BPS spectrum from resurgent topological string

Jie Gu

9 Sep 2026 Calabi-Yau at 50 @ Tsinghua University

Yau Center of Southeast University

2211.01403 (w/ Marino), 2305.19916 (w/ Kashani-Poor, Klemm, Marino), 2307.02079, 2403.04528 (w/ Guo), 2607.00880 (w/ Chen, Guo), 260?.????? (w/ Douaud, Guo, Kashani-Poor)

Introduction

Divergent string free energy and topological string

- Perturbative free energies of string theory are divergent [Gross,Periwal'88; Shenker'90]

$$\mathcal{F}(g_s) = \sum_{g=0}^{\infty} \mathcal{F}_g g_s^{2g-2}, \quad \mathcal{F}_g \sim (2g)!$$

Divergent string free energy and topological string

- Perturbative free energies of string theory are divergent [Gross,Periwal'88; Shenker'90]

$$\mathcal{F}(g_s) = \sum_{g=0}^{\infty} \mathcal{F}_g g_s^{2g-2}, \quad \mathcal{F}_g \sim (2g)!$$

- There must be non-perturbative corrections, which presumably come from D-branes. [Polchinski'94]

Divergent string free energy and topological string

- Perturbative free energies of string theory are divergent [Gross,Periwal'88; Shenker'90]

$$\mathcal{F}(g_s) = \sum_{g=0}^{\infty} \mathcal{F}_g g_s^{2g-2}, \quad \mathcal{F}_g \sim (2g)!$$

- There must be non-perturbative corrections, which presumably come from D-branes. [Polchinski'94]
- Relatively simple models like topological string theory. [Bershadsky,Cecotti,Ooguri,Vafa'94]

Divergent string free energy and topological string

- Perturbative free energies of string theory are divergent [Gross,Periwal'88; Shenker'90]

$$\mathcal{F}(g_s) = \sum_{g=0}^{\infty} \mathcal{F}_g g_s^{2g-2}, \quad \mathcal{F}_g \sim (2g)!$$

- There must be non-perturbative corrections, which presumably come from D-branes. [Polchinski'94]
- Relatively simple models like topological string theory. [Bershadsky,Cecotti,Ooguri,Vafa'94]
  - Perturbative free energy is generating series of Gromov-Witten invariants

$$\mathcal{F}(\mathbf{t}, g_s) = \sum_{g=0}^{\infty} \mathcal{F}_g(\mathbf{t}) g_s^{2g-2}, \quad \mathcal{F}_g(\mathbf{t}) = \sum_{\mathbf{d}\in H_2(X)} N_{g,\mathbf{d}} \mathrm{e}^{-\mathbf{t}\cdot\mathbf{d}}.$$

Divergent string free energy and topological string

- Perturbative free energies of string theory are divergent [Gross,Periwal'88; Shenker'90]

$$\mathcal{F}(g_s) = \sum_{g=0}^{\infty} \mathcal{F}_g g_s^{2g-2}, \quad \mathcal{F}_g \sim (2g)!$$

- There must be non-perturbative corrections, which presumably come from D-branes. [Polchinski'94]
- Relatively simple models like topological string theory. [Bershadsky,Cecotti,Ooguri,Vafa'94]
  - ▶ Perturbative free energy is generating series of Gromov-Witten invariants

$$\mathcal{F}(\mathbf{t}, g_s) = \sum_{g=0}^{\infty} \mathcal{F}_g(\mathbf{t}) g_s^{2g-2}, \quad \mathcal{F}_g(\mathbf{t}) = \sum_{\mathbf{d}\in H_2(X)} N_{g,\mathbf{d}} \mathrm{e}^{-\mathbf{t}\cdot\mathbf{d}}.$$

  - ▶ Efficient computational methods such as *holomorphic anomaly equations*.

Resurgence theory

Resurgence is a powerful framework for studying relationship between *perturbative* series and *non-perturbative* corrections. [Ecalle'81]

Resurgence theory

Resurgence is a powerful framework for studying relationship between *perturbative* series and *non-perturbative* corrections. [Ecalle'81]

- Non-perturbative corrections come in terms of *trans-series*

$$F(g_s) = F^{(0)}(g_s) + \sum_{i \geq 1} F^{(i)}(g_s), \quad F^{(i)}(g_s) = \mathrm{e}^{-\mathcal{A}_i/g_s} \sum_n g_s^n F_n^{(i)}.$$

Resurgence theory

Resurgence is a powerful framework for studying relationship between *perturbative* series and *non-perturbative* corrections. [Ecalle'81]

- Non-perturbative corrections come in terms of *trans-series*

$$F(g_s) = F^{(0)}(g_s) + \sum_{i\geq 1} F^{(i)}(g_s), \quad F^{(i)}(g_s) = \mathrm{e}^{-\mathcal{A}_i/g_s} \sum_n g_s^n F_n^{(i)}.$$

- Promotion to functions by *Borel resummation*

$$F(g_s) \to \mathscr{S} F(g_s).$$

Resurgence theory

Resurgence is a powerful framework for studying relationship between *perturbative* series and *non-perturbative* corrections. [Ecalle'81]

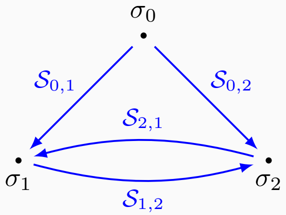
- Non-perturbative corrections come in terms of *trans-series*

$$F(g_s) = F^{(0)}(g_s) + \sum_{i \geq 1} F^{(i)}(g_s), \quad F^{(i)}(g_s) = \mathrm{e}^{-\mathcal{A}_i/g_s} \sum_n g_s^n F_n^{(i)}.$$

- Promotion to functions by *Borel resummation*

$$F(g_s) \to \mathscr{S} F(g_s).$$

- Non-pert. and pert. contributions are related by *Stokes transformation* when moving in complex g_s plane.



Non-perturbative topological string

Conjecture: Non-perturbative contributions to topological string free energies correspond to *stable D-brane bound states* in type II superstring.

- Non-perturbative actions $\mathcal{A}_i$: central charges.
- Stokes constants S_i: multiplicity — BPS or generalized DT invariants.

Non-perturbative topological string

Conjecture: Non-perturbative contributions to topological string free energies correspond to *stable D-brane bound states* in type II superstring.

- Non-perturbative actions $\mathcal{A}_i$: central charges.
- Stokes constants S_i: multiplicity — BPS or generalized DT invariants.

Evidences and arguments

- Non-trivial numerical evidences.
- Various physical arguments.

[Marino'06; Marino,Schiappa,Weiss'07; Marino'08; Pasquetti,Schiappa'09; Aniceto,Schiappa,Vonk'11; Couso,Edelstein,Schiappa,Vonk'13,'14; Grassi,JG,Marino'19; JG,Marino'21; Alim,Saha,Teschner,Tulli'21; Grassi,Hao,Neitzke'22; JG,Marino'22; JG,Kashani-Poor,Klemm,Marino'23;JG'23; Iwaki,Marino'23; Alexandrov,Marino,Pioline'23; Marino,Schwick'24; JG,Guo'24; Douaud,Kashani-Poor'24; Douaud,Kashani-Poor'26; Chen,JG,Guo'26; Douaud,JG,Guo,Kashani-Poor'26??]

Resurgence in a nutshell

Divergent power series

- Perturbative series are usually asymptotic of 1-Gevrey type
$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^{n+1}, \quad a_n \sim n!.$$
which has zero radius of convergence.

Divergent power series

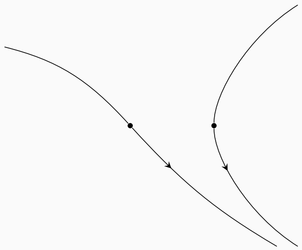
- Perturbative series are usually asymptotic of 1-Gevrey type

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^{n+1}, \quad a_n \sim n!.$$

which has zero radius of convergence.

- Origin: saddle point expansion of exponential integral

$$\phi(z) = \int_\Gamma \mathrm{d}t e^{-S(t)/z} \sim \varphi(z).$$



Divergent power series

- Perturbative series are usually asymptotic of 1-Gevrey type

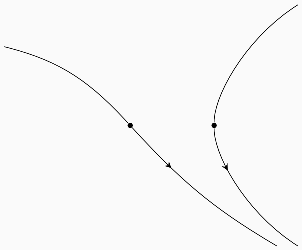
$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^{n+1}, \quad a_n \sim n!.$$

  which has zero radius of convergence.

- Origin: saddle point expansion of exponential integral

$$\phi(z) = \int_\Gamma \mathrm{d}t e^{-S(t)/z} \sim \varphi(z).$$

- Possibility to uplift to function by Lefschetz thimble.



Borel resummation

$$\boxed{\varphi(z) = \sum_{n=0}^{\infty} a_n z^{n+1}}$$

Borel resummation

$$\boxed{\varphi(z) = \sum_{n=0}^{\infty} a_n z^{n+1}} \xrightarrow[\text{Borel transform}]{} \boxed{\mathscr{B}\varphi(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n}$$

Borel resummation

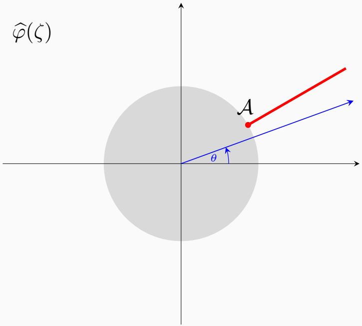
$$\begin{tikzcd}
 & \mathscr{S}^{(\theta)}\varphi(z) = \int_0^{e^{i\theta}\infty} d\zeta e^{-\zeta/z} \mathscr{B}\varphi(\zeta) \arrow[dl, "\text{expansion}"'] \\
\varphi(z) = \sum_{n=0}^{\infty} a_n z^{n+1} \arrow[r, "\text{Borel transform}"'] & \mathscr{B}\varphi(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n \arrow[u, "\text{Borel resummation}"']
\end{tikzcd}$$

Borel resummation

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\phi(z) = \int_\Gamma \mathrm{d}t\, \mathrm{e}^{-S(t)/z}} \arrow[d, dashed, "\text{saddle expansion}"'] & \boxed{\mathscr{S}^{(\theta)}\varphi(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{d}\zeta\, \mathrm{e}^{-\zeta/z}\mathscr{B}\varphi(\zeta)} \arrow[l, dashed, "\text{equal}"'] \arrow[dl, "\text{expansion}"'] \\
\boxed{\varphi(z) = \sum_{n=0}^{\infty} a_n z^{n+1}} \arrow[r, "\text{Borel transform}"'] & \boxed{\mathscr{B}\varphi(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n} \arrow[u, "\text{Borel resummation}"']
\end{tikzcd}
$$

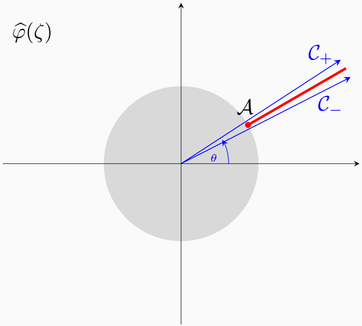
[Malgrange'80; Pham'83,'85; Berry,Howls/Howls'90s]

Ambiguity of Borel resummation



- *Singularities* of $\mathscr{B}\varphi(\zeta)$ and *Stokes rays* obstruct Borel resummation.

Ambiguity of Borel resummation

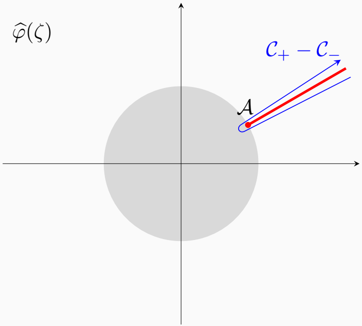


- *Singularities* of $\mathscr{B}\varphi(\zeta)$ and *Stokes rays* obstruct Borel resummation.
- *Discontinuity* across Stokes ray

$$\mathscr{S}^{(\theta_+)}\varphi(z) - \mathscr{S}^{(\theta_-)}\varphi(z) \sim \mathrm{e}^{-\mathcal{A}/z}(b_0 + b_1 z + \ldots) + \ldots$$

encode non-perturbative contributions as *trans-series*.

Ambiguity of Borel resummation

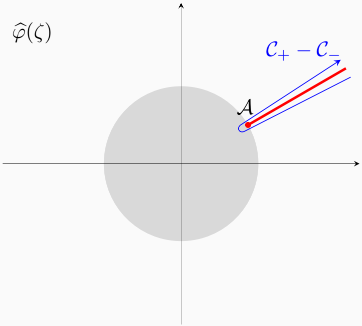


- *Singularities* of $\mathscr{B}\varphi(\zeta)$ and *Stokes rays* obstruct Borel resummation.
- *Discontinuity* across Stokes ray

$$\mathscr{S}^{(\theta_+)}\varphi(z) - \mathscr{S}^{(\theta_-)}\varphi(z) \sim \mathrm{e}^{-\mathcal{A}/z}(b_0 + b_1 z + \ldots) + \ldots$$

encode non-perturbative contributions as *trans-series*.

Ambiguity of Borel resummation



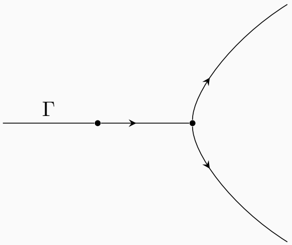
- *Singularities* of $\mathscr{B}\varphi(\zeta)$ and *Stokes rays* obstruct Borel resummation.
- *Discontinuity* across Stokes ray

$$\mathscr{S}^{(\theta_+)}\varphi(z) - \mathscr{S}^{(\theta_-)}\varphi(z) \sim \mathrm{e}^{-\mathcal{A}/z}(b_0 + b_1 z + \ldots) + \ldots$$

  encode non-perturbative contributions as *trans-series*.
- *Resurgence relation*

$$a_n \sim \frac{\Gamma(n)}{\mathcal{A}^n}\left(b_0 + \frac{b_1\mathcal{A}}{n-1} + \ldots\right), \quad n \to \infty.$$

Ambiguity of Borel resummation



- *Singularities* of $\mathscr{B}\varphi(\zeta)$ and *Stokes rays* obstruct Borel resummation.
- *Discontinuity* across Stokes ray

$$\mathscr{S}^{(\theta_+)}\varphi(z) - \mathscr{S}^{(\theta_-)}\varphi(z) \sim \mathrm{e}^{-\mathcal{A}/z}(b_0 + b_1 z + \dots) + \dots$$

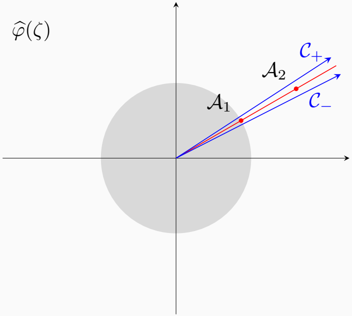
encode non-perturbative contributions as *trans-series*.

- *Resurgence relation*

$$a_n \sim \frac{\Gamma(n)}{\mathcal{A}^n}\left(b_0 + \frac{b_1\mathcal{A}}{n-1} + \dots\right), \quad n \to \infty.$$

- Discontinuity reflects jump of Lefschetz thimble Γ, which is given by thimble of a different saddle.

Stokes automorphisms



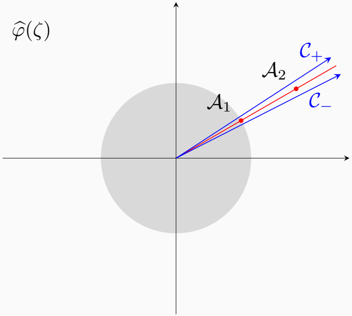
- *Stokes automorphism* of trans-series

$$\mathfrak{S}_d\varphi(z) = \varphi(z) + \sum_{\mathcal{A}\in d} \mathrm{e}^{-\mathcal{A}/z}\varphi^{(\mathcal{A})}(z).$$

so that

$$\mathscr{S}^{(\theta_+)}\varphi(z) = \mathscr{S}^{(\theta_-)}\mathfrak{S}_d\varphi(z)$$

Stokes automorphisms



- *Stokes automorphism* of trans-series

$$\mathfrak{S}_d\varphi(z) = \varphi(z) + \sum_{\mathcal{A}\in d} \mathrm{e}^{-\mathcal{A}/z}\varphi^{(\mathcal{A})}(z).$$

so that

$$\mathscr{S}^{(\theta_+)}\varphi(z) = \mathscr{S}^{(\theta_-)}\mathfrak{S}_d\varphi(z)$$

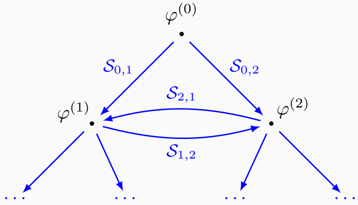
- *Alien derivatives* of perturbative series

$$\sum_{\mathcal{A}\in d} \dot{\Delta}_{\mathcal{A}} = \log \mathfrak{S}_d \;\Rightarrow\; \dot{\Delta}_{\mathcal{A}}\varphi = \mathcal{S}_{\mathcal{A}}\mathrm{e}^{-\mathcal{A}/z}\widetilde{\varphi}^{(\mathcal{A})}(z)$$

describe contributions from individual non-pert. sectors with Stokes constants $\mathcal{S}_{\mathcal{A}}$. [Ecalle'81]

Resurgence structure

- Behind perturbative series are rich non-perturbative contributions
$$\varphi \to \{\mathcal{A}_*, \varphi^{(*)}, \mathcal{S}_*\}$$
- Methods of extraction: Borel transform, resurgence relation, Stokes automorphism, alien derivatives, ...



Perturbative topological string

Topological string free energies

- Topological string is defined on Calabi-Yau threefold X with varying Kähler moduli as generating series of GW invariants $N_{g,\mathbf{d}} \in \mathbb{Q}$

$$\mathcal{F}(\mathbf{t}; g_s) = \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(\mathbf{t}), \quad \mathcal{F}_g(\mathbf{t}) = \sum_{\mathbf{d} \in H_2(X)} N_{g,\mathbf{d}} \mathrm{e}^{-\mathbf{t}\cdot\mathbf{d}}$$

Topological string free energies

- Topological string is defined on Calabi-Yau threefold X with varying Kähler moduli as generating series of GW invariants $N_{g,\mathbf{d}} \in \mathbb{Q}$

$$\mathcal{F}(\mathbf{t}; g_s) = \sum_{g=0}^{\infty} g_s^{2g-2} \mathcal{F}_g(\mathbf{t}), \quad \mathcal{F}_g(\mathbf{t}) = \sum_{\mathbf{d} \in H_2(X)} N_{g,\mathbf{d}} \mathrm{e}^{-\mathbf{t} \cdot \mathbf{d}}$$

- Gopakumar-Vafa formula with GV invariants $n_{g,\mathbf{d}} \in \mathbb{Z}$ [Gopakumar-Vafa'98]

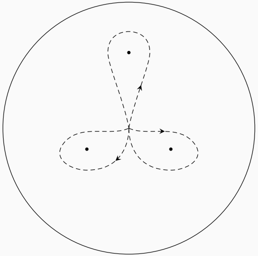
$$\mathcal{F}(\mathbf{t}; g_s) = \sum_{g \geq 0} \sum_{\omega \geq 1} \sum_{\mathbf{d} \in H_2(X)} \frac{n_{g,\mathbf{d}}}{\omega} \left(2 \sin \frac{\omega g_s}{2} \right)^{2g-2} \mathrm{e}^{-\omega \mathbf{t} \cdot \mathbf{d}}$$

Mirror description

- Alternatively, topological string is defined on mirror threefold X^* w/ varying complex structure moduli.
- Complex structure is characterised by *periods* for $A_I, B^J \in H_3(X^*), I, J = 0, 1, \ldots, b_3(X^*)$,

$$\Pi = (\mathcal{X}^I, \mathcal{P}_J)^T = (\int_{A_I} \Omega, \int_{B^J} \Omega)^T,$$

related to t^i of X by mirror map $t^i = \mathcal{X}^i / \mathcal{X}^0$.



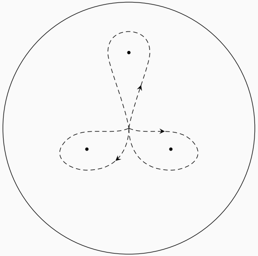
Mirror description

- Alternatively, topological string is defined on mirror threefold X^* w/ varying complex structure moduli.
- Complex structure is characterised by *periods* for $A_I, B^J \in H_3(X^*), I, J = 0, 1, \ldots, b_3(X^*)$,

$$\Pi = (\mathcal{X}^I, \mathcal{P}_J)^T = (\int_{A_I} \Omega, \int_{B^J} \Omega)^T,$$

  related to t^i of X by mirror map $t^i = \mathcal{X}^i / \mathcal{X}^0$.
- Periods undergo monodromy around singular points of moduli space

$$\Pi \to M \cdot \Pi, \quad M \in Sp(2b_3, \mathbb{Z})$$



Mirror description

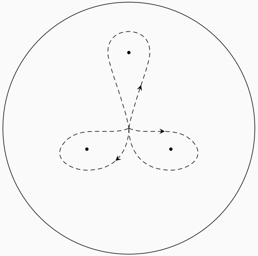
- Alternatively, topological string is defined on mirror threefold X^* w/ varying complex structure moduli.
- Complex structure is characterised by *periods* for $A_I, B^J \in H_3(X^*), I, J = 0, 1, \ldots, b_3(X^*)$,

$$\Pi = (\mathcal{X}^I, \mathcal{P}_J)^T = (\int_{A_I} \Omega, \int_{B^J} \Omega)^T,$$

  related to t^i of X by mirror map $t^i = \mathcal{X}^i / \mathcal{X}^0$.
- Periods undergo monodromy around singular points of moduli space

$$\Pi \to M \cdot \Pi, \quad M \in Sp(2b_3, \mathbb{Z})$$

- Periods are also central charges of D-branes in type II superstring on X or X^*.



Holomorphic anomaly equations

- Free energies $\mathcal{F}_g(t)$ are calculated through holomorphic anomaly equations

  [Bershadsky,Cecotti,Ooguri,Vafa'93]

$$\partial_{\bar{t}} F_g = \frac{1}{2} C_{\bar{t}}^{tt} \left(D_t^2 F_{g-1} + \sum_{h=1}^{g-1} D_t F_h D_t F_{g-h} \right), \quad \mathcal{F}_g(t) = \lim_{\bar{t} \to \infty} F_g(t, \bar{t})$$

$C_{\bar{t}}^{tt}$: an-holomorphic Yukawa coupling, D_t: covariant derivative.

Holomorphic anomaly equations

- Free energies $\mathcal{F}_g(t)$ are calculated through holomorphic anomaly equations

  [Bershadsky,Cecotti,Ooguri,Vafa'93]

$$\partial_{\bar{t}} F_g = \frac{1}{2} C_{\bar{t}}^{tt} \left(D_t^2 F_{g-1} + \sum_{h=1}^{g-1} D_t F_h D_t F_{g-h} \right), \quad \mathcal{F}_g(t) = \lim_{\bar{t} \to \infty} F_g(t, \bar{t})$$

  $C_{\bar{t}}^{tt}$: an-holomorphic Yukawa coupling, D_t: covariant derivative.

- Recursive solution in g, integration constants fixed by extra data such as behavior of $\mathcal{F}_g$ near singular points of moduli space.

Holomorphic anomaly equations

- Free energies $\mathcal{F}_g(t)$ are calculated through holomorphic anomaly equations

  [Bershadsky,Cecotti,Ooguri,Vafa'93]

$$\partial_{\bar{t}} F_g = \frac{1}{2} C_{\bar{t}}^{tt} \left(D_t^2 F_{g-1} + \sum_{h=1}^{g-1} D_t F_h D_t F_{g-h} \right), \quad \mathcal{F}_g(t) = \lim_{\bar{t}\to\infty} F_g(t,\bar{t})$$

  $C_{\bar{t}}^{tt}$: an-holomorphic Yukawa coupling, D_t: covariant derivative.

- Recursive solution in g, integration constants fixed by extra data such as behavior of $\mathcal{F}_g$ near singular points of moduli space.
- Results: all genera for local CY3; up to $g \sim 60$ for compact CY3.

  [Alexandrov, Alim, Feyzbakhsh, Fucito, Grassi, Grimm, Haghighat, Hosono, Huang, Kashani-Poor, Klemm, Krefl, Lange, Lee, Manschot, Mayr, Morales, Pioline, Poretschkin, Quackenbush, Rauch, Savelli, Schimannek, Yamaguchi, Yau, Marino, Walcher, Wang, Weiss, Westerhold-Raum, Wotschke, . . .]

Non-pert. topological string

Non-perturbative actions

- Conjecture: Borel singularities [Drukker,Marino,Putrov'11; Couso,Edelstein,Schiappa,Vonk'13,'14; JG,Marino'22; JG,Kashani-Poor,Klemm,Marino'23]

$$\mathcal{A}_\gamma = c^I \mathcal{P}_I + d_I \mathcal{X}^I, \quad c^I, d_I \in \mathbb{Z}.$$

Non-perturbative actions

- Conjecture: Borel singularities [Drukker,Marino,Putrov'11; Couso,Edelstein,Schiappa,Vonk'13,'14; JG,Marino'22; JG,Kashani-Poor,Klemm,Marino'23]

$$\mathcal{A}_\gamma = c^I \mathcal{P}_I + d_I \mathcal{X}^I, \quad c^I, d_I \in \mathbb{Z}.$$

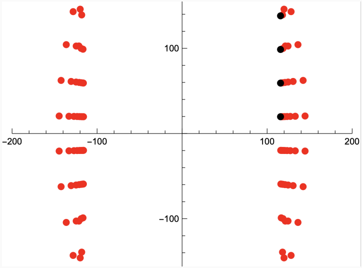
- Central charge of D6-D4-D2-D0 brane bound states with charges $\gamma = (c^0, c^i; d_i, d_0)$.

Non-perturbative actions

- Conjecture: Borel singularities [Drukker,Marino,Putrov'11; Couso,Edelstein,Schiappa,Vonk'13,'14; JG,Marino'22; JG,Kashani-Poor,Klemm,Marino'23]

$$\mathcal{A}_\gamma = c^I \mathcal{P}_I + d_I \mathcal{X}^I, \quad c^I, d_I \in \mathbb{Z}.$$

- Central charge of D6-D4-D2-D0 brane bound states with charges $\gamma = (c^0, c^i; d_i, d_0)$.



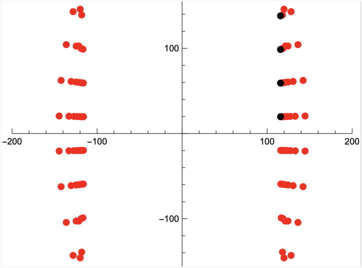
Local $\mathbb{P}^2$ near LR point

Non-perturbative actions

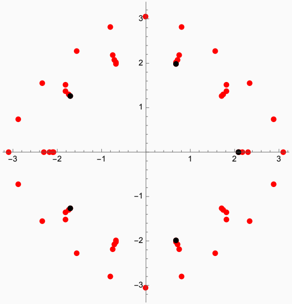
- Conjecture: Borel singularities [Drukker,Marino,Putrov'11; Couso,Edelstein,Schiappa,Vonk'13,'14; JG,Marino'22; JG,Kashani-Poor,Klemm,Marino'23]

$$\mathcal{A}_\gamma = c^I \mathcal{P}_I + d_I \mathcal{X}^I, \quad c^I, d_I \in \mathbb{Z}.$$

- Central charge of D6-D4-D2-D0 brane bound states with charges $\gamma = (c^0, c^i; d_i, d_0)$.



Local $\mathbb{P}^2$ near LR point



Quintic near orbifold point

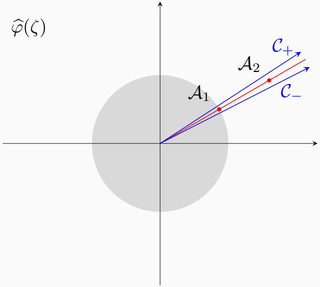
Non-perturbative amplitudes

- Master equation of holomorphic anomaly equations

$$\partial_{\bar{t}} F = \frac{g_s^2}{2} C_{\bar{t}}^{tt} \left(D_t^2 F + (D_t F)^2 \right)$$

is solved by both perturbative F and non-perturbative

$$\mathscr{S}_d F = \mathrm{e}^{\sum_{i \geq 1} \dot{\Delta}_{\mathcal{A}_i}} F = F + \dot{\Delta}_{\mathcal{A}_1} F + \ldots$$



Non-perturbative amplitudes

- Master equation of holomorphic anomaly equations

$$\partial_{\bar{t}} F = \frac{g_s^2}{2} C_{\bar{t}}^{tt} \left(D_t^2 F + (D_t F)^2 \right)$$

is solved by both perturbative F and non-perturbative

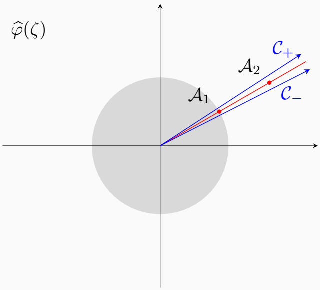
$$\mathscr{S}_d F = \mathrm{e}^{\sum_{i\geq 1} \dot{\Delta}_{\mathcal{A}_i}} F = F + \dot{\Delta}_{\mathcal{A}_1} F + \ldots$$

- Non-perturbative amplitude

$$\dot{\Delta}_\gamma \mathcal{Z} = \frac{\overline{\mathcal{S}}_\gamma}{2\pi \mathrm{i}} (1 + \mathsf{D}_\gamma) \mathrm{e}^{-\mathsf{D}_\gamma} \mathcal{Z},$$

with $\mathcal{Z} = \exp \mathcal{F}$ and

$$\mathsf{D}_\gamma = g_s c^I \partial_{\mathcal{X}^I} + g_s^{-1} d_I \mathcal{X}^I.$$



[Couso,Edelstein,Schiappa,Vonk'13,'14;JG,Marino'22; JG,Kashani-Poor,Klemm,Marino'23; Douaud,Kashani-Poor'26]

Stokes constants

- Conjecture: Stokes constants are (rational) DT invariants

$$\overline{\mathcal{S}}_\gamma = \sum_{n|\gamma} \frac{\mathcal{S}_{\gamma/n}}{n^2}, \quad \mathcal{S}_\gamma = \Omega(\gamma),$$

which are multiplicities of stable D6-D4-D2-D0 bound states with charge γ.

Stokes constants

- Conjecture: Stokes constants are (rational) DT invariants

$$\overline{\mathcal{S}}_\gamma = \sum_{n|\gamma} \frac{\mathcal{S}_{\gamma/n}}{n^2}, \quad \mathcal{S}_\gamma = \Omega(\gamma),$$

which are multiplicities of stable D6-D4-D2-D0 bound states with charge γ.

- Examples

γ	$\Omega(\gamma)$
-1D6-D2-D0	PT invariants
1D6-D2-D0	original DT invariants
D2-D0	GV invariants $n_{0,\mathbf{d}}$
D4-D2-D0	MSW invariants

[Marino'19; JG,Marino'21; JG,Marino'22; JG,Kashani-Poor,Klemm,Marino'23; Iwaki,Marino'23; Douaud,Kashani-Poor'24]

Evidence: GV invariants

- Near large radius point, GV formula implies for $g \to \infty$

$$\mathcal{F}_g(\mathcal{X}) \sim \sum_{\mathbf{d}\in H_2(X)} \sum_{d_0\in\mathbb{Z}} \frac{n_{0,\mathbf{d}}}{2\pi^2} \frac{(2g)!}{(d_i\mathcal{X}^i + d_0\mathcal{X}^0)^{2g-2}} \frac{1}{2g(2g-2)} + \ldots$$

Evidence: GV invariants

- Near large radius point, GV formula implies for $g \to \infty$

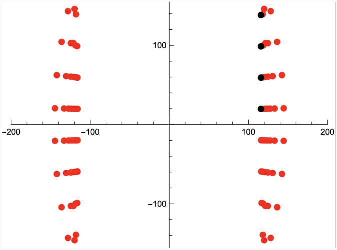
$$\mathcal{F}_g(\mathcal{X}) \sim \sum_{\mathbf{d}\in H_2(X)} \sum_{d_0\in\mathbb{Z}} \frac{n_{0,\mathbf{d}}}{2\pi^2} \frac{(2g)!}{(d_i\mathcal{X}^i + d_0\mathcal{X}^0)^{2g-2}} \frac{1}{2g(2g-2)} + \dots$$

- By comparing with resurgence relation for $\ell \to \infty$

$$a_\ell \sim \mathcal{S}\frac{\Gamma(\ell)}{\mathcal{A}^\ell}\left(b_0 + \frac{b_1\mathcal{A}}{\ell-1} + \dots\right)$$

one concludes that for $\gamma = (0; \mathbf{d}, d_0)$

$$\mathcal{A}_\gamma = d_i\mathcal{X}^i + d_0\mathcal{X}^0$$
$$\mathcal{S}_\gamma = n_{0,\mathbf{d}} = \Omega(0; \mathbf{d}, d_0)$$

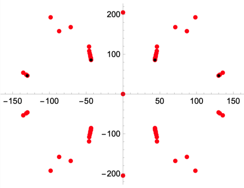


Local $\mathbb{P}^2$ near LR point

[Pasquetti,Schiappa'09; Aniceto,Schiappa,Vonk'11; Couso-Santamaria,Marino,Schiappa'16; JG,Kashani-Poor,Klemm,Marino'23]

Evidence: advanced numerical studies

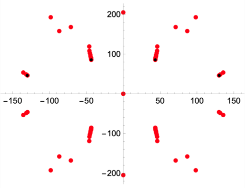
Local $\mathbb{P}^2$ close to large radius point after removing $\mathcal{A}_{(0,\mathbf{d},d_0)}$ [Douaud,Kashani-Poor'26]



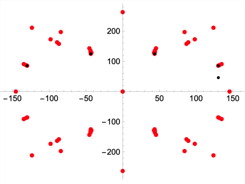
$$\mathcal{S}_{(1,0,0)} = 1$$

Evidence: advanced numerical studies

Local $\mathbb{P}^2$ close to large radius point after removing $\mathcal{A}_{(0,\mathbf{d},d_0)}$ [Douaud,Kashani-Poor'26]



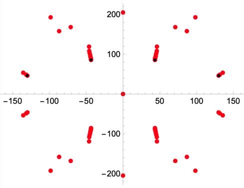
$\mathcal{S}_{(1,0,0)} = 1$



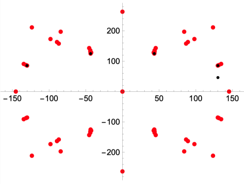
$\mathcal{S}_{(1,0,1)} = 3$

Evidence: advanced numerical studies

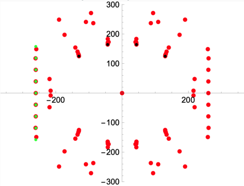
Local $\mathbb{P}^2$ close to large radius point after removing $\mathcal{A}_{(0,\mathbf{d},d_0)}$ [Douaud,Kashani-Poor'26]



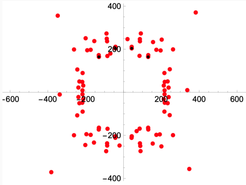
$\mathcal{S}_{(1,0,0)} = 1$



$\mathcal{S}_{(1,0,1)} = 3$



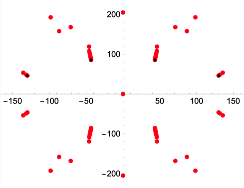
$\mathcal{S}_{(1,0,2)} = 9$



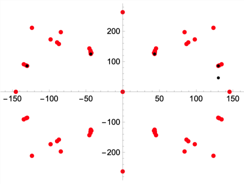
$\mathcal{S}_{(1,0,3)} = 22$

Evidence: advanced numerical studies

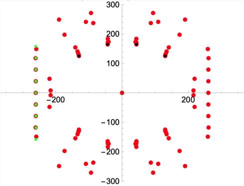
Local $\mathbb{P}^2$ close to large radius point after removing $\mathcal{A}_{(0,\mathbf{d},d_0)}$ [Douaud,Kashani-Poor'26]



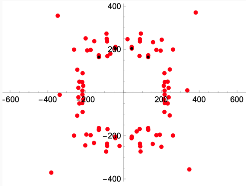
$\mathcal{S}_{(1,0,0)} = 1$



$\mathcal{S}_{(1,0,1)} = 3$



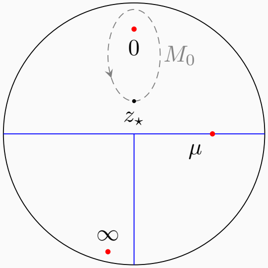
$\mathcal{S}_{(1,0,2)} = 9$



$\mathcal{S}_{(1,0,3)} = 22$

Similar results have been obtained for local $\mathbb{F}_0$ [Douaud,JG,Guo,Kashani-Poor'26??]

Property: modular invariance



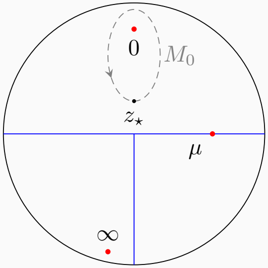
Moduli space

Property: modular invariance

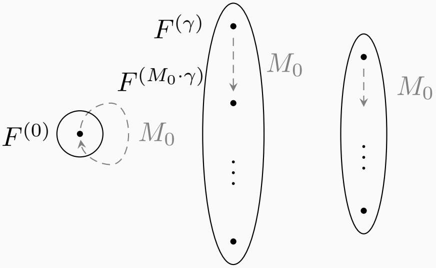
- Non-perturbative contributions are covariant under local monodromy actions.

$$(\gamma, F^{(\gamma)}, \mathcal{S}_\gamma) \to (M \cdot \gamma, F^{(M\cdot\gamma)}, \mathcal{S}_{M\cdot\gamma} = \mathcal{S}_\gamma)$$

[Chen,JG,Guo'26]



Moduli space



pert. and non-pert. sectors

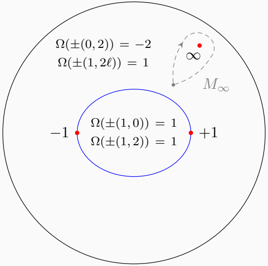
Property: modular invariance

- Non-perturbative contributions are covariant under local monodromy actions.

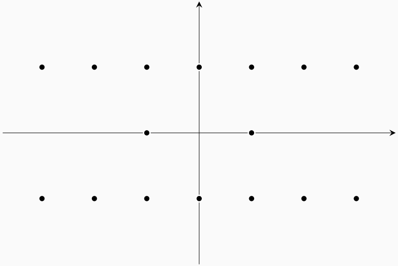
$$(\gamma, F^{(\gamma)}, \mathcal{S}_\gamma) \to (M \cdot \gamma, F^{(M\cdot\gamma)}, \mathcal{S}_{M\cdot\gamma} = \mathcal{S}_\gamma)$$

[Chen,JG,Guo'26]

- Example: Seiberg-Witten theory on self-dual Omega background



Moduli space



Borel plane

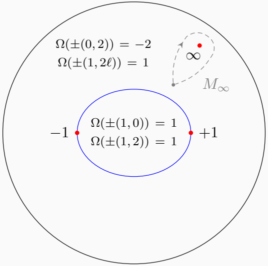
Property: modular invariance

- Non-perturbative contributions are covariant under local monodromy actions.

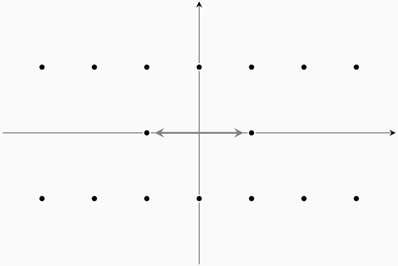
$$(\gamma, F^{(\gamma)}, \mathcal{S}_\gamma) \to (M \cdot \gamma, F^{(M\cdot\gamma)}, \mathcal{S}_{M\cdot\gamma} = \mathcal{S}_\gamma)$$

[Chen,JG,Guo'26]

- Example: Seiberg-Witten theory on self-dual Omega background



Moduli space



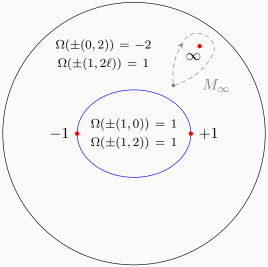
Borel plane

- Non-perturbative contributions are covariant under local monodromy actions.

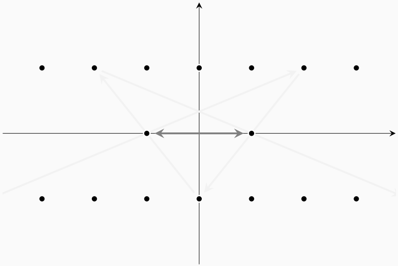
$$(\gamma, F^{(\gamma)}, \mathcal{S}_\gamma) \to (M \cdot \gamma, F^{(M\cdot\gamma)}, \mathcal{S}_{M\cdot\gamma} = \mathcal{S}_\gamma)$$

[Chen,JG,Guo'26]

- Example: Seiberg-Witten theory on self-dual Omega background



Moduli space



Borel plane

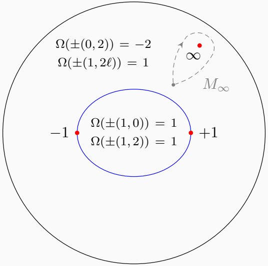
Property: modular invariance

- Non-perturbative contributions are covariant under local monodromy actions.

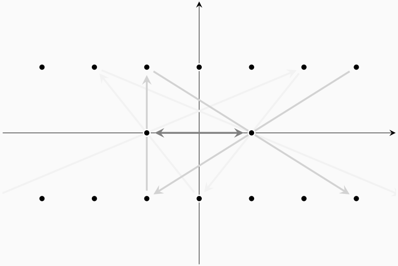
$$(\gamma, F^{(\gamma)}, \mathcal{S}_\gamma) \to (M \cdot \gamma, F^{(M\cdot\gamma)}, \mathcal{S}_{M\cdot\gamma} = \mathcal{S}_\gamma)$$

[Chen,JG,Guo'26]

- Example: Seiberg-Witten theory on self-dual Omega background



Moduli space



Borel plane

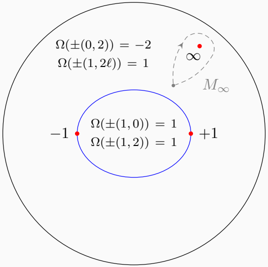
Property: modular invariance

- Non-perturbative contributions are covariant under local monodromy actions.

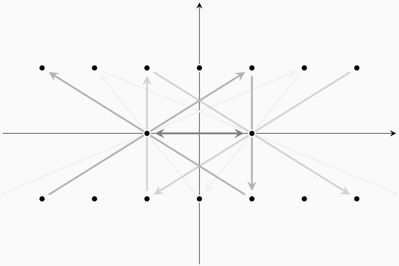
$$(\gamma, F^{(\gamma)}, \mathcal{S}_\gamma) \to (M \cdot \gamma, F^{(M\cdot\gamma)}, \mathcal{S}_{M\cdot\gamma} = \mathcal{S}_\gamma)$$

[Chen,JG,Guo'26]

- Example: Seiberg-Witten theory on self-dual Omega background



Moduli space



Borel plane

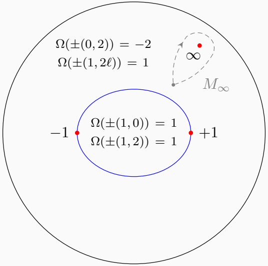
Property: modular invariance

- Non-perturbative contributions are covariant under local monodromy actions.

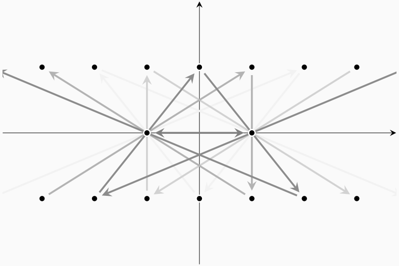
$$(\gamma, F^{(\gamma)}, \mathcal{S}_\gamma) \to (M\cdot\gamma, F^{(M\cdot\gamma)}, \mathcal{S}_{M\cdot\gamma} = \mathcal{S}_\gamma)$$

[Chen,JG,Guo'26]

- Example: Seiberg-Witten theory on self-dual Omega background



Moduli space



Borel plane

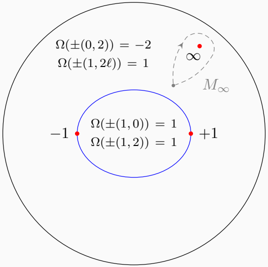
Property: modular invariance

- Non-perturbative contributions are covariant under local monodromy actions.

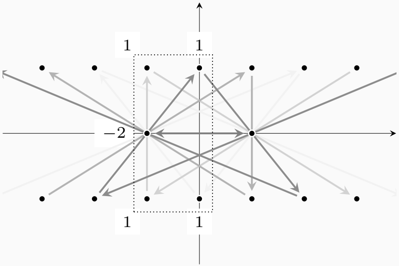
$$(\gamma, F^{(\gamma)}, \mathcal{S}_\gamma) \to (M \cdot \gamma, F^{(M\cdot\gamma)}, \mathcal{S}_{M\cdot\gamma} = \mathcal{S}_\gamma)$$

[Chen,JG,Guo'26]

- Example: Seiberg-Witten theory on self-dual Omega background



Moduli space



Borel plane

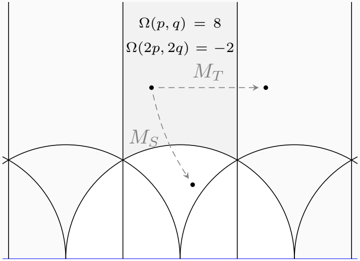
Property: modular invariance

- Non-perturbative contributions are covariant under local monodromy actions.

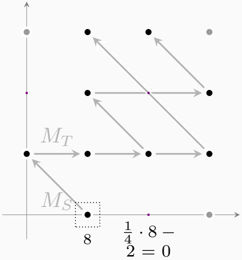
$$(\gamma, F^{(\gamma)}, \mathcal{S}_\gamma) \to (M \cdot \gamma, F^{(M\cdot\gamma)}, \mathcal{S}_{M\cdot\gamma} = \mathcal{S}_\gamma)$$

[Chen,JG,Guo'26]

- Example: Seiberg-Witten theory on self-dual Omega background
- Example: 4d $SU(2)$ massless $N_f = 4$ SQCD



Conformal manifold



Borel plane (both)

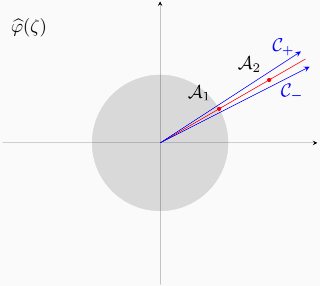
Argument: type IIB string

- The dual partition function

$$\mathcal{T}(t,\sigma;g_s) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi n\sigma/g_s}\,\mathcal{Z}(t-\mathrm{i}g_s n; g_s)$$

transforms across a Stokes ray by

$$\mathcal{T}(t,\sigma;g_s) \mapsto \mathrm{e}^{\frac{\mathcal{S}_\gamma}{2\pi\mathrm{i}}\mathrm{Li}_2(\mathrm{e}^{2\pi c\sigma/g_s})}\,\mathcal{T}\left(t+\frac{\mathcal{S}_\gamma}{2\pi}g_s c\log\left(1-\mathrm{e}^{2\pi c\sigma/g_s}\right),\sigma;g_s\right)$$



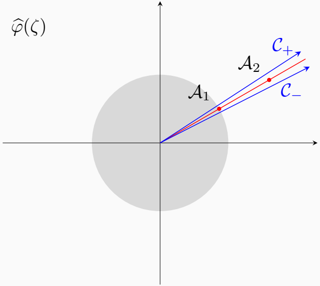
Argument: type IIB string

- The dual partition function

$$\mathcal{T}(t,\sigma;g_s)=\sum_{n\in\mathbb{Z}}\mathrm{e}^{2\pi n\sigma/g_s}\mathcal{Z}(t-\mathrm{i}g_s n;g_s)$$

transforms across a Stokes ray by

$$\mathcal{T}(t,\sigma;g_s)\mapsto \mathrm{e}^{\frac{\mathcal{S}_\gamma}{2\pi\mathrm{i}}\mathrm{Li}_2(\mathrm{e}^{2\pi c\sigma/g_s})}\mathcal{T}\left(t+\frac{\mathcal{S}_\gamma}{2\pi}g_s c\log\left(1-\mathrm{e}^{2\pi c\sigma/g_s}\right),\sigma;g_s\right)$$



- Flat coordinate t transforms like quantum period whose Stokes constants are identified with BPS invariants.

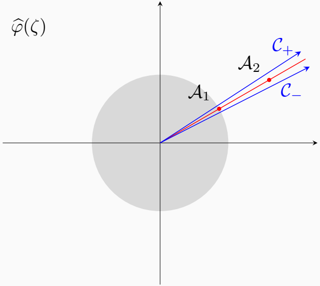
Argument: type IIB string

- The dual partition function

$$\mathcal{T}(t,\sigma;g_s) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi n\sigma/g_s}\,\mathcal{Z}(t-\mathrm{i}g_s n;g_s)$$

transforms across a Stokes ray by

$$\mathcal{T}(t,\sigma;g_s) \mapsto \mathrm{e}^{\frac{\mathcal{S}_\gamma}{2\pi\mathrm{i}}\mathrm{Li}_2(\mathrm{e}^{2\pi c\sigma/g_s})}\mathcal{T}\left(t+\frac{\mathcal{S}_\gamma}{2\pi}g_s c\log\left(1-\mathrm{e}^{2\pi c\sigma/g_s}\right),\sigma;g_s\right)$$



- Flat coordinate t transforms like quantum period whose Stokes constants are identified with BPS invariants.
- $\mathcal{T}(t,\sigma;g_s)$ is identified with five-brane instanton corrections to $\mathcal{M}_H$ as a section of line bundle, and Stokes automorphism with proper gluing conditions if $\mathcal{S}_\gamma = \Omega(\gamma)$.

[Iwaki,Marino'23; Alexandrov,Marino,Pioline'23]

Argument: Kontsevich-Soibelman wall-crossing formula

- Kontsevich-Soibelman Lie algebra with basis elements (e_γ) for $\gamma \in \Gamma$ the charge lattice

$$[e_\gamma, e_{\gamma'}] = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle \, e_{\gamma+\gamma'}$$

Argument: Kontsevich-Soibelman wall-crossing formula

- Kontsevich-Soibelman Lie algebra with basis elements (e_γ) for $\gamma \in \Gamma$ the charge lattice

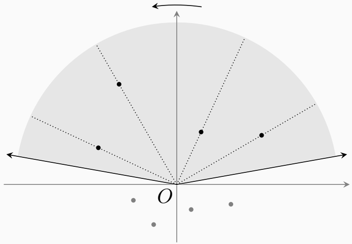
$$[e_\gamma, e_{\gamma'}] = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle \, e_{\gamma+\gamma'}$$

- Wall-crossing invariant: The ordered product of Lie group elements associated to cone V

$$K_V = \prod_{\gamma : Z_\gamma \in V}^{\rightarrow} K_\gamma, \quad K_\gamma = \exp\left(\Omega(\gamma) \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right).$$

is invariant if no central charges enter or exit the cone.

[Kontsevich,Soibelman'08; Joyce,Song'12]



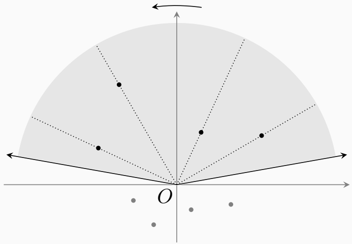
Argument: Kontsevich-Soibelman wall-crossing formula

- Recall

$$\dot{\Delta}_\gamma \mathcal{Z} = \overline{\mathcal{S}}_\gamma \mathcal{D}_\gamma \mathcal{Z},$$

with

$$\mathcal{D}_\gamma = \frac{1}{2\pi \mathrm{i}}(1 + \mathrm{D}_\gamma)\mathrm{e}^{-\mathrm{D}_\gamma}, \quad \overline{\mathcal{S}}_\gamma = \sum_{n|\gamma} \frac{\mathcal{S}_{\gamma/n}}{n^2}$$



Argument: Kontsevich-Soibelman wall-crossing formula

- Recall

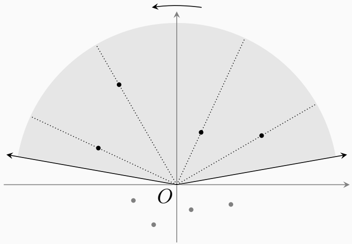
$$\dot{\Delta}_\gamma \mathcal{Z} = \overline{\mathcal{S}}_\gamma \mathcal{D}_\gamma \mathcal{Z},$$

with

$$\mathcal{D}_\gamma = \frac{1}{2\pi \mathrm{i}}(1 + \mathrm{D}_\gamma)\mathrm{e}^{-\mathrm{D}_\gamma}, \quad \overline{\mathcal{S}}_\gamma = \sum_{n|\gamma} \frac{\mathcal{S}_{\gamma/n}}{n^2}$$

- The alien derivatives satisfy

$$[\mathcal{D}_\gamma, \mathcal{D}_{\gamma'}] = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle \mathcal{D}_{\gamma+\gamma'}$$



Argument: Kontsevich-Soibelman wall-crossing formula

- Recall

$$\dot{\Delta}_\gamma \mathcal{Z} = \overline{\mathcal{S}}_\gamma \mathcal{D}_\gamma \mathcal{Z},$$

with

$$\mathcal{D}_\gamma = \frac{1}{2\pi \mathrm{i}}(1 + \mathrm{D}_\gamma)\mathrm{e}^{-\mathrm{D}_\gamma}, \quad \overline{\mathcal{S}}_\gamma = \sum_{n|\gamma} \frac{\mathcal{S}_{\gamma/n}}{n^2}$$

- The alien derivatives satisfy

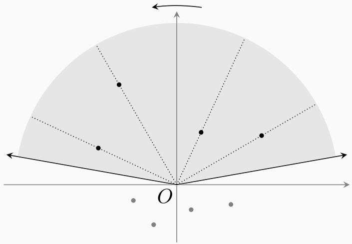
$$[\mathcal{D}_\gamma, \mathcal{D}_{\gamma'}] = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle \mathcal{D}_{\gamma+\gamma'}$$

- The *sectorial Stokes automorphism* is analogue of *wall-crossing invariant*:

$$\mathscr{S}_V = \prod_{\gamma:\mathcal{A}_\gamma \in V}^{\rightarrow} \mathscr{S}_\gamma, \quad \mathscr{S}_\gamma = \exp\left(S_\gamma \sum_{n=1}^{\infty} \frac{\mathcal{D}_{n\gamma}}{n^2} \right)$$

invariant if no Borel singularities enter or exit the sector.

[Kontsevich-Soibelman'20; Douaud,Kashani-Poor'26; Chen,JG,Guo'26]



What have we learned?

Conclusions

- For free energies of topological string on Calabi-Yau threefolds, non-perturbative corrections correspond conjecturally to stable D-brane bound states.
  - Non-perturbative *actions* are central charges.
  - *Stokes constants* are multiplicities (DT invariants).
- Non-perturbative amplitudes can be written down in closed form, and they enjoy nice modular properties.
- Non-trivial numerical evidence, and supportive arguments: string moduli space, wall-crossing formula, and etc.

What have we learned?

Conclusions

- For free energies of topological string on Calabi-Yau threefolds, non-perturbative corrections correspond conjecturally to stable D-brane bound states.
  - Non-perturbative *actions* are central charges.
  - *Stokes constants* are multiplicities (DT invariants).
- Non-perturbative amplitudes can be written down in closed form, and they enjoy nice modular properties.
- Non-trivial numerical evidence, and supportive arguments: string moduli space, wall-crossing formula, and etc.

Further questions

- Prove rigorously that Stokes constants are stable D-brane counts.
- Explore resurgence structures of defect partition functions. [Gu,Guo'24; Grassi,Hao,Neitzke'22]

Thank you for your attention!