabla^{L.C.}) = 0$

2° (strict) Calabi-Yau manifold $\quad Hol(Y^n, \nabla^{L.C.}) = SU(n)$

$$\cup\!\!|$$

3° Hyperkähler manifold $\quad Hol(Y^n, \nabla^{L.C.}) = Sp(n/2)$

2° Calabi-Yau $\sim$ (2d) Mirror Symmetry

$\mathrm{Hol}(\nabla^{L.C.}) = SU(n)$

(Y^n, J, ω, Ω), $\omega^n/n! = \Omega\overline{\Omega}$

holom. vol. form (for Ω) e.g. $dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$

$J \xleftrightarrow{\text{up to scaling}} \Omega$

3° Hyperkähler $\sim$ 3d Mirror Symmetry

$\mathrm{Hol}(\nabla^{L.C.}) = Sp(n)$

$(Y^{2n}, J, \omega, \omega_{\mathbb{C}})$, $\Omega = \omega_{\mathbb{C}}^n$

holom. sympl. form (for $\omega_{\mathbb{C}}$) e.g. $\sum_{i=1}^{n} dz_i \wedge dw_i$

$$\left.\begin{aligned} \omega_{\mathbb{C}} &= \omega_I + i\,\omega_K \\ \omega &= \omega_J \end{aligned}\right\} \Longrightarrow \mathbb{H}\text{-structure}$$

§ 2d Mirror Symmetry for Calabi-Yau's

$$
\begin{array}{ccc}
Y^{2n} & \xleftrightarrow{\text{2d MS}} & Y^{\vee 2n} \\
\omega & \longleftrightarrow & \Omega^{\vee} \\
\Omega & \longleftrightarrow & \omega^{\vee}
\end{array}
$$

Generalization:

Log CY

$Y = \overline{Y} \setminus D, \quad D \in |-K_{\overline{Y}}|$

LG model

$(Y^\vee, W^\vee)$

$Y^\vee \xrightarrow{W^\vee} \mathbb{C}$

Eg $\mathbb{C}^\times = \mathbb{CP}^1 \setminus \{0, \infty\} \xleftrightarrow{\text{2d MS}} \mathbb{C}^\times, \; W^\vee = z + \frac{q}{z}$

§ 3d Mirror Symmetry setting

hyperkähler manifold

$(X, \omega_{\mathbb{C}} = \omega_I + i\omega_K)$ J-holomorphic symplectic,

$\omega_X = \omega_J \in \mathcal{K}_X$ Kähler parameter

$\rho_X \in T_X \overset{\text{Cartan}}{\Longleftarrow} \mathrm{Hamil}(X, \omega_{\mathbb{C}})$ equivariant parameter

3d Mirror Symmetry for holom. symplectic manifolds

$$\begin{tikzcd}
(X, \omega_{\mathbb{C}}) \arrow[rr, leftrightarrow, "\text{3d MS}"] & & (X^!, \omega_{\mathbb{C}}^!) \\
\omega_X \arrow[rrd, leftrightarrow] & & \omega_{X^!} \\
\rho_X \arrow[rru, leftrightarrow] & & \rho_{X^!}
\end{tikzcd}$$

$$T^*\mathbb{P}^{n-1} \xleftrightarrow{\text{3d MS}} \widetilde{\mathbb{C}^2 /\!\!/ \mathbb{Z}_n}$$

<u>Proposal</u> (Kifung Chan - L.)

Geometric explanation of 3d MS via MS

$$(Y^{2n}, \omega, \Omega) \xleftrightarrow[\ A \qquad\qquad B\ ]{2d\ MS} (Y^{\vee 2n}, \omega^\vee, \Omega^\vee)$$

Kontsevich HMS $\quad \mathrm{Fuk}(Y, \omega) \simeq D^b(Y^\vee, J^\vee)$

object level :
(2d A/B-branes)

$$\underline{L} = (L, E) \longleftrightarrow \underline{L}^\vee = (L^\vee, E^\vee)$$

Lagr. submfd. $\quad$ flat bundle $/L$ $\qquad$ complex submfd. $\quad$ holomorphic bundle $/L^\vee$

morphism level :

$$HF_Y(\underline{L}, \underline{L}') \overset{\text{vector space}}{=\!=} \mathrm{Ext}^{\bullet}_{Y^\vee}(\underline{L}^\vee, \underline{L}^{\vee\prime})$$

§ 3d A/B-branes

2d A/B-branes

E family of vector spaces

$$\begin{array}{c} E \\ \Big\downarrow \ \text{flat / holom.} \\ L \end{array} \qquad L \underset{\text{Lagr./cpx.}}{\subset} Y \quad CY$$

3d A/B-branes

$\mathcal{Y}$ family of categories

$$\begin{array}{c} \mathcal{Y} \\ \Big\downarrow \ \text{flat / holom.} \\ C \end{array} \qquad C \underset{\substack{\text{cpx. / cpx.} \\ \text{Lagr. / Lagr.}}}{\subset} X \quad HK$$

$$\begin{tikzcd} & Y \arrow[d, "\pi"', "\text{holom.}"] & \text{Kähler} \\ \mathcal{B}r(X) \ni \underline{C} : & C \arrow[r, hook, "\wr", "\omega_{\mathbb{C}}\text{-Lagrangian}"'] & (X, \omega_{\mathbb{C}}) \end{tikzcd}$$

$\rightsquigarrow$ family of categories / C $<$ $\{\mathrm{Fuk}(Y_c)\}_{c \in C}$ flat (3d A-brane)

$\{D^b(Y_c)\}_{c \in C}$ holom. (3d B-brane)

$${}_A\mathrm{Hom}(\underline{C}, \underline{C}') \approx \mathrm{Fuk}(\underline{C} \underset{X}{\times} \underline{C}')$$

$${}_B\mathrm{Hom}(\underline{C}, \underline{C}') \approx D^b(\underline{C} \underset{X}{\times} \underline{C}')$$

i.e. intersection theory. "$\mathrm{Hom}(\underline{C}, \underline{C}') = \underline{C} \underset{X}{\times} \underline{C}'$"

$$(X, \omega_{\mathbb{C}}, \omega_X, \rho_X) \xleftrightarrow[\boxed{A \quad B}]{3d\ MS} (X^!, \omega^!_{\mathbb{C}}, \omega_{X^!}, \rho_{X^!})$$

object level: $\underline{C} = (C, Y) \longleftrightarrow \underline{C}^! = (C^!, Y^!)$

morphism $\quad {}_A\mathrm{Hom}(\underline{C}, \underline{C}') \overset{\text{category}}{=\!=} {}_B\mathrm{Hom}(\underline{C}^!, \underline{C}'^!)$

$$\begin{array}{ccc} {}_A\mathrm{Hom}(\underline{C}, \underline{C}') & & {}_B\mathrm{Hom}(\underline{C}^!, \underline{C}'^!) \\ \| & & \| \\ \mathcal{F}uk(\text{“}\underline{C} \underset{X}{\times} \underline{C}'\text{”}) & & D^b(\underline{C}^! \underset{X^!}{\times} \underline{C}'^!) \end{array}$$

i.e. $\quad$ “$\underline{C} \underset{X}{\times} \underline{C}'$” $\xleftrightarrow[\boxed{A \quad B}]{2d\ MS}$ $\underline{C}^! \underset{X^!}{\times} \underline{C}'^!$

(knowing enough 2d MS $\Longrightarrow$ 3d MS)

3d MS for $X = pt.$, $X' = pt.$

$\equiv$ 2d MS

3d MS for $X = pt/\!/\mathbb{C}^\times$, $X^! = T^*\mathbb{C}^\times$

$\equiv$ equivariant MS / Teleman conjecture.

$$\mathbb{C}^\times \curvearrowright Y$$

$$\begin{tikzcd}
\rightsquigarrow & Y/\mathbb{C}^\times \arrow[d] & & \rightsquigarrow & \text{3d brane in } X = pt/\!/\mathbb{C}^\times \\
& pt/\mathbb{C}^\times \arrow[r] & pt/\!/\mathbb{C}^\times & &
\end{tikzcd}$$

3d MS for $X = T^*\mathbb{C} /\!\!/ \mathbb{C}^\times$, $\quad X^! = T^*\mathbb{C}$

$\equiv$ functorial MS for Log CY / Auroux conj.

- $(\bar{Y}, D) \sim \mathbb{C} \longrightarrow \mathcal{L} \overset{s}{\leftrightarrows} \bar{Y} \Rightarrow D = \{s = 0\}$

$$
\rightsquigarrow \quad
\begin{tikzcd}
\mathcal{L}^* \arrow[d, "p\cdot s"'] & \\
\mathbb{C} & \mathbb{C}^\times \arrow[ul, bend right] \arrow[l, bend right]
\end{tikzcd}
\qquad \rightsquigarrow \quad
\begin{tikzcd}
(\mathcal{L}^* \setminus \bar{Y})/\mathbb{C}^\times \arrow[d, "ps"'] & \\
\mathbb{C}/\mathbb{C}^\times \arrow[r] & T^*\mathbb{C}/\!\!/\mathbb{C}^\times
\end{tikzcd}
$$

$\rightsquigarrow$ 3d brane in $X = T^*\mathbb{C}/\!\!/\mathbb{C}^\times$

§ 3d MS for hypertoric varieties

$$T^*\mathbb{C}^n /\!/ \mathbb{C}^{\times k} \xleftrightarrow{\text{3d MS}} T^*\check{\mathbb{C}}^n /\!/ \check{\mathbb{C}}^{\times(n-k)}$$

dual seq.

$$1 \to \mathbb{C}^{\times k} \xrightarrow{\Delta} \mathbb{C}^{\times n} \longrightarrow \mathbb{C}^{\times(n-k)} \to 1 \qquad 1 \to \check{\mathbb{C}}^{\times(n-k)} \longrightarrow \check{\mathbb{C}}^{\times n} \longrightarrow \check{\mathbb{C}}^{\times k} \to 1$$

eg $k=1$ $\qquad T^*\mathbb{P}^{n-1} \xleftrightarrow{\text{3d MS}} \widetilde{\mathbb{C}^2 /\!/ \mathbb{Z}_n}$

eg $(n,k)=(1,1)$ $\qquad T^*\mathbb{C} /\!/ \mathbb{C}^\times \xleftrightarrow{\text{3d MS}} T^*\mathbb{C}$

eg $(n,k)=(0,1)$ $\qquad pt /\!/ \mathbb{C}^\times \xleftrightarrow{\text{3d MS}} T^*\mathbb{C}^\times$

§ Kähler and equivariant parameters

$$\begin{array}{ccccc} Y^{2n} & \xleftrightarrow{\text{2d MS}} & Y^{\vee 2n} & & \\ \omega & \times & \omega^{\vee} & \text{symplectic} & \\ \Omega & & \Omega^{\vee} & \text{complex} & \end{array}$$

$\uparrow$ 2d parameters

$$\begin{array}{ccccc} & (X, \omega_{\mathbb{C}}) & \xleftrightarrow{\text{3d MS}} & (X^!, \omega_{\mathbb{C}}^!) \\ \text{Kähler} & \omega_X & \times & \omega_{X^!} \\ \text{equivar.} & \rho_X & & \rho_{X^!} \end{array}$$

$\uparrow$ 3d parameters

1° Acting on $\mathcal{B}r(X)$

$$(\omega_X, \rho_X) \bullet \left(\begin{tikzcd} (Y, \omega_Y) \arrow[d, "\pi"'] & \\ C \arrow[r, hook, "\iota"] & X \end{tikzcd} \right) = \left(\begin{tikzcd} (Y, \omega_Y + \pi^* \circ \iota^* \omega_X) \arrow[d, "\pi"'] & \\ C \arrow[r, hook, "\rho_X \circ \iota"] & X \end{tikzcd} \right)$$

2° $\quad {}_A\mathrm{Br}(X) \ni \underline{C} \xleftrightarrow{\text{3d mirror brane}} \underline{C}^! \in {}_B\mathrm{Br}(X^!)$

$$\begin{tikzcd}
\mathcal{K}_X \ni [\omega_X] \arrow[r, leftrightarrow, "\text{3d mirror map}"] \arrow[d, squiggly] \arrow[dr, phantom, "\circlearrowright"] & \rho^! \in \mathcal{T}_{X^!} \arrow[d, squiggly] \\
{}_A\mathrm{Hom}_X(\omega_X \bullet \underline{C}_1, \underline{C}_2) \arrow[r, leftrightarrow, "\text{2d mirror map}"] & {}_B\mathrm{Hom}_{X^!}(\rho^! \bullet \underline{C}_1^!, \underline{C}_2^!)
\end{tikzcd}$$

vary sympl. str. $\qquad\qquad$ vary complex str.

i.e. 3d mirror maps compatible with 2d mirror maps

§ SYZ proposal for 2d MS

$$(Y, \omega, \Omega) \underset{A \qquad\qquad B}{\overset{2d\ MS}{\longleftrightarrow}} (Y^\vee, \omega^\vee, \Omega^\vee)$$

1° SYZ construction

$$\{\underline{L} = (L, \underset{\text{rank } 1}{E})\} = Y^\vee$$

count holom. disks bound L $\rightsquigarrow$ $W^\vee : Y^\vee \longrightarrow \mathbb{C}$

dual fibrations

$$\begin{tikzcd} Y \arrow[dr, "T"'] & & Y^\vee \arrow[dl, "T^\vee"] \\ & B & \end{tikzcd}$$

when $L \simeq T$

Note: Mirror is $dCrit(\check{W}) = T^*[-1]_W Y^\vee$ (-1)-shifted sympl$_{\mathbb{C}}$ manifold.

2° SYZ transf. / family Floer theory / mirror functor

$$\Phi : \mathrm{Fuk}(Y) \longrightarrow D^b(Y^\vee)$$

$$\begin{tikzcd}
\underline{\tilde{L}} \arrow[r, mapsto] & \Phi(\underline{\tilde{L}}) \arrow[d] & \supset & HF(\underline{L}, \underline{\tilde{L}}) \arrow[d, mapsto] \\
 & Y^\vee & \supset & \underline{L}
\end{tikzcd}$$

§ 3d SYZ

Theorem (Chan - L.)

X : hypertoric

$$\Rightarrow \quad X^! = T^*(\{\text{3d A-branes in } X\})$$

i.e. 3d SYZ construction of $X^!$

Next, 3d SYZ transform via family Fukaya theory

§ MS in 3d in 3d MS (Joint w/ Mengqi Yang)

$$(Y, \omega, \Omega) \underset{A| \qquad |B}{\overset{2d\ MS}{\longleftrightarrow}} (Y^\vee, \omega^\vee, \Omega^\vee)$$

$$\omega \in \mathcal{M}_{sympl}(Y) = \mathcal{M}_{cpx}(Y^\vee) \ni \Omega$$

$$\qquad\quad \cong \qquad\qquad\qquad\qquad \cong$$

$$\sigma_\omega \in \mathrm{Stab}(D^b(Y)) \qquad \mathrm{Stab}(\mathcal{F}uk(Y^\vee)) \ni \sigma_\Omega$$

central charge $\quad Z(E) = \int_Y ch(E)\,\Gamma(Y)\, e^\omega \qquad\qquad Z(L) = \int_L \Omega$

$+$ quantum correct^{n}.

Universal intermediate Jacobian for $\underline{CY3}$ Y

$$\begin{tikzcd} \mathcal{J}_{cpx}(Y) \arrow[d] & \supset & H^3(Y,\mathbb{Z})\backslash H^3(Y,\mathbb{C})/H^{3,0}(Y)\oplus H^{2,1}(Y) \arrow[d, mapsto] \\ \mathcal{M}_{cpx}(Y) & \ni & [\Omega] \end{tikzcd}$$

$$\mathcal{J}_{cpx}(Y) = T^*_{\mathbb{Z}}\mathcal{M}_{cpx}(Y) \backslash T^*\mathcal{M}_{cpx}(Y)$$

2d MS $\quad \mathcal{J}_{sympl}(Y) \overset{HK}{=\!=\!=} \mathcal{J}_{cpx}(Y^\vee)$

$\rightsquigarrow$ 3d A/B on $\mathcal{J}(Y)$ or $T^*\mathcal{M}(Y)$

Physical 3d branes (family of cat. w/ stab. cond.)

$$\begin{array}{c} (Y, \Omega_Y, \omega_Y) \\ \downarrow \pi \\ C \xhookrightarrow[\omega_{\mathbb{C}}\text{-}\mathrm{Lagr.}]{\iota} X \end{array} \rightsquigarrow$$

family of categories	$\mathrm{Fuk}(Y_c, \omega_c)$	(flat)	$D^b(Y_c, \Omega_c)$	(holo.)
w/ stab. condit$^{\underline{n}}$	σ_{Ω_c}	(holo.)	σ_{ω_c}	(flat)
	3d A-branes		3d B-branes	

3d A/B-branes

Y family of categories w/ stab. cond.

flat/holom. holom./flat

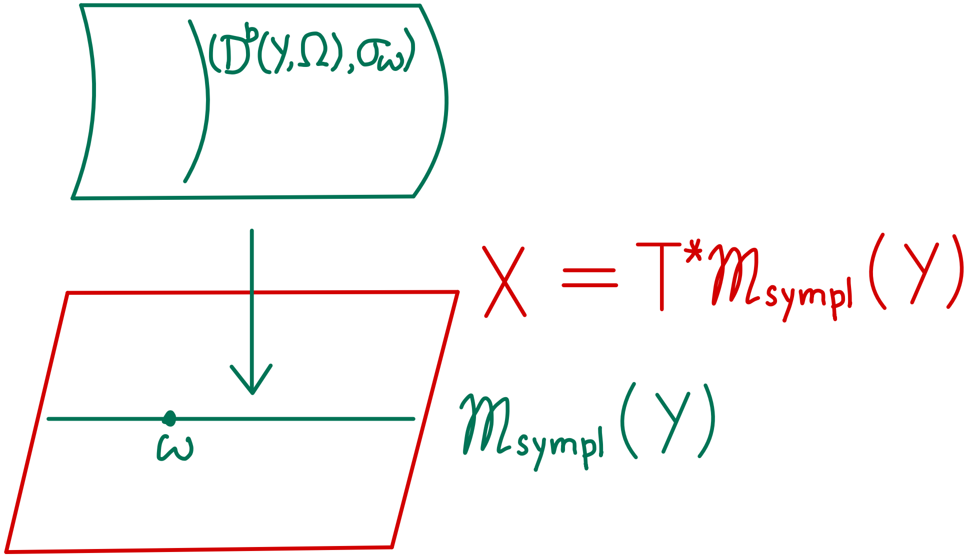
$\downarrow$

C cpx./cpx. Lagr./Lagr. X HK

Y CY_n

holo. sympl.

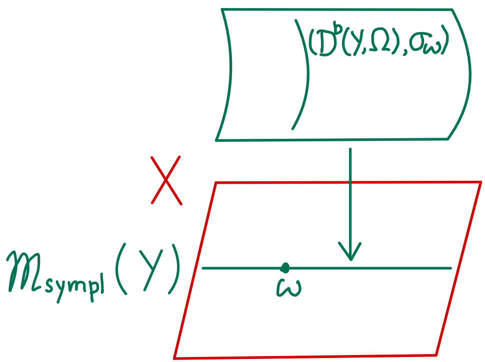
$\forall\, \Omega \in \mathcal{M}_{cpx}(Y) \rightsquigarrow$ 3d A-brane in $X = T^*\mathcal{M}_{sympl}(Y)$

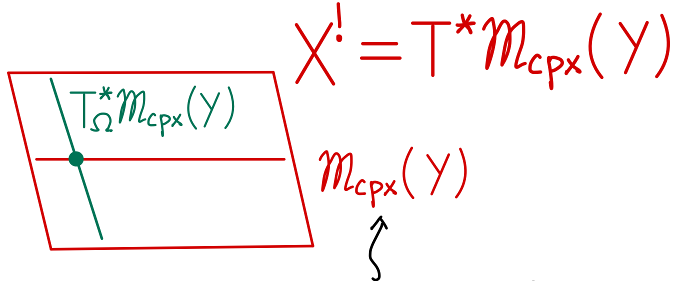


Expect: moduli of such 3d A-branes is $\{\Omega\} = \mathcal{M}_{cpx}(Y)$

$\rightsquigarrow$ 3d SYZ transform

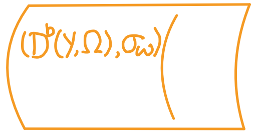
$\forall\, \Omega \in \mathcal{M}_{cpx}(Y)$ $\qquad \mathcal{A} \xleftrightarrow{\text{3d MS}} \mathcal{B}$

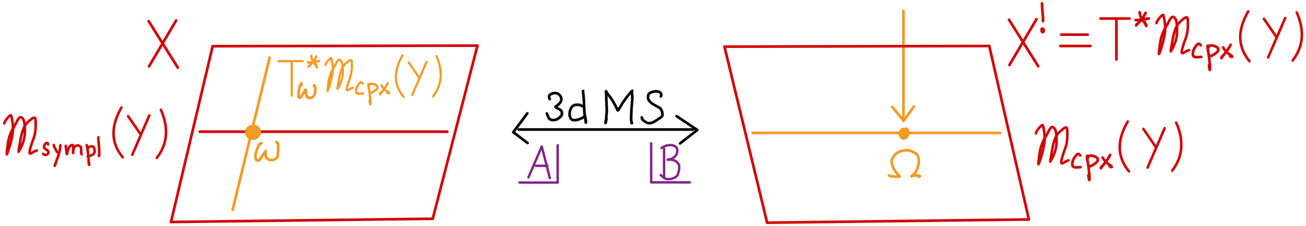




moduli of A-branes.

Claim: $\forall\ \omega \in \mathcal{M}_{sympl}(Y)$





3d SYZ transf., check Hom:

$$\begin{array}{ccc} A\text{-Hom} \big\downarrow & & B\text{-Hom} \big\downarrow \\ (D^b(Y,\Omega),\sigma_\omega) & = & (D^b(Y,\Omega),\sigma_\omega) \end{array}$$

Thus, $T^*\mathcal{M}_{sympl}(Y)$ and $T^*\mathcal{M}_{cpx}(Y)$ exhibit certain 3d SYZ MS

Legacy : CY in MS

Future : CY in 3d MS