Geometric Engineering of SCFTs from Canonical Threefold Singularities

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Motivations

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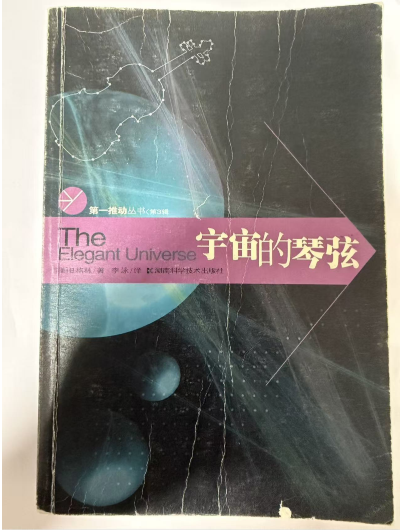
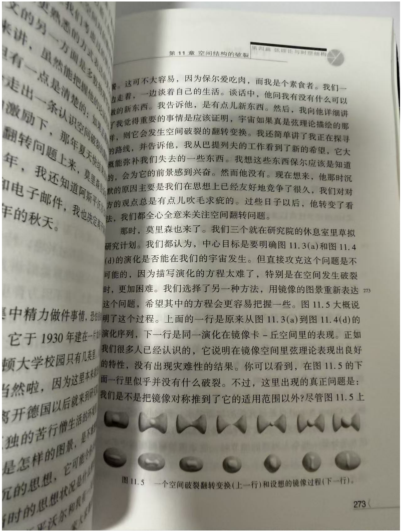
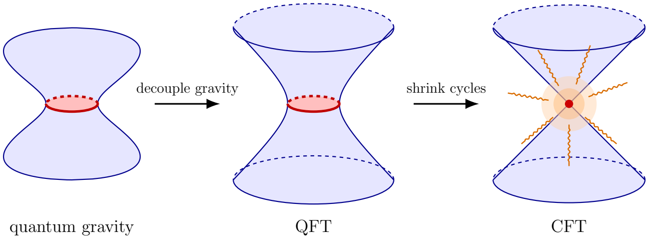




Motivations

• String/M-theory on compact CY space $\rightarrow$ Quantum Gravity theory

• This talk: zoom into singularities in the CY space, what are the local degrees of freedom at a canonical singularity X?



• Desingularization of singularities (resolution/deformation) $\leftrightarrow$ Deformation of the conformal field theory

• D-brane/membrane objects wrapping exceptional cycles on the smooth space $\rightarrow$ Massless, tensionless objects at the singularity/SCFT point

Motivations

• The geometric engineering of SCFTs is a useful application of string theory!

(1) CFTs are fixed points of quantum field theories.

(2) CFTs are also useful to describe critical phenomenon in e. g. condensed matter physics.

(3) From AdS/CFT correspondence, the classification and study of CFTs helps to classify and understand quantum gravity as well.

Motivations

• For higher dimensional SCFTs, the full operator spectrum, OPEs ... are not known

• In $d \geq 5$, the definition of the SCFTs comes from string theory!

• 6d (2,0) SCFTs: IIB on $\mathbb{C}^2/\Gamma$

• 6d (1,0) SCFTs: F-theory on local elliptic CY3

• 5d SCFTs: M-theory on 3-fold singularity X ...

• 4d SCFTs: type II superstring constructions are also very useful!

• 4d string phenomenology motivations:

(1) String vacua with small cosmological constants often appear near singularity in compact CY3

(2) (S)CFT as dark sector, "unparticle physics"

Motivations

• In this talk, X =canonical threefold singularity, consider its crepant resolution $\widetilde{X}$ and miniversal deformation $\widehat{X}$

(Note that $\widetilde{X}$ may have at worst terminal singularities)

• type II superstring/M-theory on $X \rightarrow$ SCFT with eight supercharges

• Such SCFT typically has flat moduli spaces with $V = 0$, e.g. Coulomb branch/Higgs branch, corresponds to the desingularizations of X

(1) $\mathcal{T}_X^{\mathrm{5d}}$ from M-theory on X: CB from $\widetilde{X}$, HB from $\widehat{X}$

(2) $\mathscr{T}_X^{\mathrm{4d}}$ from IIB on X: CB from $\widehat{X}$, HB from $\widetilde{X}$

5d SCFTs from M-theory - Coulomb branch

Basics of 5d $\mathcal{N} = 1$ gauge theories

- 8 supercharges, supermultiplets:

(1) Vector multiplet with gauge group G: A_μ, λ, real scalar ϕ

(2) Hypermultiplets in rep. R: ψ, two complex scalars $h \oplus h^c$

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- Super Yang-Mills action

$$S = \int d^5x \left[\frac{1}{g_{\mathrm{YM}}^2} \mathrm{Tr}(F^{\mu\nu} F_{\mu\nu}) + i\bar{\psi}\not{D}\psi + \dots \right]$$

- Action is non-renormalizable
- For some 5d $\mathcal{N} = 1$ supersymmetric gauge theories, they can be UV completed to a strongly coupled SCFT when $g_{\mathrm{YM}} \to \infty$ (Seiberg 96')(Intriligator, Morrison, Seiberg 96').
- Has a non-trivial UV flavor symmetry G_F

Example: rank-1 theories

- Rank-1 Seiberg $E_N(N \leq 8)$ theories (Seiberg 96')

(1) IR gauge theory description $SU(2) + (N-1)\mathbf{F}$.

(2) IR classical flavor symmetry $G_{F,IR} = SO(2N-2)$ enhanced to $G_F = E_N$.

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- For $N = 1$, there are two theories with different discrete θ-angles

(1) E_1: $SU(2)_0$, $G_F = SU(2)$

(2) $\tilde{E}_1$: $SU(2)_\pi$, $G_F = U(1)$.

- For $N = 0$, E_0 theory does not have a gauge theory description, $G_F = \varnothing$.

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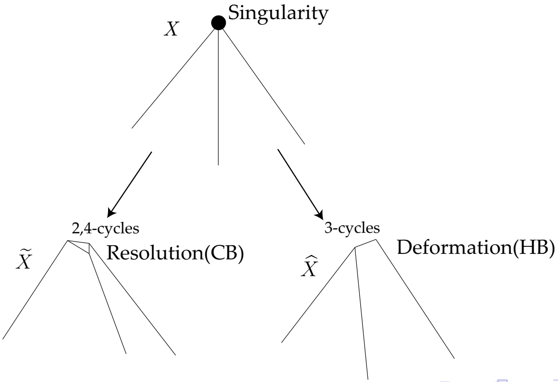
- Exact 1-1 correspondence with the classification of Fano surfaces: dP_N, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$. Rational curves on (g)dP_N has exceptional E_N structure!

M-theory interpretation

• Rank-1 Seiberg E_N theory is defined as M-theory on X = local dP_N singularity!

(1) Coulomb branch: scalars in the vector multiplets have non-zero vev: the crepant resolution $\widetilde{X}$

(2) Higgs branch: scalars in the hypermultiplets have non-zero vev: the deformation $\widehat{X}$



Coulomb branch

- Effective theory: $U(1)^r$ gauge theory+matter fields
- Dimension of Coulomb Branch r: rank of 5d $\mathcal{N}=1$ SCFT
- Coulomb branch parameter: $\langle\phi^i\rangle$

Coulomb branch

- Effective theory: $U(1)^r$ gauge theory+matter fields
- Dimension of Coulomb Branch r: rank of 5d $\mathcal{N}=1$ SCFT
- Coulomb branch parameter: $\langle\phi^i\rangle$
- Effective Lagrangian on CB:

$$\mathcal{L}=G_{ij}(F^i\wedge\star F^j+d\phi^i\wedge\star d\phi^j)+\frac{c_{ijl}}{24\pi^2}A^i\wedge F^j\wedge F^l+\ldots$$

$$G_{ij}=\frac{\partial^2\mathcal{F}}{\partial\phi^i\partial\phi^j}\ ,\ c_{ijl}=\frac{\partial^3\mathcal{F}}{\partial\phi^i\partial\phi^j\partial\phi^l}$$

- Prepotential $\mathcal{F}$ is 1-loop exact:

$$\mathcal{F}=\frac{1}{2g_{\mathrm{YM}}^2}h_{ij}\phi^i\phi^j+\frac{k}{6}d_{ijl}\phi^i\phi^j\phi^l+\frac{1}{12}\left(\sum_{\alpha\in\Phi_G}|\alpha_i\phi^i|^3-\sum_{\lambda\in W_R}|\lambda_i\phi^i+m_f|^3\right).$$

Coulomb branch in M-theory

• M-theory on the (crepant) resolved space $\widetilde{X}$, with new compact 4-cycles S_i

$$C_3 = \sum_{i=1}^{r} A_i \wedge \omega_i \ \leftarrow \text{Poincaré dual to } S_i \text{ in } \widetilde{X}$$

• Rank $r = \#$ compact 4-cycles S_i.

• $\langle \phi^i \rangle \sim$ Kähler parameters of $\widetilde{X}$

• M2-brane wrapping compact 2-cycles $C \subset \widetilde{X}$: 5d particle BPS states with mass $m \propto$Area(C), $U(1)_i$ charge: $C \cdot S_i$.

(1) $N_{C|\widetilde{X}} = \mathcal{O} \oplus \mathcal{O}(-2)$: vector multiplet

(2) $N_{C|\widetilde{X}} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$: hypermultiplet

• c_{ijl} in the 5d $\mathcal{N} = 1$ prepotential: triple intersection number $S_i \cdot S_j \cdot S_l$

• $\mathbb{P}^1$-fibration structure of $\widetilde{X} \rightarrow$ non-abelian gauge theory in the IR

• Partial classification of 5d SCFTs based on the geometries of smooth S_is (IMS)(Xie, Yau)(Bhardwaj, Jefferson, Katz, Kim, Tarazi, Vafa, Zafrir. . .)

Non-isolated singularities

- Resolutions of non-minimal Weierstrass models in 6d F-theory(Apruzzi, Lawrie, Ling, Schafer-Nameki, YNW 19')
- Example: 6d (1,0) rank-1 E-string theory

$$[E_8] - 1$$

- Start with a non-isolated singularity

$$y^2 = x^3 + u^4 x + u^5 v\,.$$

- non-compact elliptic CY3 with base $B = \mathbb{C}^2$
- After the resolution, a single compact 4-cycle S_1 and a number of new non-compact 4-cycles D_α.
- Topology of S_1 depends on the resolution sequence

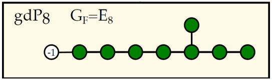
Combined Fiber Diagram

- Using the intersection numbers between non-compact and compact 4-cycles after the resolution, define a Combined Fiber Diagram (CFD)

(1) The non-Abelian UV flavor symmetry $G_{F,nA}$ is given by the Dynkin diagram in green color

(2) Possible IR gauge theory descriptions are constrained by the subgraph structure of CFD;

- Example: rank-1 Seiberg E_8 theory $\sim SU(2) + 7\mathbf{F}$



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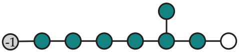
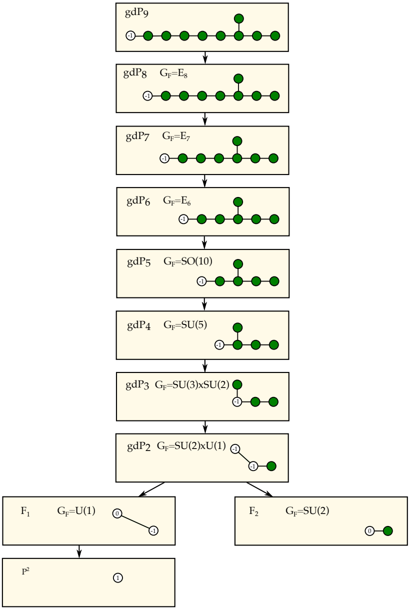


(3) *CFD transition* (mass deformation of IR gauge theory): removing (-1)-vertex, transform the others (corresponds to flops from S_i into D_α)

Combined Fiber Diagram for rank-1 theories

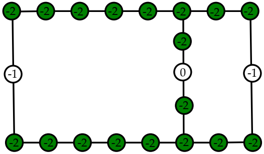


More cases

- S^1 reduction of the minimal (E_8, E_8) Conformal Matter

$$[E_8] - 1 - 2 - \overset{\mathfrak{su}(2)}{2} - \overset{\mathfrak{g}_2}{3} - 1 - \overset{\mathfrak{f}_4}{5} - 1 - \overset{\mathfrak{g}_2}{3} - \overset{\mathfrak{su}(2)}{2} - 2 - 1 - [E_8] \,.$$

- From the topology of 21 compact surfaces $\rightarrow$ CFD for the KK theory



- Many other cases! (Apruzzi, Lawrie, Ling, Schafer-Nameki, YNW 19')(Apruzzi, Schafer-Nameki, YNW 19').

General threefold singularities

- For a general canonical threefold singularity X, after the crepant resolution, $\widetilde{X}$ may still have terminal singularities. The exceptional divisors S_i could also be singular

Case (I): $\widetilde{X}$ is smooth, S_i are all smooth;

Case (II): $\widetilde{X}$ is smooth, some of S_i are singular, with at worst log canonical singularities;

Case (III): $\widetilde{X}$ is singular;

- Classify what happened, for different classes of singularities

Partial Classifications

(1) Isolated hypersurface singularities (IHS)

• IHS threefold singularities classified by (Yau, Yu 03')

• We build computer algorithm to generate systematic "large resolution" of X, by (weighted) blowing up the toric 4-fold ambient space of X, with weights $(1,1,1,1)$, $(1,1,1,2)$, $(1,1,2,3)$, $(1,1,1)$ or $(1,1)$ (Closset, Schafer-Nameki, YNW 20', 21')(Closset, Giacomelli, Schafer-Nameki, YNW 20')

• Classified Case (I) with smooth $\widetilde{X}$ and smooth divisors, up to $r = 10$

• Classified Case (II) with smooth $\widetilde{X}$ and singular divisors, up to $r = 4$

• Infinite classes in Case (III), coupled to terminal singularities

Partial Classifications

(2) Isolated complete intersection singularities (ICIS)

• Classified by (Chen, Xie, Yau, Yau, Zuo 16')

• Similarly build an algorithm to generate the resolved $\widetilde{X}$, by (weighted) blowing up the toric 5-fold ambient space of X (Mu, YNW, Zhang 23')

• Classified Case (I) with smooth $\widetilde{X}$ and smooth divisors, up to $r = 10$

(3) $\mathbb{C}^3/\Gamma$ orbifold singularities, $\Gamma \subset SU(3)$

• Useful in the 3d McKay correspondence

• They by default have a smooth $\widetilde{X}$, either Case (I) or (II), systematically studied the resolution for the cases with either hypersurface, complete intersection or toric descriptions (Tian, YNW 22')

5d SCFT from M-theory - Higgs branch

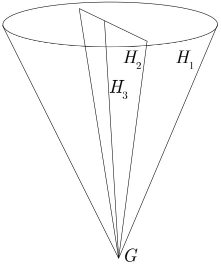
Higgs branch

• vev. of the scalars in hypermultiplets $\langle h \rangle \neq 0$: Higgs branch (HB), hyper-Kähler cone (symplectic singularity)

• Quaternionic dimension: d_H

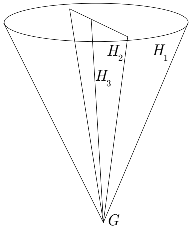
• On a generic point the gauge symmetry G is completely broken, but at special subloci a subgroup $H_i \subset G$ remains:

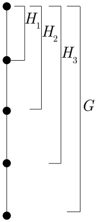
$$\varnothing \subset H_1 \subset H_2 \cdots \subset G$$



Higgs branch

- Hasse diagram: illustrate the layered structure of HB (Bourget, Cabrera, Grimminger, Hanany, Sperling, Zajac, Zheng 19')





- Each layer: elementary slice:

(1) Minimal nilpotent orbit of simple Lie algebra a_n, b_n, c_n, d_n, e_n, f_4, g_2;
$d_H = h^v ee - 1$

(2) ADE singularity $\mathbb{C}^2/\Gamma$, e. g. Kleinian A_N singularity; $d_H = 1$

- $G_{F,nA}$: read off from the bottom layer

HB from an isolated threefold singularity

- Question: can we directly study the HB of 5d SCFT from geometry?
- Answer: deformation of isolated threefold singularity X, from a IIB mirror perspective!

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- IIB on $X \to$ 4d $\mathcal{N} = 2$ SCFT $\mathscr{T}_X^{\mathrm{4d}}$ was studied by (Shapere, Vafa 96')(Xie, Yau 15', 16')(Wang, Xie, Yau, Yau 16')...
- Define a 3d $\mathcal{N} = 4$ theory EQ$^{(4)}$:= reduction of $\mathscr{T}_X^{\mathrm{4d}}$ on S^1, flow to IR
- $\mathscr{T}_X^{\mathrm{4d}}$ and EQ$^{(4)}$ have CB dimension $\widehat{r}_{4d}$.

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- Define a 3d $\mathcal{N}=4$ theory EQ$^{(4)}$:= reduction of $\mathscr{T}_X^{\rm 4d}$ on S^1, flow to IR
- $\mathscr{T}_X^{\rm 4d}$ and EQ$^{(4)}$ have CB dimension $\widehat{r}_{4d}$.

- 5d-4d Correspondence Conjecture (Closset, Schafer-Nameki, YNW 20'): the magnetic quiver of the 5d SCFT MQ$^{(5)}$ is obtained from the gauging of EQ$^{(4)}$ by $U(1)^f$
- f is the flavor rank preserved on the HB of $\mathcal{T}_X^{\rm 5d}$ and $\mathscr{T}_X^{\rm 4d}$

HB from an isolated threefold singularity

• $\mathrm{MQ}^{(5)}$ is another 3d $\mathcal{N}=4$ quiver gauge theory (Ferlito, Hanany, Mekareeya, Zafrir 18')..., related by the T^2 reduction of $\mathcal{T}_X^{\mathrm{5d}}$ by 3d mirror symmetry.

• HB of $\mathcal{T}_{\mathrm{5d}}$ = CB of $\mathrm{MQ}^{(5)}$

• Derivation of the Higgs branch Hasse diagram: quiver subtractions from $\mathrm{MQ}^{(5)}$ (Cabrera, Hanany 18')

Isolated hypersurface singularity X

• Isolated hypersurface singularity (IHS) in $\mathbb{C}^4$

$$F(x_1, x_2, x_3, x_4) = 0\ ,\ \partial_{x_i} F = 0\,.$$

• Quasi-homogeneous: each variable x_i has a weight q_i, such that $F(x)$ has weight 1.

• Example: Seiberg rank-1 E_6 theory:

$$F(x) = x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\,.$$

Deformation of X

- Deformation of quasi-homogeneous IHS $F(x) = 0$: $\widehat{X}$:

$$F(x) \to F(x) + \sum_{l=1}^{\mu} t_l x^{m_l} \,.$$

x^{m_l} are the monomial generators of the Milnor ring

$$\mathcal{M}(F) = \mathbb{C}[x_1, x_2, x_3, x_4]/(\partial_i F)$$

- Total number of generators $\mu =$ the Milnor number of X.

$$\mu = \prod_{i=1}^{4} (\frac{1}{q_i} - 1) \,.$$

Deformation of X

- In Coulomb branch of $\mathscr{T}_X^{\mathrm{4d}}$, the monomials x^{m_l} correspond to operators with scaling dimension

$$\Delta = \frac{1 - q(x^{m_l})}{\sum_i q_i - 1}\,.$$

(1) # of operators with $\Delta > 1$: $\widehat{r}_{4d}$, CB parameters of $\mathscr{T}_X^{\mathrm{4d}}$

(2) # of operators with $\Delta = 1$: f

- In the example of

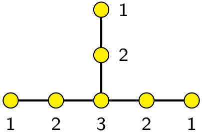
$$F(x) = x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\,.$$

Spectrum:

Δ	1	2	3
#	6	4	1

- Corresponds to the degree of Casimir operator of the gauge group of $\mathscr{T}_X^{\mathrm{4d}}$

- From the spectrum: 4d $SU(N)$ quiver description of $\mathscr{T}_X^{\mathrm{4d}}$, same as $\mathrm{EQ}^{(4)}$:



Higgs branch of 5d SCFT from deformation

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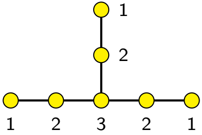
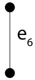


- 5d magnetic quiver $\mathrm{MQ}^{(5)} \cong \mathrm{EQ}^{(4)}/U(1)^6$, $\mathrm{MQ}^{(5)}$ is a $U(N)$ quiver with the same shape, mod out a common U(1)
- Quiver subtraction Affine E_6 Dynkin diagram $\rightarrow$ Hasse diagram



- The HB of rank-1 Seiberg E_6 theory is the minimal nilpotent orbit of E_6

More general cases

• We also studied many cases with fractional 4d Coulomb branch spectrum (Closset, Giacomelli, Schafer-Nameki, YNW 20')(Closset, Schafer-Nameki, YNW 21')

• Conjectures relating the crepant resolution and deformation of X:

(1) Case (I) with smooth $\widetilde{X}$ and smooth divisors, the 4d CB spectrum is all integral

(2) Case (III) with singular $\widetilde{X}$ has a non-integral 4d CB spectrum

4d SCFT from IIB - Higgs branch

Higgs branch of $\mathscr{T}_X^{\rm 4d}$

• Crepant resolution $\widetilde{X}$ of X, new 2/4-cycles $\to$ HB of $\mathscr{T}_X^{\rm 4d}$ (from IIB on X)

• Quaternionic dimension of HB of $\mathscr{T}_X^{\rm 4d}$, $\widehat{d}_{H,4d}$=# of Kähler parameters of $\widetilde{X}$, i.e. $h_{1,1}(\widetilde{X})$.

• Related to the rank r_{5d} and flavor rank f of the 5d SCFT from M-theory on the X (Closset, Schafer-Nameki, YNW 20')...:

$$\widehat{d}_{H,4d} = r_{5d} + f$$

• How to derive the Higgs branch of the 4d $\mathcal{N} = 2$ SCFT from the geometry?

Explored many examples, in our recent paper (YNW, Yan, Yang 26')

• How to relate the CB and HB of the same 4d $\mathcal{N} = 2$ SCFT?

The bridge: vertex operator algebra (VOA)

VOA

- The vertex operator algebra (VOA) is an important information for 4d SCFT, corresponding to a 2d non-unitary CFT with $c_{2d} = -12c_{4d}$ (Beem, Lemos, Liendo, Peelaers, Rastelli, van Rees 13')...
- HB of 4d SCFT = Associated Variety of the VOA
- Schur index of 4d SCFT $\leftrightarrow$ Vacuum character $X_V(\tau)$ of the VOA, can be computed from the Schur index of the BPS quiver of $\mathscr{T}_X^{\mathrm{4d}}$, which is a CB data!
- a, c and CB spectrum of $\mathscr{T}_X^{\mathrm{4d}}$ also contain the information of VOA

$$a - c = -\frac{1}{48}\mathcal{G}(V)$$

- $\mathcal{G}(V)$ is the asymptotic growth of the character of the VOA

$$X_V(\tau) \sim \exp\left(\frac{\pi i \mathcal{G}(V)}{12\tau}\right).$$

Examples 1: generalized conifold

(1) Generalized conifold, $\mathscr{T}_X^{\rm 4d} = (A_1, A_{2N-1})$ Argyres-Douglas theory

$$X : x_1^2 + x_2^2 + x_3^2 + x_4^{2N} = 0\,.$$

• After the small resolution, the exceptional curve $C \cong \mathbb{P}^1$ has normal bundle $N_{C|\widetilde{X}} = \mathcal{O} \oplus \mathcal{O}(-2)$. There is a multiplicity of N 2-cycles shrinking to zero at the singularity, in the same homology class.

• Expand the IIB fields and Kähler form along the single $\omega_1^{(1,1)}$ as

$$B_2 = b_1\omega_1^{(1,1)}\,,\ C_2 = c_1\omega_1^{(1,1)}\,,\ C_4 = d_{2,1}\wedge\omega_1^{(1,1)}\,,\ J = J_1\omega_1^{(1,1)}$$

• Hypermultiplet scalars: $(b_1, c_1, d_1 = (\text{EM dual of } d_{2,1}), J_1)$.

Examples 1: generalized conifold

- The metric of scalars in the hypermultiplet (Ooguri, Vafa 96')(Saueressig, Vandoren 07'):

$$\mathrm{d}s^2 = \tau_2^{-2}[V^{-1}(\mathrm{d}d_1 - \vec{A}\cdot \mathrm{d}\vec{y})^2 + V|\mathrm{d}\vec{y}|^2].$$

with $\vec{y} = (-(c_1 - \tau_1 b_1), z\tau_2, \bar{z}\tau_2)$, $z = b_1 + iJ_1$

$$V = N\left(\frac{1}{4\pi}\ln\left(\frac{1}{z\bar{z}}\right) + \frac{1}{2\pi}\sum_{m\neq 0} K_0(2\pi\tau_2|mz|)\mathrm{e}^{2\pi\mathrm{i}m(c_1-\tau_1 b_1)}\right)$$

- Singular limit $\to \mathcal{M}_H(\mathscr{T}_X^{4\mathrm{d}}) = \mathbb{C}^2/\mathbb{Z}_N$.
- Consistent with the result from magnetic quiver (Xie 12')

$$\mathrm{MQ}^{(4)} = \underset{1}{\bullet}\underset{N}{\text{———}}\underset{1}{\bullet}$$

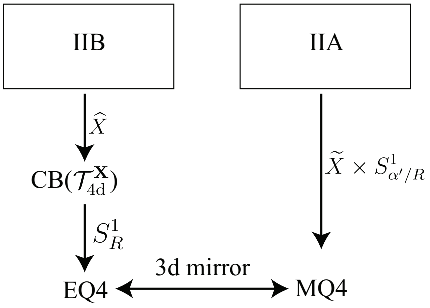
- Associated VOA is known(Song, Xie, Yan 17')(Beem, Rastelli 17')

$$W_{-\frac{N^2}{N+1}}(\mathfrak{sl}_N, [N-1, 1])\,.$$

Examples 2: terminal singularities

(2) General terminal singularities

- $\widetilde{X}$ has f 2-cycles and no compact 4-cycle
- Read off the magnetic quiver of $\mathscr{T}_X^{4\mathrm{d}}$, $\mathrm{MQ}^{(4)}$ from Gopakumar-Vafa (GV) invariants of $\widetilde{X}$! Following T-duality arguments in (Seiberg, Shenker 96')(Hori, Ooguri, Vafa 97')



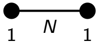
Examples 2: terminal singularities

• $\mathrm{MQ}^{(4)}$ is a $U(1)^f$ gauge theory with charged hypermultiplets from D2-brane wrapping 2-cycles C, with charge $q_\alpha = C \cdot D_\alpha$ under the α-th $U(1)$ gauge group

• These modes are exactly characterized by the $g = 0$ GV invariants of the degree q_α!

• E.g. in the example (A_1, A_{2N-1}) Argyres-Douglas theory

• non-zero GV invariant: $n^{g=0}_{\mathbf{d}=1} = N$, giving rise to

$\mathrm{MQ}^{(4)} =$ 

• Can be generalized to compute the $\mathrm{MQ}^{(4)}$ for many X, following (Collinucci, Sangiovanni, Valandro 21')(Collinucci, De Marco, Sangiovanni, Valandro 21', 22')(De Marco, Sangiovanni, Valandro 22')..., including AD (A_m, A_n) , (A_m, D_n) and (A_{2n-1}, E_7) theories. $\mathrm{MQ}^{(4)}$ reproduced

Associated VOA from geometry

- For 4d $\mathcal{N}=2$ SCFT with a trivial, 0-dimensional HB, corresponds to the associated variety of a lisse VOA. X is a Q-factorial terminal singularity with $r_{5\mathrm{d}}=f=0$.

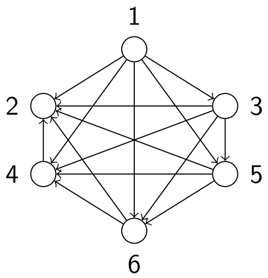
j	b	Singularity	condition
A_{N-1}	N	$x_1^2+x_2^2+x_3^N+z^k=0$	$\mathrm{gcd}(k,N)=1$
D_N	$2N-2$	$x_1^2+x_2^{N-1}+x_2x_3^2+z^k=0$	$\frac{2N-2}{\gcd(k,2N-2)}$ even, $\gcd(k,2N-2)$ odd
	N	$x_1^2+x_2^{N-1}+x_2x_3^2+z^kx_3=0$	$\frac{N}{\gcd(k,N)}$ is even
E_6	12	$x_1^2+x_2^3+x_3^4+z^k=0$	$k\neq 3n$
	9	$x_1^2+x_2^3+x_3^4+z^kx_3=0$	$k\neq 9n$
E_7	18	$x_1^2+x_2^3+x_2x_3^3+z^k=0$	$k\neq 2n$
	14	$x_1^2+x_2^3+x_2x_3^3+z^kx_3=0$	$k\neq 2n$
E_8	30	$x_1^2+x_2^3+x_3^5+z^k=0$	$k\neq 30n$
	24	$x_1^2+x_2^3+x_3^5+z^kx_3=0$	$k\neq 24n$
	20	$x_1^2+x_2^3+x_3^5+z^kx_2=0$	$k\neq 20n$

- Derive the BPS quiver, from the intersection of S^3, $n_{ij}=\Sigma_i\cdot\Sigma_j$ on $\widehat{X}$
- Following Orlik and Randell's conjecture (1977), applicable for

$$f(z_0,\ldots,z_n)=z_0^{a_0}+z_0z_1^{a_1}+\ldots+z_{n-1}z_n^{a_n},\quad n\geq 1.$$

Associated VOA from geometry

- $D_N^N[k]$ singularity given by $x^2 + zy^2 + z^{N-1} + w^k y = 0$
- For the example of $N = 8$, $k = 1$, the BPS quiver is given by



- We can compute the Schur index $\mathcal{I}(q)$ from the BPS quiver following (Cordova, Shao 15')(Cordova, Gaiotto, Shao 16')

Associated VOA from geometry

- For general N and $k = 1$,

$$\mathcal{I}(q) = (q;q)_\infty^{N-2} \sum_{n_i=0}^{\infty} \prod_{i=1}^{N-2} \frac{(-q)^{n_i}}{(q;q)_{n_i}^2} q^{n_i \cdot A \cdot n_i^T}$$

- A is given by the intersection form of 3-cycles in $\widehat{X}$
- For $N = 8$, $k = 1$:

$$\mathcal{I}(q) = 1 - 9q^2 + 7q^3 + 18q^4 - 27q^5 + q^6 + 57q^7 - 45q^8 + \mathcal{O}(q^9)\,.$$

- Such kinds of VOA has no standard name, just the "Calabi-Yau VOA" V_X associated to the singularity X

Example 3: Cases with compact 4-cycles in $\widetilde{X}$

- We studied cases with one-dimensional 4d HB, $f = 0$ in Case (II) (Closset, Schafer-Nameki, YNW 21'):

sing.	$F(x)$	f	r_{5d}	$d_{H,5d}$	$\widehat{r}_{4d}$	$\widehat{d}_{H,4d}$
sing(E_8)	$x_3^7 + x_4^5 x_3 + x_2^3 + x_1^2$	0	1	29	29	1
sing(E_7)	$x_2^5 + x_3^3 x_2 + x_3 x_4^3 + x_1^2$	0	1	17	17	1
sing(E_6)	$x_2^4 + x_3^2 x_2 + x_1^3 + x_3 x_4^2$	0	1	11	11	1

- The Higgs branch of $\mathcal{T}_X^{\rm 5d}$ is the minimal nilpotent orbit of E_n
- The magnetic quiver of $\mathcal{T}_X^{\rm 5d}$ and $\mathscr{T}_X^{\rm 4d}$ are 3d mirror to each other
- For the Higgs branch of $\mathscr{T}_X^{\rm 4d}$, using inversion (Grimminger, Hanany 20')(symplectic duality), they are $\mathbb{C}^2/\Gamma_{E_n}$!

Example 3: Cases with compact 4-cycles in $\widetilde{X}$

- The associated VOA is conjectured to be

$$\mathcal{W}_{k_{2d}}(E_n, \mathcal{O}_{\text{subregular}}), \quad k_{2d} = -h^\vee + \frac{h^\vee}{h^\vee + 1}\,.$$

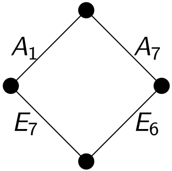
- Central charge matches that of the 4d SCFT, from deformed geometry

$$c_{2d} = -\frac{1342}{13}, -\frac{3706}{19}, -\frac{13978}{31} \text{ for } E_6, E_7, E_8\,.$$

- The associated variety is indeed $\mathbb{C}^2/\Gamma_{E_n}$!

Example 3: Cases with compact 4-cycles in $\widetilde{X}$

- Studied similar cases with 2-dimensional Higgs branch
- We also found a number of new VOAs with no W-algebra descriptions, but with associated varieties e.g.



- Just "Calabi-Yau" VOA

Conclusions

• Extremely rich geometric, algebraic and physical structures from superstring/M-theory on canonical threefold singularities X

• Connection between the resolution/deformation from the VOA of the 4d $\mathcal{N}=2$ SCFT

• More adventurous: into the realm of 4-fold singularities, with less supersymmetry

(1) M-theory $\rightarrow$ 3d $\mathcal{N}=2$ theories (van Beest, Gould, Schafer-Nameki, YNW 22')(Najjar,Tian,YNW 23')

(2) Type II superstring $\rightarrow$ 2d $(2,2)$ or $(0,4)$ theories (Ongoing work with Cvetić, Hübner, Tian, Zhang)

• Generalized symmetries not covered in this talk (see Mirjam Cvetic's talk)

• Learn more about quantum field theories/many body physics from string theory!

• Thanks!