Calabi–Yau Geometry and F-theory: From Anomaly to Finiteness

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In celebration of 50 years of Calabi-Yau manifolds

Part II: joint work with Hee-Cheol Kim and Cumrun Vafa

Gifts in both directions

Calabi–Yau geometry is one of the richest meeting points of mathematics and physics.

Mathematics to physics. Yau's theorem produces Ricci-flat Kähler metrics and made Calabi–Yau manifolds natural internal spaces for supersymmetric string compactification —a profound gift from mathematics to physics.

Physics to mathematics. Mirror symmetry returned the gift: physical arguments revealed unexpected equivalences between Calabi–Yau spaces and made exact predictions in enumerative geometry, opening major new directions in mathematics.

Sources: Yau, *Commun. Pure Appl. Math.* 31 (1978); Candelas–Horowitz–Strominger–Witten, *Nucl. Phys. B* 258 (1985); Candelas–de la Ossa–Green–Parkes, *Nucl. Phys. B* 359 (1991).

The two parts of this talk

This two-way exchange is the theme of the talk.

Part I returns from Calabi–Yau geometry to physics by deriving six-dimensional gravitational anomaly cancellation for F-theory compactifications.

Part II uses anomaly cancellation and the structure of BPS strings to prove finiteness of six-dimensional supergravity and constrain Calabi–Yau geometry.

Both parts combine ideas from geometry and physics. Together they give a partial answer to a finiteness conjecture of Yau.

Sources: Kim–Vafa–Xu, arXiv:2411.19155.

Yau's finiteness conjecture

1984: Witten's question: *How many Calabi–Yau threefolds are there?*

Yau recalled that he then knew only two constructions. He soon constructed at least 10,000, and many more examples followed.

He conjectured that in each dimension there are only finitely many topological types of Calabi–Yau manifolds. In particular, their Hodge numbers should be bounded. The largest value currently realized for a known Calabi–Yau threefold is 491.

Sources: Yau–Nadis, *The Shape of a Life* (2019), pp. 177–178; Yau, Fields Institute lecture (2011).

From Calabi–Yau geometry to quantum gravity

String compactification turns Calabi–Yau geometry into quantum-gravity vacua. In particular, an elliptically fibered Calabi–Yau threefold defines a six-dimensional F-theory vacuum.

Vafa and Douglas broadened Yau's geometric conjecture into a conjecture that the landscape of consistent string and quantum-gravity vacua is finite.

For elliptic Calabi–Yau threefolds, the results discussed here give the sharp bound

$$\boxed{h^{1,1}(X), h^{2,1}(X) \leq 491.}$$

Sources: Douglas, arXiv:hep-th/0303194; Vafa, arXiv:hep-th/0509212; Kim–Vafa–Xu, arXiv:2411.19155.

How large is the class of elliptic Calabi-Yau?

The analogous bound for arbitrary Calabi–Yau threefolds remains conjectural.

There is nevertheless strong evidence that the class of elliptic Calabi-Yau is very large. Among the Kreuzer–Skarke reflexive four-polytopes, 99.994% contain a reflexive two-polytope. This indicates an elliptic phase in the associated birational class.

Thus the sharp elliptic bound gives substantial evidence for the general finiteness picture.

Sources: Taylor, arXiv:1205.0952; Huang–Taylor, arXiv:1907.09482.

Physical proofs and consistency tests

As Brian Greene emphasized yesterday, physicists distinguish different levels of evidence.

Mirror symmetry comes with what physicists regard as a physical proof, while its mathematical proofs proceed by different methods.

For the nonperturbative existence of string theory—and F-theory in particular—no comparable first-principles construction is known. The evidence instead comes from stringent consistency tests.

Gravitational anomaly cancellation is one of the strongest such tests.

Sources: Vafa, arXiv:hep-th/9602022; Grassi–Morrison, arXiv:1109.0042.

Where can gravitational anomalies occur?

A gravitational anomaly is a quantum failure of the path-integral measure to remain invariant under spacetime diffeomorphisms; its cancellation is required for general covariance of the quantum theory.

Local gravitational anomalies can occur in dimensions $D = 4k + 2$; below eleven dimensions these are $D = 2, 6, 10$.

In ten dimensions, Green–Schwarz cancellation was decisive for perturbative superstring theory.

In two dimensions, the Einstein–Hilbert action is topological and gravity is degenerate.

Sources: Alvarez–Gaumé–Witten, *Nucl. Phys. B* 234 (1984); Green–Schwarz, *Phys. Lett. B* 149 (1984).

Why six dimensions?

Six dimensions is therefore the only remaining case and the richest. A uniform geometric proof for F-theory compactifications gives strong evidence for nonperturbative string theory—another gift from mathematics to physics.

Every elliptic Calabi–Yau threefold supplies a separate exact test for the consistency of nonperturbative string theory.

Sources: Vafa, arXiv:hep-th/9602022; Grassi–Morrison, arXiv:1109.0042.

F-theory maps geometry to 6d data

Let $\pi : X \to B$ be an elliptically fibered Calabi–Yau threefold. F-theory associates to it

$$(X \to B) \longmapsto (\Lambda, \mathfrak{g}, R),$$

Here Λ is string charge lattice, $\mathfrak{g}$ is the gauge algebra and R is the hypermultiplet spectrum.

Sources: Vafa, arXiv:hep-th/9602022.

Gauge and matter from the fibration

String charge lattice is easily defined from geometry as $H_2(B, \mathbb{Z})/\mathrm{tors}$

The nonabelian part of $\mathfrak{g}$ comes from codimension-one fibers and monodromy; the free part of Mordell–Weil group gives the abelian part of $\mathfrak{g}$.

Matter R is subtler, and Part I constructs its charged sector intrinsically from BPS sheaves and cohomological Hall algebra.

Different geometries may describe the same low-energy physics. Conversely, not every consistent six-dimensional quantum gravity has an F-theory realization.

Sources: Morrison–Park, arXiv:1208.2695; Weigand, arXiv:1806.01854.

Matter and the gravitational anomaly

Write $T = rk\ \Lambda - 1, V = dim\ \mathfrak{g}$ and $H = dim\ R$. They are number of massless fields in the 6d F-theory.

Cancellation of gravitational anomaly requires

$$\boxed{H - V + 29T = 273.}$$

There are also anomaly from gauge and mixed sector, and they are better understood from geometry.

Sources: Green–Schwarz–West, *Nucl. Phys. B* 254 (1985); Sadov, arXiv:hep-th/9606008; Grassi–Morrison, arXiv:1109.0042.

The six-dimensional spectrum is bounded

In joint work with Hee-Cheol Kim and Cumrun Vafa, we proved that the massless spectrum of six-dimensional $\mathcal{N} = (1,0)$ supergravity is uniformly bounded.

$$\boxed{T \leq 193, \qquad \operatorname{rk}\mathfrak{g} \leq 480, \qquad T + \operatorname{rk}\mathfrak{g} \leq 489.}$$

Here $\operatorname{rk}\mathfrak{g}$ is the total continuous gauge rank, including abelian factors.

Sources: Kim–Vafa–Xu, *SciPost Phys.* 20 (2026) 016, arXiv:2411.19155.

Sharp Hodge-number bounds

For a elliptic Calabi–Yau threefold,

$$h^{1,1}(B) = T + 1, \qquad h^{1,1}(X) = T + \operatorname{rk}\mathfrak{g} + 2,$$

and, for geometric F-theory models,

$$H_0 = h^{2,1}(X) + 1 \leq 492.$$

Consequently,

$$\boxed{h^{1,1}(B) \leq 194, \qquad h^{1,1}(X) \leq 491, \qquad h^{2,1}(X) \leq 491.}$$

These bounds are sharp.

Birkar–Lee independently proved the algebraic bound $h^{1,1}(B) \leq 568$.

Sources: Wazir, arXiv:math/0112259; Kim–Vafa–Xu, arXiv:2411.19155; Taylor, arXiv:1205.0952; Birkar–Lee, arXiv:2507.06317.

Geometry supplies the BPS-string structure

One group of ingredients is visible directly on the base surface:

- the Mori cone, interpreted as the BPS-string cone;
- nonnegative intersections of distinct primitive generators;
- the surface minimal model program: blowdowns and a ruling $B \to \mathbb{P}^1$;
- the classification of superconformal field theory and little string theory configurations, as higher dimensional analogues of Kodaira's classification of singular elliptic fibers.

Sources: Kim–Vafa–Xu, arXiv:2411.19155; Heckman–Morrison–Rudelius–Vafa, arXiv:1502.05405; Bhardwaj et al., arXiv:1511.05565.

Geometry reconstructed from BPS strings

A general quantum gravity theory has no base surface, so these structures must instead be derived from six-dimensional strings and the tensor-branch moduli space. Consistent quantum gravity thus reconstructs Calabi-Yau structures even in absence of geometric compactification.

Hee-Cheol Kim will discuss our past and ongoing work on the BPS-string cone, also joint with Hamada and Jeon, including its block structure and intersection bounds.

Sources: Hamada–Jeon–Kim, arXiv:2607.05496; Kim–Xu, arXiv:2607.26197; Hamada–Kim–Xu.

Physics supplies matter and anomaly equations

The complementary ingredients are the field contents and anomaly cancellation we just discussed, which are basic ingredients in six-dimensional physics, but subtler to construct intrinsically from the Calabi–Yau threefold.

The massless fields must form representations of the gauge algebra. The physical argument uses not only the weighted dimension of the matter sector, but its actual representation structure and the exact anomaly equations.

Sources: Green–Schwarz–West, *Nucl. Phys. B* 254 (1985); Sadov, arXiv:hep-th/9606008; Kim–Vafa–Xu, arXiv:2411.19155.

Part I: construct matter representation

To derive anomaly cancellation, we first need a mathematical construction of the charged matter representation.

Suppose that at a codimension-two point p, the generic gauge algebra $\mathfrak{g}$ enhances to $\tilde{\mathfrak{g}}_p$. If $\mathfrak{h}$ is its centralizer, Katz–Vafa decomposes

$$\mathrm{ad}\,\tilde{\mathfrak{g}}_p = \mathrm{ad}(\mathfrak{g} \oplus \mathfrak{h}) \oplus \bigoplus_a R_a \otimes S_a.$$

The R_a predict the local matter types, but multiplicities can be complicated.

Sources: Katz–Vafa, arXiv:hep-th/9606086.

Earlier approaches

Grassi–Morrison related representations, Euler characteristics, and anomaly cancellation under assumptions on codimension-two fibers. Arras–Grassi–Weigand and Grassi–Weigand later studied selected $\mathbb{Q}$-factorial terminal models using Milnor and deformation data. In those models, the Milnor contribution counts localized uncharged hypermultiplets.

Sources: Grassi–Morrison, arXiv:math/0005196, arXiv:1109.0042; Arras–Grassi–Weigand, arXiv:1612.05646; Grassi–Weigand, arXiv:1804.02424.

Our intrinsic construction

We will give a uniform construction for the charged matter representation from BPS sheaves and cohomological Hall algebra for general elliptic Calabi-Yau threefold, and derive gravitational anomaly cancellation by geometric transition.

Three related compactifications

Let W be the singular Weierstrass model and $X \to W$ a crepant resolution. The same geometry appears in three related compactifications:

$$\text{F-theory on } W \ (6d) \xrightarrow{S^1} \text{M-theory on } X \ (5d),$$

$$\text{M-theory on } X \ (5d) \xrightarrow{S^1} \text{type IIA on } X \ (4d).$$

In type IIA, D2-branes on fibral curves are BPS particles, described mathematically by stable objects in $D^b\,\mathrm{Coh}(X)$.

Sources: Vafa, arXiv:hep-th/9602022; Witten, arXiv:hep-th/9603150.

The BPS algebra recovers six-dimensional data

Harvey–Moore defined an algebra on BPS states through scattering and bound-state formation. Kontsevich–Soibelman give a mathematical construction of 4d BPS algebra as cohomological Hall algebra.

$$D^b\,\mathsf{Coh}(X) \longrightarrow \mathrm{CoHA}$$

Sources: Harvey–Moore, arXiv:hep-th/9609017; Kontsevich–Soibelman, arXiv:1006.2706.

Holomorphic Chern–Simons theory

For a vector bundle E (or more generally, a chain complex) on a Calabi–Yau threefold (X, Ω), let $A \in \Omega^{0,1}(X, \operatorname{End} E)$. The holomorphic Chern–Simons functional is the holomorphic Maurer-Cartan equation

$$S_{\mathrm{hCS}}(A) = \int_X \Omega \wedge \operatorname{Tr}\left(\frac{1}{2} A \wedge \bar{\partial} A + \frac{1}{3} A \wedge A \wedge A\right).$$

Its critical-point equation is

$$\bar{\partial} A + A \wedge A = 0,$$

which says that $\bar{\partial} + A$ defines a holomorphic structure.

This is the complex analogue of ordinary Chern–Simons theory in real dimension 3, whose critical points are flat connections.

Sources: Witten, arXiv:hep-th/9207094.

Vanishing cycles give BPS sheaves

The moduli space of coherent sheaves is a critical locus of the holomorphic Chern–Simons functional. This gives a Lefschetz fibration and its vanishing-cycle sheaf records the local BPS states.

More formally, the Calabi–Yau condition gives the derived moduli stack a (-1)-shifted symplectic structure. A Hall-compatible, or strong associative, orientation glues the local critical charts and makes the vanishing-cycle complex $\varphi_{\mathfrak{M}_X}$ algebraic.

Sources: Pantev–Toën–Vaquié–Vezzosi, arXiv:1111.3209; Brav–Bussi–Joyce, arXiv:1305.6302; Joyce–Upmeier, arXiv:2001.00113.

One-particle and multiparticle BPS states

After passage to the good moduli space, the cohomological integrality theorem identifies a distinguished BPS factor in the lowest perverse piece. Its hypercohomology gives the one-particle BPS states; the total DT cohomology is reconstructed from it by the corresponding multiparticle symmetric algebra.

Sources: Davison–Meinhardt, arXiv:1601.02479.

Exact triangles define the CoHA

Exact triangles $E_1 \to E \to E_2$ give the Hall correspondence

$$\mathfrak{M}_X \times \mathfrak{M}_X \longleftarrow \mathfrak{E}_X \longrightarrow \mathfrak{M}_X.$$

It is compatible with the holomorphic Chern–Simons functional and hence with the vanishing-cycle sheaf.

With the chosen strong associative orientation, pull–push along this correspondence defines an associative cohomological Hall algebra. It records extensions in the derived category and, physically, the scattering and formation of BPS bound states.

This CoHA is the four-dimensional BPS algebra.

Sources: Harvey–Moore, arXiv:hep-th/9510182, arXiv:hep-th/9609017; Kontsevich–Soibelman, arXiv:1006.2706; Kinjo–Park–Safronov, arXiv:2406.12838; Bu et al., arXiv:2502.04253.

From the 4d algebra to the 5d algebra

The full CoHA records the four-dimensional BPS algebra of type IIA on X.

To pass to five dimensions, consider only the D2-brane sector and adjoin the torus extension associated with $H^4(X, \mathbb{Z})$:

$$\boxed{\mathcal{A}_{5d} = \text{the } H^4(X, \mathbb{Z})\text{-torus extension of } \mathcal{A}_{\mathrm{D2}}.}$$

The torus keeps track of the curve-charge grading.
The appropriate zero-mode sector of this five-dimensional BPS algebra will then recover the six-dimensional gauge algebra and matter representation.

Sources: Harvey–Moore, arXiv:hep-th/9609017; Kontsevich–Soibelman, arXiv:1006.2706; Kinjo–Park–Safronov, arXiv:2406.12838.

The fibral category contains 6d states

Let $\pi : X \to B$ be an elliptic Calabi–Yau threefold and define

$$\mathcal{D}_{\mathrm{fib}}(X) = \left\{E \in D^b \operatorname{Coh}(X) : \dim \pi(\operatorname{Supp} E) = 0\right\}.$$

Its objects are supported on finite unions of elliptic fibers. The condition is preserved by shifts, cones, and extensions; for proper support, Calabi–Yau Serre duality makes this a 3-Calabi–Yau subcategory.

Its moduli stack is stratified by the support cycle in B. These support strata may meet when support points collide. This fibral category contains the BPS particles from which the six-dimensional gauge and matter sectors will be extracted.

Sources: Pantev–Toën–Vaquié–Vezzosi, arXiv:1111.3209; Kinjo–Park–Safronov, arXiv:2406.12838.

The support map: three sectors (I)

Generic-base sector. Objects on a smooth elliptic fiber over $B \setminus \Delta$. This gives the universal sector, together with its KK excitations.

Gauge-curve sector. Objects on the reducible fiber over a general point of a discriminant component Σ. The exceptional rational curves generate the gauge-root states and the bulk charged sector.

Sources: Weigand, arXiv:1806.01854.

The support map: three sectors (II)

Defect sector. Objects over a special point $p \in \Sigma$, where the transverse singularity enhances, the gauge curve is singular, or discriminant components meet. Their reduced vanishing cycles give localized matter.

The construction is local on B, so it applies simultaneously to several gauge curves and their intersections.

Sources: Katz–Vafa, arXiv:hep-th/9606086; Grassi–Morrison, arXiv:math/0005196.

ADE roots live over the gauge curve

Let

$$\Sigma^\circ = \Sigma \setminus P,$$

where P is the finite set of enhancement points. Transversely to Σ°, the singular model has a fixed ADE surface singularity.

In a crepant resolution, let C_j be the exceptional fiber curves and D_i the corresponding Cartan divisors. Their pairing is

$$-D_i \cdot C_j = A_{ij}.$$

Thus the classes of the C_j represent the simple roots of the gauge algebra.

Sources: Bershadsky et al., arXiv:hep-th/9605200; Katz–Vafa, arXiv:hep-th/9606086; Grassi–Morrison, arXiv:1109.0042.

Root objects and monodromy

On a local monodromy cover, choose one object for each simple exceptional curve:

$$S_i = \mathcal{O}_{C_i}(-1), \qquad \mathrm{ch}_2(S_i) = [C_i] = \alpha_i, \\ \chi(S_i) = 0.$$

Their curve charges are the simple roots.

With monodromy, the roots form a local system along Σ°. Descending from the cover folds the ADE diagram and gives the physical algebra $\mathfrak{g}$.

Sources: Bridgeland, arXiv:math/0508257; Katz–Vafa, arXiv:hep-th/9606086.

The local quiver

On a transverse ADE surface, their Ext^1-quiver is the doubled Dynkin quiver. Restoring the direction along Σ° adds one loop at each vertex and the canonical cubic potential, giving the tripled quiver.

Dimensional reduction identifies its critical CoHA with the preprojective CoHA. The varying cDV family is obtained by gluing these local models.

Sources: Yang–Zhao, arXiv:1604.01477, arXiv:1604.01865; Davison, arXiv:1311.7172, arXiv:2007.03289; Hennecart–Jindal, arXiv:2602.18102.

The CoHA gives the Serre relations

For Dynkin type, the primitive BPS Lie algebra is $\mathfrak{n}_+$. Its generators satisfy

$$[e_i, e_j] = 0 \quad (A_{ij} = 0), \qquad (\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0.$$

Disjoint simple classes commute. For adjacent vertices, the successive Hall correspondences produce the adjacent-root class, while precisely the Serre combination vanishes.

Thus the positive CoHA gives $U(\mathfrak{n}_+)$. This is the positive half of the gauge algebra constructed directly from the geometry of extensions.

Sources: Yang–Zhao, arXiv:1604.01477; Davison, arXiv:2007.03289; Hennecart–Jindal, arXiv:2602.18102.

The Drinfeld double recovers the gauge algebra

Adjoining the Cartan and forming the reduced Drinfeld double supplies the negative generators f_i and the diagonal relation

$$[e_i, f_j] = \delta_{ij} h_i.$$

The zero-current modes therefore recover $U(\tilde{\mathfrak{g}})$ on the local monodromy cover.

Physically, the shifts $S_i[1]$ represent antibranes; the reduced double supplies the negative half. Descent from the monodromy cover then folds the simply-laced algebra and gives the physical gauge algebra $\mathfrak{g}$.

Sources: Yang–Zhao, arXiv:1407.7994, arXiv:1604.01865; Davison, arXiv:2209.05971; Hennecart–Jindal, arXiv:2602.18102.

Extracting the six-dimensional zero modes

The local CoHA has several independent directions:

$$u = c_1(\mathcal{L}) \in H^2(B\mathbb{G}_m)$$

is the determinant-line descendant; external equivariant parameters describe the Ω-background; $\chi(E)$ records D0/KK charge in the chosen duality frame; and the coordinate on Σ is separate.

These directions are sometimes summarized schematically by a two-variable current algebra, but they have different origins.

Sources: Davison, arXiv:2007.03289, arXiv:2209.05971.

The finite zero-mode reduction

For the six-dimensional reduction, we only need to:

- take primitive BPS cohomology and the normalized zero-KK sector;
- retain the u^0 mode and turn off the external equivariant weights;
- take the monodromy-invariant zero current mode along Σ.

The reduced double on this finite sector is $U(\mathfrak{g})$.

Sources: Davison, arXiv:2209.05971; Hennecart–Jindal, arXiv:2602.18102.

Several fibers form a factorization algebra

Write, in the multiplicity-sensitive colored sense,

$$\mathrm{Ran}_{\mathrm{eff}}(\Sigma) = \bigcup_{n\geq 0} \mathrm{Sym}^n(\Sigma), \qquad s_n : \mathfrak{M}_{\Sigma,n} \to \mathrm{Sym}^n(\Sigma).$$

Push the DT vanishing-cycle complex forward along s_n; its primitive part is the BPS sheaf.

Sources: Davison–Meinhardt, arXiv:1601.02479; Kapranov–Vasserot, arXiv:1901.07641.

Disjoint fibers and collisions

For objects with disjoint supports,

$$\mathrm{RHom}(E_1, E_2) = \mathrm{RHom}(E_2, E_1) = 0.$$

Thom–Sebastiani identifies the sheaf over a configuration of distinct points with the exterior product of the one-point sheaves. When points collide, the Hall correspondence supplies multiplication. These maps form the required factorization structure.

Sources: Kinjo–Park–Safronov, arXiv:2406.12838; Bu et al., arXiv:2502.04253.

Bulk factorization gives the gauge algebra

Restrict the factorization object to a small real disk $D \subset \Sigma^\circ$. Local constancy makes its reduced zero-mode sector an E_2-algebra.

The local ADE calculation and the factorization–Hall comparison identify it with

$$\mathcal{A}_D \simeq_{E_2} C^\bullet(\mathfrak{g}) \simeq \mathrm{RHom}_{U(\mathfrak{g})}(\mathbf{1}, \mathbf{1}).$$

Thus the bulk factorization algebra is Koszul dual to the enveloping algebra of the six-dimensional gauge algebra.

Sources: Beilinson–Drinfeld, *Chiral Algebras*; Kapranov–Vasserot, arXiv:1901.07641; Kinjo–Park–Safronov, arXiv:2406.12838.

Defects give localized matter

At $p \in P$, local constancy fails and the special fiber becomes a factorization defect. After Koszul duality, it gives a module for $U(\mathfrak{g})$: this is the localized matter representation.

Sources: Ayala–Francis–Tanaka, arXiv:1409.0848; Kinjo–Park–Safronov, arXiv:2406.12838.

Vanishing cycles define the local matter space

For one smooth local branch with parameter z, define

$$R_{p,\lambda} = \mathbb{H}^\bullet\big(\phi_z \mathcal{F}^{\mathrm{BPS}}_{\Sigma,\lambda}\big)^{6d}_p, \qquad R_p = \bigoplus_\lambda R_{p,\lambda}.$$

Here the superscript $6d$ means that we take the primitive, normalized zero-KK and zero-current sector and turn off the external Ω-background. For a singular or multibranch point, we use the corresponding multibranch vanishing-cycle complex.

Thus R_p records the new charged BPS states at the defect.

Sources: Davison–Meinhardt, arXiv:1601.02479; Kinjo–Park–Safronov, arXiv:2406.12838.

Geometry defines $\mathfrak{g}$ and R

The constructible factorization structure gives both parts of the six-dimensional charged sector:

- the generic gauge-curve sector gives the gauge algebra $\mathfrak{g}$ and the bulk charged representation R_{bulk}, including genus and monodromy contributions;
- the reduced vanishing cycles at each codimension-two defect give a localized representation R_p;

Sources: Grassi–Morrison, arXiv:1109.0042.

Hall collisions assemble the representation

- the Hall collision maps assemble these into
$$R_{\mathrm{ch}} = R_{\mathrm{bulk}} \oplus \bigoplus_{p \in P} R_p,$$
with its canonical $\mathfrak{g}$-action.

The important point is not only the dimension: the charged states form an actual representation of the gauge algebra.

Sources: Grassi–Morrison, arXiv:1109.0042.

From the representation to anomaly data

For anomaly counting, record the matter representations in the formal weighted class

$$[R_{\mathrm{ch}}]_{\mathrm{anom}} = \sum_{\rho} x_\rho [\rho],$$

$$V = \dim \mathfrak{g}, \qquad H_{\mathrm{nontriv}} = \sum_{\rho} x_\rho \dim \rho, \qquad H = H_0 + H_{\mathrm{nontriv}}.$$

Here $x_\rho = 1$ for a full hypermultiplet and $x_\rho = \frac{1}{2}$ for a half-hypermultiplet. The actual $\mathfrak{g}$-representation was constructed on the preceding slide; the half-weight is used only in the anomaly count.

Sources: Heckman–Morrison–Rudelius–Vafa, arXiv:1502.05405; Bhardwaj et al., arXiv:1511.05565.

The representation supplies the anomaly data

The gravitational anomaly sees the weighted dimension H, while the gauge and mixed anomaly equations see the representation-theoretic indices of each ρ. Our construction therefore supplies precisely the representation and multiplicity data needed in the anomaly calculation.

Sources: Heckman–Morrison–Rudelius–Vafa, arXiv:1502.05405; Bhardwaj et al., arXiv:1511.05565.

The anomaly becomes an Euler identity

In the semisimple, Mordell–Weil rank-zero setting treated by Grassi and Morrison, gravitational anomaly cancellation is equivalent to

$$\boxed{\frac{1}{2}\chi(X) + 30K_B^2 = -\left(\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g}\right) + H_{\mathrm{ch}}.}$$

Sources: Grassi–Morrison, arXiv:1109.0042, Sections 5–6.

The charged dimension in the Euler identity

Here $X \to B$ is the resolved elliptic Calabi–Yau threefold and

$$H_{\mathrm{ch}} = \sum_R x_R \big(\dim R - \dim R[0]\big),$$

with a half-hypermultiplet weighted by $\frac{1}{2}$.

Thus it remains to identify the left-hand side with the geometric BPS count which defines the right-hand side. For nonzero Mordell–Weil rank, the abelian charged sector must be treated separately.

Sources: Grassi–Morrison, arXiv:1109.0042, Sections 5–6.

cDV singularities in the resolvable setting

Let $W \to B$ be the Weierstrass model of an elliptic fibration with section, and let

$$\rho : X = W^{\mathrm{res}} \longrightarrow W$$

be a crepant resolution.

In the crepant-resolvable setting considered here, the relevant singularities of W are cDV.

Sources: Grassi–Morrison, arXiv:math/0005196; Katz–Morrison, arXiv:alg-geom/9202002.

Weierstrass cDV strata

Along a nonabelian discriminant curve, the singularity of W is generically a transverse Du Val, or ADE, surface singularity. At finitely many special points, this transverse type or its monodromy changes.

If one starts instead with a genus-one fibration, its Jacobian is a different space and need not be crepantly resolved by the original threefold. Here we work in the sectioned Weierstrass setting.

Sources: Grassi–Morrison, arXiv:math/0005196; Katz–Morrison, arXiv:alg-geom/9202002.

The left side is an Euler jump

Suppose that a smooth Calabi–Yau Weierstrass model $W_{\mathrm{sm}} \to B$ over the same base exists. Grassi and Morrison show

$$\chi(W_{\mathrm{sm}}) = -60K_B^2.$$

Therefore

$$\frac{1}{2}\chi(X) + 30K_B^2 = \frac{1}{2}\Big(\chi(X) - \chi(W_{\mathrm{sm}})\Big).$$

In this case, the left-hand side of the anomaly formula is precisely the Euler jump of the geometric transition

$$X \longrightarrow W \rightsquigarrow W_{\mathrm{sm}}.$$

Sources: Grassi–Morrison, arXiv:math/0005196, Definition 2.1 and Theorem 2.2.

When the global smoothing does not exist

If a global smoothing does not exist, the left-hand side remains a well-defined invariant, but it should not be called the Euler characteristic of an actual global deformation. Our comparison is then formulated locally.

The cDV singularities admit the local deformations needed below, so the proof only uses local data at the relevant strata.

Sources: Katz–Morrison, arXiv:alg-geom/9202002; Nabijou–Wemyss, arXiv:2109.13289.

Bulk terms and codimension-two defects

Along the smooth part of a gauge curve, the transverse ADE type is constant. For rank r, the exceptional and Milnor fibers have the same Euler characteristic:

$$\chi(E) = r + 1 = \chi(F).$$

Thus there is no new *local defect* at a general point of the curve.

Sources: Elementary ADE calculation; see Katz–Morrison, arXiv:alg-geom/9202002, for the cDV setting.

Bulk matter and codimension-two defects

The gauge curve still contributes bulk matter through its genus and, in the non-simply-laced case, its monodromy cover.
After the bulk genus and monodromy contributions are separated, the remaining contribution is concentrated where the transverse cDV singularity changes at codimension two.

Sources: Grassi–Morrison, arXiv:math/0005196, Theorem 8.2.

Deform to isolated cDV pieces

At each special point, we deform the local configuration on the resolved threefold X so that the remaining contribution separates into finitely many isolated cDV flopping contractions.

Both the localized Euler contribution and the corresponding BPS count are preserved under this deformation. It is therefore enough to establish one local identity for each isolated cDV piece.

The passage from the original non-isolated singularity along a gauge curve to these isolated pieces is a localization step in this work. The Katz–Morrison and Nabijou–Wemyss results apply after this reduction, in the isolated small-contraction setting.

Sources: Katz–Morrison, arXiv:alg-geom/9202002; Nabijou–Wemyss, arXiv:2109.13289.

The isolated cDV equality

For every isolated cDV piece, the Nabijou–Wemyss calculation, following Katz–Morrison, identifies two descriptions of the same local number:

- the Euler-characteristic jump between resolution and deformation;
- the charged BPS multiplicity, equivalently the matter contribution minus the charged vector contribution.

Sources: Katz–Morrison, arXiv:alg-geom/9202002; Nabijou–Wemyss, arXiv:2109.13289.

The local equality proves the anomaly

The BPS construction identifies the isolated cDV curve counts with the localized charged matter contribution. Adding the bulk genus and monodromy terms and summing over all special points gives

$$\boxed{\frac{1}{2}\chi(X) + 30K_B^2 = -\left(\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g}\right) + H_{\mathrm{ch}}.}$$

When a global smooth Weierstrass model exists, the left side is equivalently $\frac{1}{2}\left(\chi(X) - \chi(W_{\mathrm{sm}})\right)$. Thus gravitational anomaly cancellation is an intrinsic consequence of the BPS geometry of the Calabi–Yau threefold.

Sources: Grassi–Morrison, arXiv:math/0005196, arXiv:1109.0042; Nabijou–Wemyss, arXiv:2109.13289.

Part II: the BPS-string cone

We now take anomaly cancellation as the fixed numerical budget of a general six-dimensional theory and combine it with the geometry of its BPS strings.

An effective curve $C \subset B$ gives a D3-brane string whose tension is proportional to its area:

$$\overline{NE}(B) \longleftrightarrow \mathcal{C}_{\mathrm{BPS}}, \qquad T_C \propto J \cdot C.$$

For two distinct irreducible curves on a smooth surface,

$$C_i \cdot C_j \geq 0.$$

Geometrically, this follows immediately from local intersection theory.

Sources: Kim–Vafa–Xu, arXiv:2411.19155, Sections 3.1–3.2.

Positivity directly from supergravity

For an abstract supergravity theory, no surface is given.

The same positivity of distinct primitive BPS-string generators must instead be derived from:

- anomaly equations;
- worldsheet unitarity;
- the behavior of primitive strings and their bound states.

This positivity allows the blowdown algorithm to be reconstructed directly in the string-charge lattice.

Sources: Kim–Vafa–Xu, arXiv:2411.19155, Sections 3.1–3.2; under the SCFT/LST completeness assumptions stated there.

The MMP produces the H-string

A primitive (-1) generator gives an E-string transition: geometrically its curve is contracted; physically the E-string becomes tensionless. Repeating this is the supergravity analogue of the surface minimal model program.

This reconstructs the blowdown process directly in the string-charge lattice.

Sources: Kim–Vafa–Xu, arXiv:2411.19155, Sections 3.1–3.3.

The ruling and the H-string

For a rational F-theory base with $h^{1,1}(B) > 1$, this process produces a ruling

$$p : B \longrightarrow \mathbb{P}^1$$

with fiber class f satisfying

$$f^2 = 0, \qquad -K_B \cdot f = 2.$$

The D3-brane wrapping f is the distinguished H-string. Moreover,

$$C \cdot f = 0 \quad \Longleftrightarrow \quad C \text{ lies in a special fiber of } p.$$

Sources: Kim–Vafa–Xu, arXiv:2411.19155, Sections 3.1–3.3 and 5.

The H-string organizes LST fibers

The fiber class gives a nested fibration

$$X \xrightarrow{\pi} B \xrightarrow{p} \mathbb{P}^1.$$

A singular fiber of p has the form

$$p^{-1}(t_0) = \sum_a d_a C_a \sim f, \qquad f \cdot C_a = 0.$$

Sources: Kim–Vafa–Xu, arXiv:2411.19155, Section 3.3.

The singular fibers are LST sectors

Its components have negative-semidefinite intersection form with the single null class f. They form the tensor branch of an LST whose little-string charge is f.

Thus the BPS generators orthogonal to the H-string are not arbitrary. Geometry assembles them into singular fibers, and physics assembles them into classified LST sectors.

Sources: Bhardwaj et al., arXiv:1511.05565; Kim–Vafa–Xu, arXiv:2411.19155, Section 3.3.

SCFT sectors lie inside LST fibers

A proper negative-definite subconfiguration of a singular fiber can be contracted to an SCFT point.

If

$$q : B \longrightarrow \overline{B}$$

contracts these curves while the Calabi–Yau threefold remains resolved, the induced map

$$X \longrightarrow \overline{B}$$

has a two-dimensional fiber over the contraction point. This is the non-flat locus associated with the SCFT sector.

Sources: Kim–Vafa–Xu, arXiv:2411.19155, Sections 3.3–3.4.

SCFTs sit inside LST fibers

In this precise sense, negative-definite SCFT configurations sit inside the negative-semidefinite singular fibers that carry LSTs. This containment turns the local classification into a global constraint on the BPS-string cone.

Sources: Kim–Vafa–Xu, arXiv:2411.19155, Sections 3.3–3.4.

SCFT and LST fibers have atomic blocks

The classification is organized by a finite list of **nodes and links**.

D- and E-type atoms are nodes $\mathfrak{g}_i$. Finite chains of the remaining atoms joining adjacent nodes are interior links $L_{i,i+1}$; side links may occur at the ends. Schematically,

$$\mathfrak{g}_1 - L_{12} - \mathfrak{g}_2 - \cdots - L_{k-1,k} - \mathfrak{g}_k.$$

Sources: Heckman–Morrison–Rudelius–Vafa, arXiv:1502.05405; Bhardwaj et al., arXiv:1511.05565.

The atomic classification is finite

This is a higher-dimensional analogue of the Kodaira classification. Negative-definite configurations describe SCFT tensor branches; negative-semidefinite configurations with one null direction describe LST tensor branches.

The finite list of nodes, links, and allowed gluings is the main discrete input in the finiteness theorem.

Sources: Kim–Vafa–Xu, arXiv:2411.19155, Appendix A.

Every node–link unit has positive cost

For a sector S, define its contribution to the gravitational anomaly budget by

$$\Delta(S) = H_S + 29\,T_S - V_S.$$

A D- or E-type node can contribute negatively by itself. The correct interior unit is a link together with half of each adjacent node:

$$\Delta(L_{i,i+1}) = \Delta_{\mathrm{link}} + \frac{1}{2}\Delta_{\mathfrak{g}_i} + \frac{1}{2}\Delta_{\mathfrak{g}_{i+1}}.$$

The classification gives the uniform positive bound

$$\boxed{\Delta(L_{i,i+1}) \geq 27 > 0.}$$

Sources: Kim–Vafa–Xu, arXiv:2411.19155, equations (38)–(40), Sections 3.4–4.1 and Appendix A.

Repeatable blocks cannot evade the budget

The repeatable exceptional E_6, E_7, and E_8 conformal-matter units have anomaly cost 30.

The long repeatable $\mathfrak{so}$ conformal-matter units also have cost 30.

Therefore the anomaly cost of an LST fiber grows linearly with its length. No arbitrarily long chain can evade the fixed anomaly budget, so the local node–link classification gives a global finiteness statement.

Sources: Kim–Vafa–Xu, arXiv:2411.19155, Sections 3.4–4.1 and Appendix A.

From anomaly cancellation to finiteness

Part I explains geometrically why F-theory models satisfy anomaly cancellation. For abstract supergravity, the same identity becomes the finite global budget.

Gauge sectors orthogonal to the H-string lie in classified LST fibers; the H-string current algebra bounds the remaining sectors, and blowdowns control the tensor sector. Together with $H + 29T - V = 273$, these ingredients give

$$\boxed{T \leq 193, \qquad \operatorname{rk}\mathfrak{g} \leq 480, \qquad T + \operatorname{rk}\mathfrak{g} \leq 489.}$$

Through the F-theory dictionary, these are the sharp Hodge-number bounds stated at the beginning.

Sources: Kim–Vafa–Xu, arXiv:2411.19155.

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