Movable cones from mirror symmetry of Calabi-Yau manifolds

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Calabi-Yau 50: Legacy and Future
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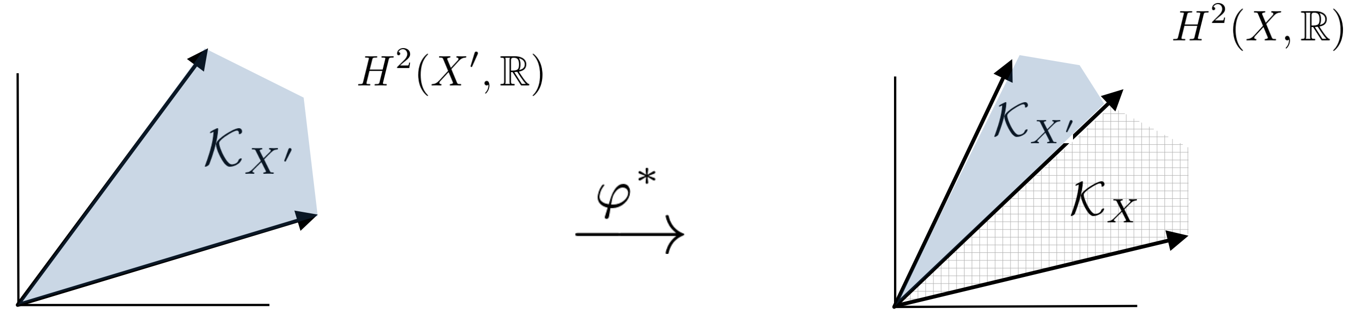
§1. Movable cones and mirror symmetry

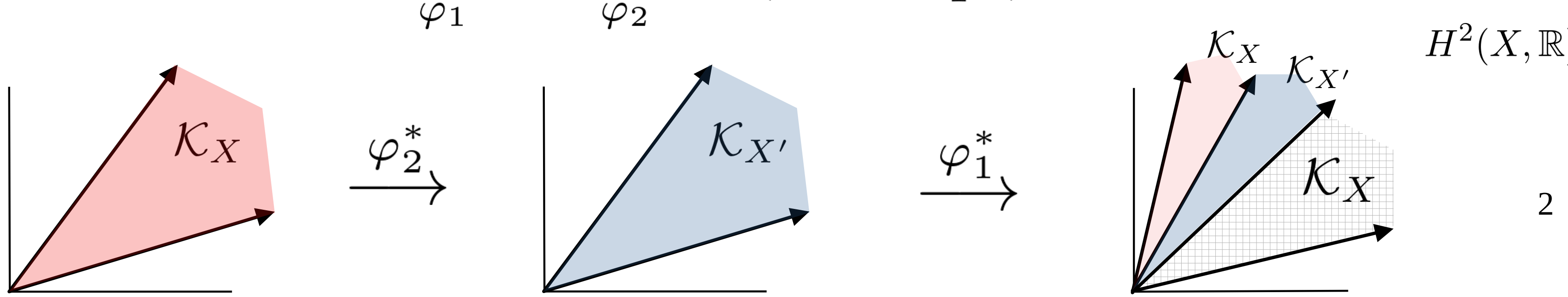
Definitions

- X : Calabi-Yau 3-fold $\Leftrightarrow$
  - $\cdot$ smooth projective
  - $\cdot\ K_X \simeq \mathcal{O}_X$
  - $\cdot\ H^i(X, \mathcal{O}_X) = 0 \quad (0 < i < 3)$
- X, X': Birational $\Leftrightarrow \exists \varphi : X \overset{\sim}{\dashrightarrow} X'$ (cf. bi-holomorphic)

If $X \underset{\varphi}{\overset{\sim}{\dashrightarrow}} X'$, we can glue the Kähler cone



If we have further $X \underset{\varphi_1}{\overset{\sim}{\dashrightarrow}} X' \underset{\varphi_2}{\overset{\sim}{\dashrightarrow}} X$ $(\varphi_2 \neq \varphi_1^{-1})$,

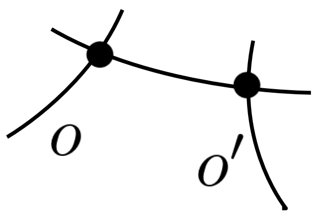


- Our interest today: the cases where we have

$$|\mathrm{BirAut(X)/Aut(X)}| = \infty$$

- Mirror symmery

$$X,\ X'\text{: Birational CY mfds} \quad \overset{\text{ms}}{\longleftrightarrow} \quad \begin{array}{c}\mathcal{Y}\\ \downarrow\\ \mathcal{M}\end{array}$$



Observations (Results):

1. $X_1, X_2, \cdots$: Birational CYs $\overset{\text{ms}}{\longleftrightarrow}$ $o_1, o_2, \cdots \in \mathcal{M}$

2. $\varphi_{ji} : X_i \overset{\sim}{\dashrightarrow} X_j \overset{\text{ms}}{\longleftrightarrow} \check{\varphi}_{ji}$: connection matrix (from o_i to o_j)

3. Kähler cone $\mathcal{K}_i \overset{\text{ms}}{\longleftrightarrow}$ Monodromy nilpotent cone $\mathcal{N}_i$

§2. Examples

Ex.1 $X_1 = \begin{pmatrix} \mathbb{P}^4_z | 11111 \\ \mathbb{P}^4_w | 11111 \end{pmatrix}^{2,52}$

$$\begin{cases} f_1(z,w) = \sum z_i a^{(1)}_{ij} w_j \\ \cdots \\ f_5(z,w) = \sum z_i a^{(5)}_{ij} w_j \end{cases}$$

$$\begin{tikzcd}
 & \mathbb{P}^4_\lambda \times \mathbb{P}^4_z & & \mathbb{P}^4_z \times \mathbb{P}^4_w & & \mathbb{P}^4_w \times \mathbb{P}^4_\lambda & \\
 & \cup & & \cup & & \cup & \\
{} & X_3 \arrow[l, dashed, "\overset{\varphi_{23}}{\sim}"'] \arrow[dl, "50\mathbb{P}^1"] \arrow[dr, "50\mathbb{P}^1"] & & X_1 \arrow[ll, dashed, "\overset{\varphi_{31}}{\sim}"'] \arrow[rr, dashed, "\overset{\varphi_{21}}{\sim}"] \arrow[dl, "50\mathbb{P}^1"] \arrow[dr, "50\mathbb{P}^1"] & & X_2 \arrow[r, dashed, "\overset{\varphi_{32}}{\sim}"] \arrow[dl, "50\mathbb{P}^1"] \arrow[dr, "50\mathbb{P}^1"] & {} \\
Z_3 & & Z_1 & & Z_2 & & Z_3
\end{tikzcd}$$

$$Z_1 = \left\{ \det\left(\sum_k z_k a^{(i)}_{kj} \right) = 0 \right\}, \quad Z_2 = \left\{ \det\left(\sum_k a^{(i)}_{jk} w_k \right) = 0 \right\}, \quad Z_3 = \left\{ \det\left(\sum_k \lambda_k a^{(k)}_{ij} \right) = 0 \right\}$$

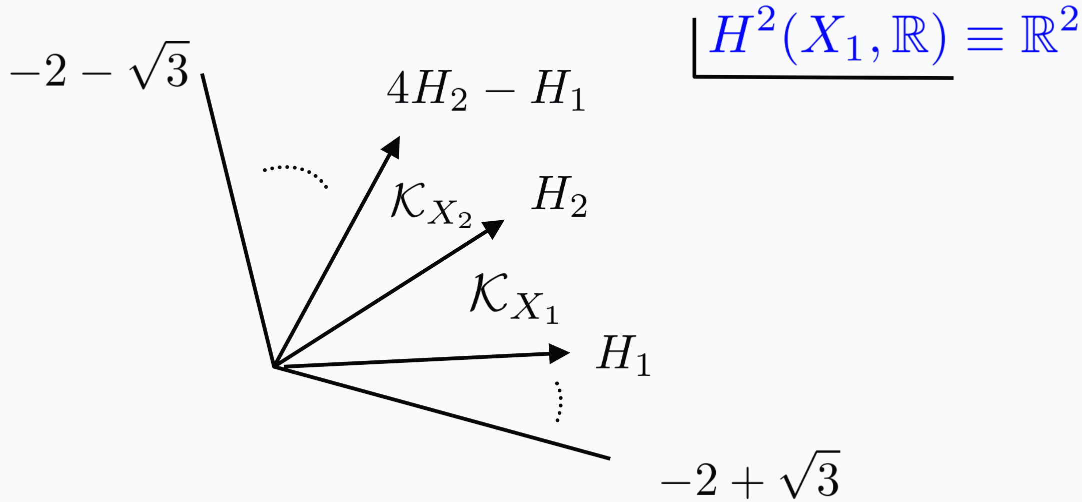
Z_1, Z_2, Z_3 are determinantal quintics with 50 nodes

We naturally have X_2 and X_3, which are birational to X_1.

We arrange the three CY manifolds with their generators of Kähler cones, birational maps φ_{ij} between them as

$$\begin{tikzcd}
 & \{H_1, H_2\} & \\
 & X_1 \arrow[dl, dash, "\varphi_{21}"'] \arrow[dr, dash, "\varphi_{13}"] & \\
\{H_1', H_2'\}\ X_2 \arrow[rr, dash, "\varphi_{32}"'] & & X_3\ \{H_1'', H_2''\}
\end{tikzcd}$$

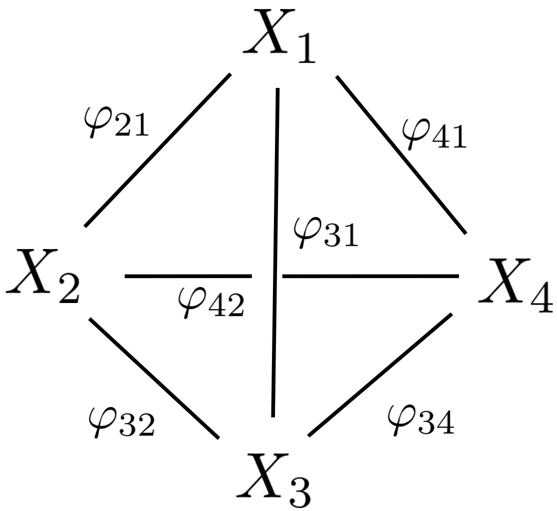
Having relations e.g. $\begin{cases} \varphi_{21}^*(H_2') = H_2 \\ \varphi_{21}^*(H_1') = 4H_2 - H_1 \end{cases}$, we obtain movable cone of X_1;



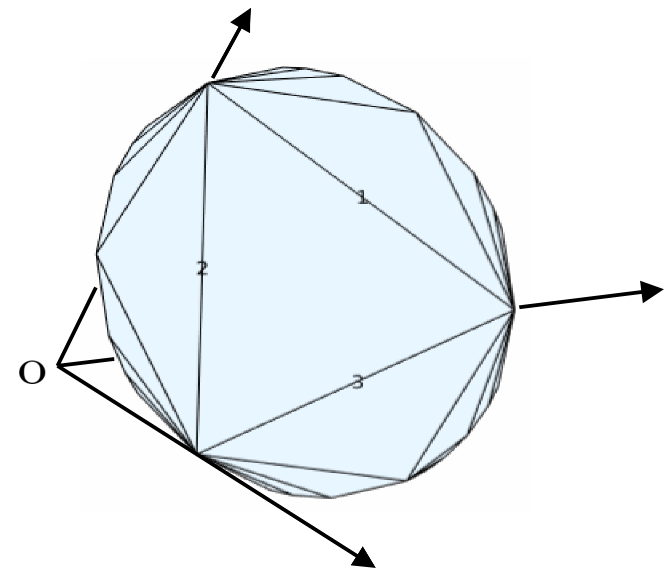
Movable cone of $X_1 = \{x^2 + 4xy + y^2 > 0\}^+ \subset \mathbb{R}^2$

Ex.2 (studied by Knapp, and McGovern (2025) in term of GLSM)

$$X = \begin{pmatrix} \mathbb{P}^2_z | 111 \\ \mathbb{P}^2_w | 111 \\ \mathbb{P}^2_u | 111 \end{pmatrix}^{3,48} \xrightarrow[\pi]{36\mathbb{P}^1} Z = \left\{ \det(a^{(i)}_{jmn} w_m u_n) = 0 \right\} \subset \mathbb{P}^2_w \times \mathbb{P}^2_u$$



Birational models of X



Movable cone of X_1
$= \{xy + yz + zx > 0\}^+ \subset \mathbb{R}^3$

Coxeter Group $\left(\mathbb{Z}^3, \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}\right)$

Ex.3

$$X = \begin{pmatrix} \mathbb{P}^1 | 11 \\ \mathbb{P}^1 | 11 \\ \mathbb{P}^1 | 11 \\ \mathbb{P}^1 | 11 \\ \mathbb{P}^1 | 11 \end{pmatrix}^{5,45}$$

Called as Hulek-Verrill Calabi-Yau manifold, and studied in detail by Candelas, de la Ossa, Kuusela, and McGovern (2021).

Lukas and Ruehle (2023) also study other examples of CICYs.

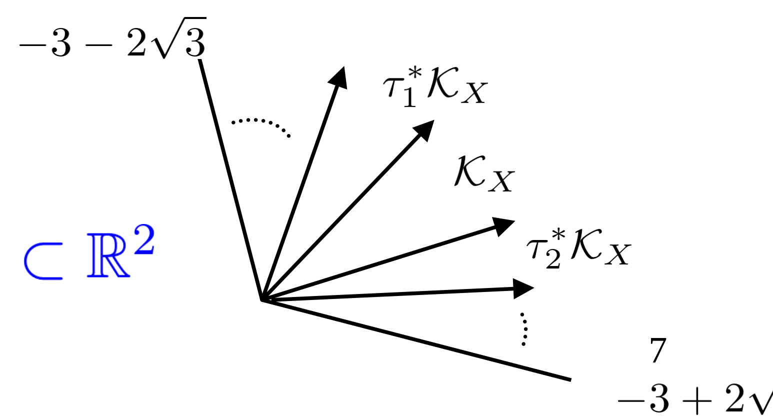
Ex.1′

$$X = \begin{pmatrix} \mathbb{P}^3 | 2111 \\ \mathbb{P}^3 | 2111 \end{pmatrix}^{2,66}$$

$$\begin{tikzcd}
 & X \arrow[dl] & & X \arrow[ll, dashed, "\overset{\tau_2}{\sim}"'] \arrow[dr, "\phi_1"] \arrow[dl, "\phi_2"'] \arrow[ddl, "\pi_2"'] \arrow[ddr, "\pi_1"] \arrow[rr, dashed, "\overset{\tau_1}{\sim}"] & & X \arrow[dr] & \\
W_2^+ \arrow[rr, phantom, "\overset{\tilde{\tau}_2}{\simeq}"] \arrow[drr, "2:1"'] & & W_2 \arrow[d] & & W_1 \arrow[d] \arrow[rr, phantom, "\overset{\tilde{\tau}_1}{\simeq}"] & & W_1^+ \arrow[dll, "2:1"] \\
 & & \mathbb{P}^3 & & \mathbb{P}^3 & &
\end{tikzcd}$$

Proposition 1. W_1 and W_2 are smooth Calabi-Yau threefold, given as double covers of $\mathbb{P}^3$ branched along octics.

Proposition 2. (Oguiso (2014))
(1) $\tau_i : X \dashrightarrow X$ are birational, but not bi-holomorphic.
(2) $\phi_i : X \to W_i$ contract 80 lines and 4 conics.
(3) $\tau_i^2 = \mathrm{id}$ and $\tau_1\tau_2$ has infinite order.



Movable cone of $X = \{x^2+6xy+y^2>0\}^+ \subset \mathbb{R}^2$

§3. Mirror families

For all of our examples, we have mirror families

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \\ \mathcal{M} \end{array}$$

over projective spaces

$$\mathcal{M} = \begin{cases} \mathbb{P}^2 & \text{for } \overset{5}{\cap}(1,1) \text{ in } \mathbb{P}^4 \times \mathbb{P}^4 \\ & \text{and } (2,2) \cap (1,1) \cap (1,1) \text{in } \mathbb{P}^3 \times \mathbb{P}^3 \\ \mathbb{P}^3 & \text{for } \overset{3}{\cap}(1,1,1) \text{ in } \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \\ \mathbb{P}^5 & \text{for } (1,1,\overset{2}{1},1,1) \text{ in } \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \end{cases}$$

We will focus on Ex.1 (and Ex.1′).

• Example: mirror family of $\overset{5}{\cap}(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^4$

We start with a special family of $X \subset \mathbb{P}^4_z \times \mathbb{P}^4_w$;

$$X_{sp} := \left\{ \begin{aligned} & z_1w_1 + az_1w_2 + bz_2w_1 = 0 \\ & \cdots \\ & z_5w_5 + az_5w_1 + bz_5w_1 = 0 \end{aligned} \right\} \subset \mathbb{P}^4_z \times \mathbb{P}^4_w$$

Proposition (HT 2014). For general $a, b \in \mathbb{C}$, it holds

(1) $X_{sp}$is singular along 20 lines of A_1 singularity,

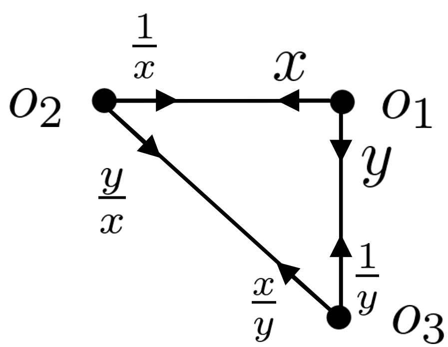
(2) there exists a crepant resolution $Y \to X_{sp}$, giving a Calabi-Yau manifold with $h^{1,1}(Y) = 52$ and $h^{2,1}(Y) = 2$.

(3) the parameter space is natually compactified by $[a^5, b^5, 1] \in \mathbb{P}^2$.

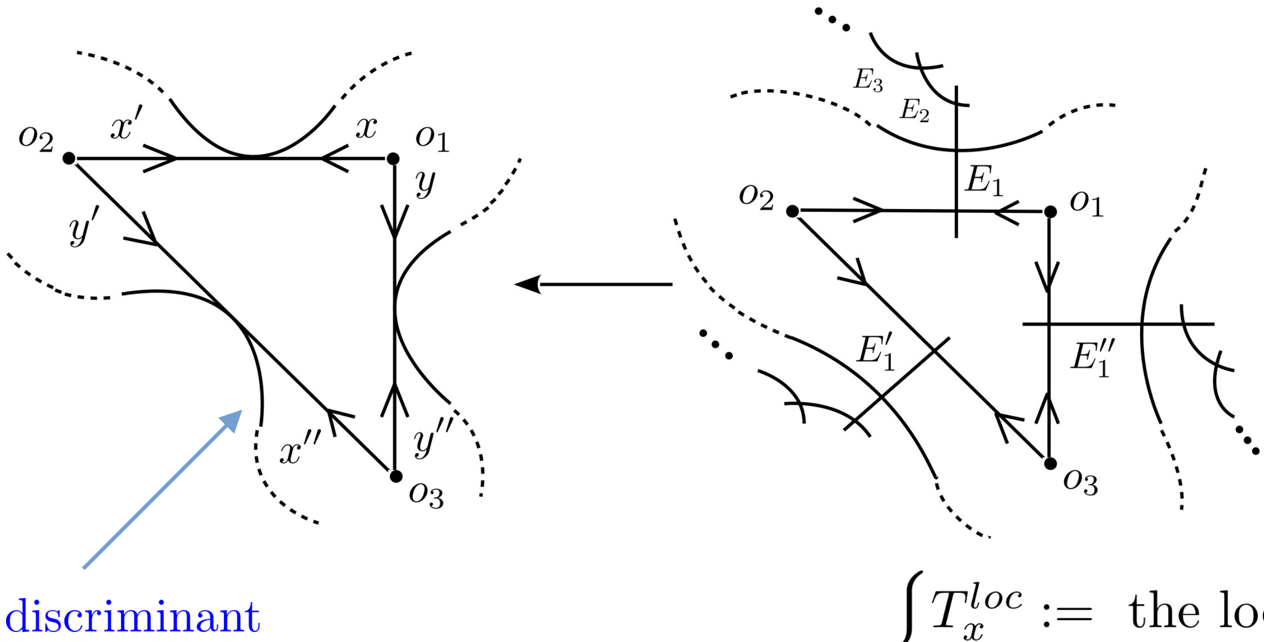
(4) the main component of the discriminants is given by

$$dis_0 = (1-x-y)^5 - 5^4 xy(1-x-y)^2 + 5^5 xy(xy - x - y)$$

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \\ \mathcal{M} \subset \overline{\mathcal{M}} = \mathbb{P}^2 \end{array}$$

$\mathbb{P}^2 =$ 

• **moduli space $\mathcal{M} = \mathbb{P}^2$ (of the mirror family of $\overset{5}{\cap}(1,1)$ in $\mathbb{P}^4 \times \mathbb{P}^4$)**



$\cdot\, o_1, o_2, o_3$ are all LCSLs!

·The blow-up coordinate is

$$\left((x-1), \frac{y}{(x-1)^5}\right) \text{ for } E_1$$

$$\begin{cases} T_x^{loc} := \text{ the local monodromy around } x = 0 \\ T_y^{loc} := \text{ the local monodromy around } y = 0 \end{cases}$$

Definition. (**monodromy nilpotent cone at o_k, $k = 1, 2, 3$**)

$$N_x^{loc} := \log T_x^{loc} = -(1 - T_x^{loc}) - \frac{1}{2}(1 - T_x^{loc})^2 - \frac{1}{3}(T_x^{loc} - 1)^3$$

$$N_y^{loc} := \log T_y^{loc}$$

Then

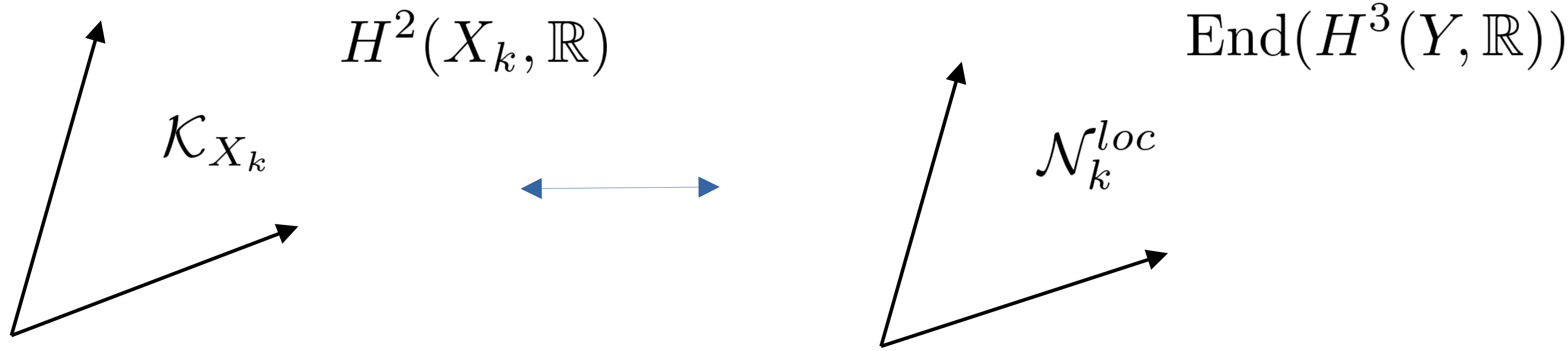
$$\boxed{\mathcal{N}_1^{loc} := \mathbb{R}_{>0} N_x^{loc} + \mathbb{R}_{>0} N_y^{loc}}$$

Similarly for o_2 and o_3, we define $\mathcal{N}_2^{loc}$ and $\mathcal{N}_3^{loc}$

Due to mirror symmetry, we have the following correspondence:

$$\text{semi-ample divisors } H_1, H_2 \quad \leftrightarrow \text{nilpot. matrices } N_x^{loc}, N_y^{loc}$$

With this correspondence, for each k, we naturally have



Remark

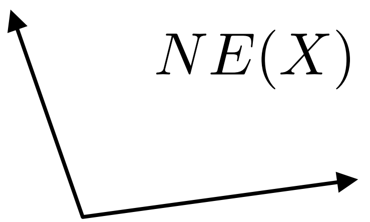
Not only a correspondence, we also have an isomorphism of the products;

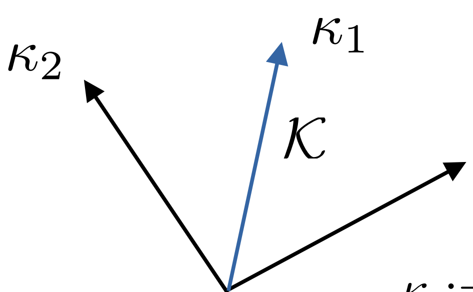
$$H_i H_j H_k = \kappa_{ijk} \, \mathrm{Vol} \quad \leftrightarrow \quad N_i^{loc} N_j^{loc} N_k^{loc} = \kappa_{ijk} \, V_0$$

(Candelas, de la Ossa, Font, Katz, and Morrison (1993))

• A- and B-structures of Calabi-Yau manifolds

$\boxed{\text{A-structure of } X}$





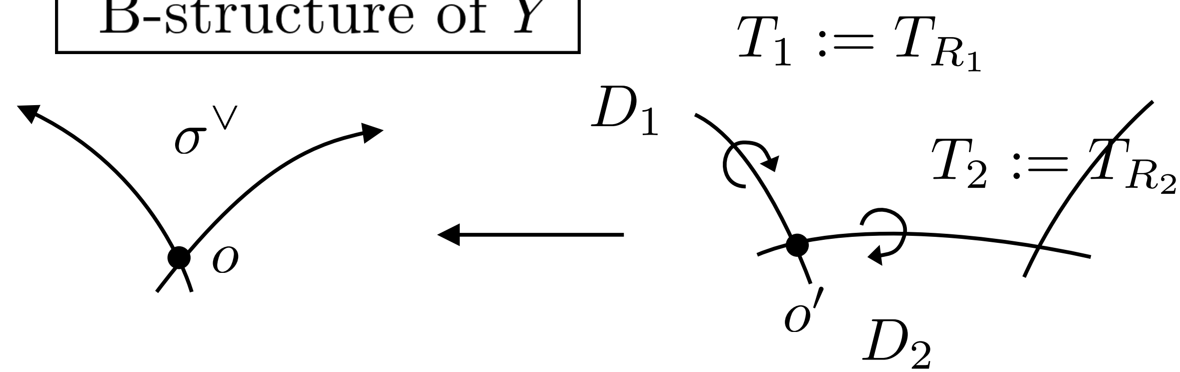
$\kappa := \sum \alpha_i \kappa_i$

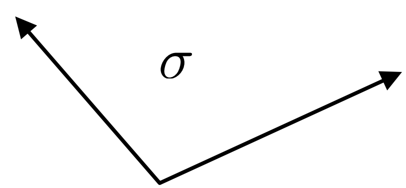
$\boxed{L_\kappa := \kappa \wedge}$

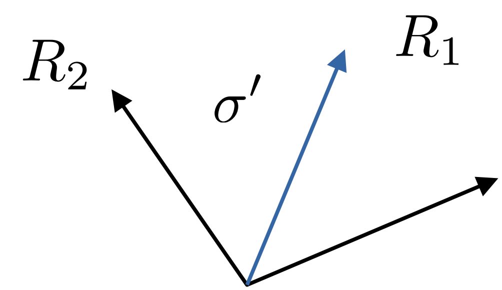
$H^{even}(X) := H^{0,0} \oplus H^{1,1} \oplus H^{2,2} \oplus H^{3,3)}$

Lefschetz action

$\boxed{\text{B-structure of } Y}$







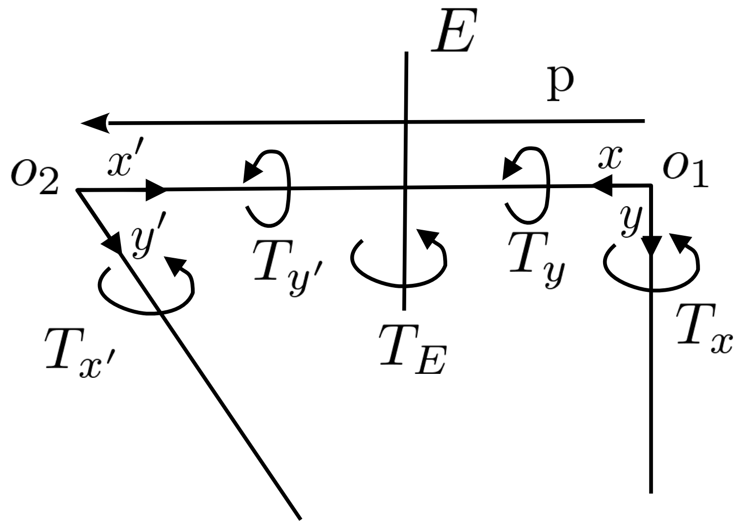
If $T_1, \cdots, T_r$: unipotent

$\boxed{N_\lambda := \sum \lambda_i \log T_i}$

$H^3(Y, \mathbb{R})$

$W^0 \subset W^2 \subset W^4 \subset W^6 = H^3$

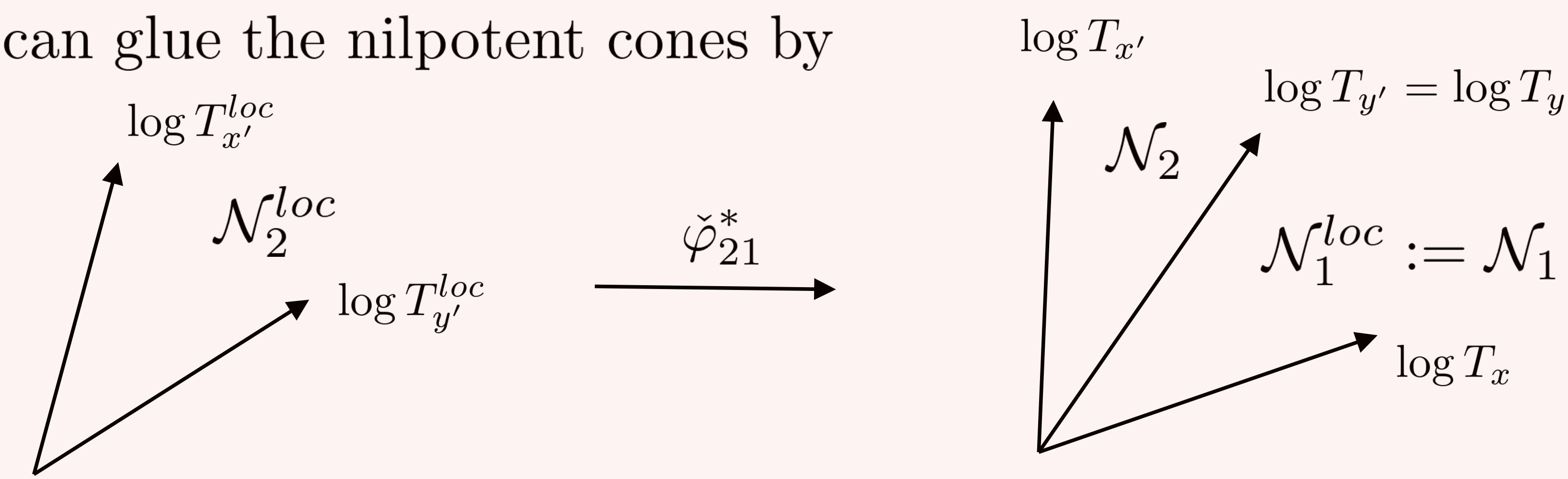
Definition (Gluing monodromy nilpotent cones):



Take a path connecting o_1 and o_2.

$\check{\varphi}_{21}$: connection matrix of the local system

Then we can glue the nilpotent cones by



$$\mathcal{N}_2 := \check{\varphi}_{21}^*(\mathcal{N}_2^{loc}) = \check{\varphi}_{21}^{-1}\mathcal{N}_2^{loc}\check{\varphi}_{21}$$

By analytic continuation $\check{\varphi}_{21}$, we define (global) monodromy matrices by

$$\begin{cases} T_{x'} := \check{\varphi}_{21}^{-1} T_{x'}^{loc} \check{\varphi}_{21} \\ T_{y'} := \check{\varphi}_{21}^{-1} T_{y'}^{loc} \check{\varphi}_{21} \end{cases} \quad \begin{cases} T_{x} := T_{x}^{loc} \\ T_{y} := T_{y}^{loc} \end{cases} \quad \text{and } T_E = \check{\varphi}_{e1}^{-1} T_E^{loc} \check{\varphi}_{e1}$$

Proposition.

We have the following monodromy relations and a property:

$$(1) \begin{cases} T_{x'} = T_E^{-1} T_x^{-1} T_y^4 \\ T_{y'} = T_y \end{cases}$$

$$(2) \begin{cases} \log T_{x'} = 4 \log T_y - \log T_x + \Delta_E \\ \log T_{y'} = \log T_y \end{cases} \qquad \text{cf.} \begin{cases} \varphi_{21}^* H_1' = 4H_2 - H_1 \\ \varphi_{21}^* H_2' = H_2 \end{cases}$$

(3) Over E, we identify the local system of **the mirror quintic**.

(this is mirror to the contraction $X_1 \to Z_2$.)

Remarks:

1. If it were not T_E^{-1} in $T_{x'} = T_E^{-1} T_x^{-1} T_y^4$, we have

$$N_{x'} = 4N_y - N_x \; + \cancel{\Delta_E}$$

But in this case, the mirror symmetry relation $(\tilde{\kappa}_{ijk} = \kappa_{ijk})$

$$\tilde{N}_i \tilde{N}_j \tilde{N}_k = \tilde{\kappa}_{ijk} V_0 \quad \left(\overset{ms}{\longleftrightarrow} \; H_i H_j H_k = \kappa_{ijk} \mathrm{Vol} \right)$$

does not hold (for $\tilde{N}_1 = N_{x'}, \tilde{N}_2 = N_{y'}$).

(However, we have $\kappa_{111} + 50\dfrac{q_1}{1-q_1} = (\tilde{\kappa}_{111} + 50) + 50\dfrac{\tilde{q}_1}{1-\tilde{q}_1}$ with $(\tilde{q}_1 = 1/q_1)$ for the flop)

[Aspinwall,Greene,Morrison 94; Morrison 94]

2. **Picard-Lefschetz monodromy**;

$\{\, \mathrm{A}_0, A_1, A_2, B_2, B_1, B_0 \}$: symplectic basis

$(A_0 \approx T^3, B_0 \approx S^3)$

$$\begin{pmatrix} \int_{A_0} \Omega = 1 + \cdots \\ \int_{A_1} \Omega = \log x + \cdots \\ \int_{A_2} \Omega = \log y + \cdots \\ \vdots \end{pmatrix}$$

Then, we have

$$T_E : \begin{cases} A_1 \to A_1 \\ B_1 \to B_1 - 50 A_1 \end{cases}$$

Picard-Lefschetz monodromy showing the flopping curves!!

§4. **Conclusions:**

1. Birational geometry is encoded nicely in the moduli spaces of mirror Calabi-Yau manifolds!

2. I still do not have a complete picture of the global properties of moduli spaces of Calabi-Yau manifolds.

References:
(1) S.H. and H. Takagi, *Movable vs monodromy nilpotent cones of Calabi-Yau manifolds*, SIGMA (2018).
(2) S.H., *Families of Calabi-Yau manifolds and mirror symmetry*, Surveys in Diff. Geom. (2026), a special volume celebrating Prof. Yau's 75th birthday.

Congratulations on the 50th anniversary of Professor Yau's breakthrough proof of the Calabi conjecture!