Generalized Riemann-Hilbert correspondence

Yan Soibelman

KANSAS STATE UNIVERSITY

September 8, 2026

Title of the conference: "Calabi-Yau at 50" is quite remarkable linguistically. It stresses that the topic of the conference is not about particular area of mathematics or physics, like. e.g. "Calabi-Yau manifolds" (geometry) or " Calabi-Yau compactifications" (String Theory). There are also "Calabi-Yau algebras", "Calabi-Yau categories", and so on. One could speak about Calabi-Yau structures in general, but this term is already taken. As in the case of any big discovery, the influence of Yau's proof of Calabi conjecture goes far beyond the original aim.

From this general perspective my talk is about quantum geometry of hyperkähler manifolds or, more generally, complex symplectic manifolds.

Riemann-Hilbert correspondence for holonomic D-modules

Modern version of the Riemann-Hilbert correspondence for D-modules says that the derived category of holonomic D-modules on a complex manifold X is equivalent to the derived category of constructible sheaves on X.
This formulation goes back to Kashiwara. He also proved in the 80's that for holonomic D-modules with regular singularities the equivalence functor of taking "derived solutions" $\mathcal{M} \mapsto \mathbf{R}Hom_{D_X}(\mathcal{M}, \mathcal{O}_X)$ identifies the abelian category of holonomic D_X-modules with regular singularities to the abelian category of perverse constructible sheaves on X. More recently D'Agnolo and Kashiwara generalized this result to holonomic D-modules with irregular singularities.

HFT and GRHC

In 2014 together with Maxim Kontsevich we started the program Holomorphic Floer Theory. As a part of the program we formulated the global generalized RH-correspondence (global GRHC). Among other things it unifies existing RH correspondences for differential, difference, q-difference and elliptic difference equations. One lesson of GRHC is that the "right" RH correspondence is not about the relation of equations and monodromies of their solutions (Hilbert's 21st problem). It is about the relation of deformation quantization of complex symplectic manifolds and their Floer theory. These are two different ways to quantize the complex symplectic geometry, and GRHC roughly says that the results are equivalent.

HMS and GRHC

GRHC resembles Homological Mirror Symmetry, so one can speak about some kind of RH duality. Two sides of this duality are called de Rham side and Betti side. They are respectively deformations of B and A sides of HMS. The main difference with HMS is that both sides of GRHC live on the same complex symplectic manifold (i.e. no mirror duality at the level of manifolds).

More recently we also formulated local GRHC (the one associated with a fixed complex Lagrangian subset of a complex symplectic manifold). I am not going to discuss it today. Let me just say that the relation between local and global GRHC is a kind of "instanton corrected" inductive limit which involves a non-trivial generalization of the WKB method as well as certain wall-crossing formulas.

Set up for GRHC

Similarly to HMS one of the problems is the very definitions of the categories which appear on the de Rham and Betti sides of GRHC. In a short talk I can only describe them very briefly.

Let $(M, \omega^{2,0})$ be a complex symplectic manifold of complex dimension $2n$. Choose $P \supset M$, a Poisson compactification by a normal crossing divisor, and let $P \supset P_{log} \supset M$ be a partial log compactification (we call it a log extension of M), i.e. on $D_{log} = P_{log} - M$ the holomorphic symplectic form $\omega^{2,0}$ has poles of order 1 (D_{log} can be empty).
Both sides of GRHC will depend on P_{log}, so different choices lead to different categorical equivalences. Roughly, a particular choice of P_{log} corresponds to a choice of singular behavior of solutions of differential, q-difference, etc. equations.

Categories

After a choice of P_{log} (and after making some other technical assumptions and choices) one defines two categories:
a) the $\mathbb{C}\{\hbar\}$-linear category Hol_{glob} of holonomic DQ-modules, (here DQ stands for deformation quantization; examples of DQ-modules are $\hbar$-connections or modules over the quantum torus $xy = e^{\hbar}yx$);
b) the A_∞-category F_{glob} obtained from the family of partially wrapped Fukaya categories associated with the family of real symplectic manifolds endowed with B-fields: $(M, Re(\omega^{2,0}/\hbar) + i\, Im(\omega^{2,0}/\hbar))$. We assume that F_{glob} is linear over the ring of analytic germs on the punctured disc $\mathcal{A} = \lim_{\varepsilon \to 0} \mathcal{O}^{an}(0 < |\hbar| < \varepsilon)$.

Global GRHC

Conjectural global GRHC says:

After extension of scalars to $\mathcal{A}$ there is a derived equivalence $Hol_{glob} \simeq \mathcal{F}_{glob}$. Under this equivalence the t-structure with the heart consisting of holonomic DQ-modules corresponds to the subcategory of $\mathcal{F}_{glob}$ in which objects are supported on complex Lagrangian analytic subsets.

Short formulation: de Rham and Betti sides of GRHC are equivalent.

Example: differential equations

Differential equation $\frac{df}{dx} = A(x)f$ is understood invariantly as a meromorphic connection $(\mathcal{E}, \nabla)$ on a marked smooth complex curve $(X, x_1, ..., x_n)$ which can be singular at x_i's. By Hukuhara-Levelt-Turrittin (HLT) theorem near each x_i we have the formal decomposition $(\mathcal{E}, \nabla) = \oplus_{i,\alpha} e^{c_\alpha^{(i)}} \otimes \nabla_\alpha^{(i),RS}$. Here finite collection of singular terms $c_\alpha^{(i)} = \sum_{\lambda \in \mathbb{Q}_{\leq 0}} c_{\alpha,\lambda}^{(i)} (x - x_i)^\lambda$ controls the irregular behavior (Stokes phenomenon) of flat sections at x_i, and $\nabla_\alpha^{(i),RS}$ are connections which are regular singular at x_i.

Define $Conn(X,(c_{\alpha}^{(i)})_{i,\alpha})$ to be the category of bundles with meromorphic connections on X such that their formal types at the marked points are given by the above HLT decompositions with fixed singular terms. It follows from Deligne-Malgrange results (about 1982) that $Conn(X,(c_{\alpha}^{(i)})_{i,\alpha})$ is equivalent to the category of locally-constant sheaves on $X-\{x_1,...,x_n\}$ endowed with explicitly described "Stokes filtrations" on fibers close to the marked points (filtrations control exponential growth).

Theorem

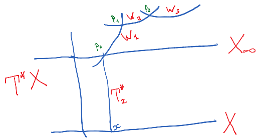
Deligne-Malgrange RH-correspondence is equivalent to the GRHC after the choices made on the next slide.

Proof is based on the combination of the Deligne-Malgrange result with more recent works, especially the one by Shende-Treumann-Zaslow on Legendrian knots and constructible sheaves.

The partial log-compactification P_{log} is obtained from the fiberwise compactification $\overline{T^*X} = T^*X \cup X_\infty$ by a finite sequence of blow-ups and then by adding to $M = T^*X$ divisors $D_\alpha^{(i)} \subset D_{log}$ such that $\omega^{2,0}_{T^*X}$ has pole of order 1 on $D_\alpha^{(i)}$. Simple theorem says that one can uniquely choose a sequence of blow-ups so that the components $D_\alpha^{(i)} \simeq \mathbb{C}$ bijectively correspond to the singular terms $c_\alpha^{(i)}$. Then GRHC relates meromorphic connections with prescribed singular terms $(c_\alpha^{(i)})_{i,\alpha}$ with a subcategory of the Fukaya category of T^*X generated by certain real conic Lagrangians $L_{c_\alpha^{(i)}}$. On the figure below we illustrate blow-ups in the case of one point $x = x_i$, and use the notation $W_\alpha := D_\alpha^{(i)}$.

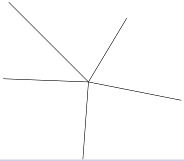
Figure: blow-ups

Thursday, October 29, 2015
9:02 AM



Example: q-difference equations

q-difference equations $f(qx) = C(x)f(x), x \in \mathbb{C}^*$ are examples of holonomic q-D-modules. Equivalent language: they are holonomic modules over the quantum torus $A_q, q \in \mathbb{C}^*$. Quantum torus is a $\mathbb{C}$-algebra with invertible generators $\hat{x}, \hat{y}$ subject to the relation $\hat{x}\hat{y} = q\hat{y}\hat{x}$. It turns out that the analog of singular terms in this case is given by a fixed collection of rational rays $(l_i)_{1\le i\le m}$ (this follows from the q-analog of HLT). Then one defines the abelian category $Hol_{(l_i),q}((\mathbb{C}^*)^2)$ of holonomic A_q-modules with "singularities controlled by $(l_i)_{1\le i\le m}$". Poisson geometry: $M = (\mathbb{C}^*)^2, \omega^{2,0} = \frac{dx}{x} \wedge \frac{dy}{y}$, and $P_{log} \subset P_\Sigma$, where P_Σ is the toric surface with the fan Σ given by the collection of rational rays $(l_i)_{1\le i\le m}$ and sectors between them . More precisely, P_{log} is obtained from P_Σ by removing the intersection points of toric divisors. Then D_{log} is a disjoint union of $\mathbb{C}^*$'s.



In the case $|q| \neq 1$ we can state and prove GRHC using a result Ramis-Sauloy-Zhang (2009). For that we need the notion of anti-Harder-Narasimhan filtration (AHN filtration) of a coherent sheaf on a curve:

AHN filtration of a coherent sheaf $\mathcal{E}$ is a finite filtration $\mathcal{E} \supset \mathcal{E}_1 \supset \mathcal{E}_2 \supset \ldots \supset \mathcal{E}_k$ such that $F_i := \mathcal{E}_i/\mathcal{E}_{i+1}$ are semistable and their slopes increase: $\frac{deg(F_i)}{rk(F_i)} < \frac{deg(F_{i+1})}{rk(F_{i+1})}$. Differently from the HN filtration the AHN filtration is non-unique.

Fix a finite collection of rational rays $(l_i)_{1\leq i\leq m}$ as on the previous slide as well as a straight line which contains the origin and splits the set of rays into two subsets: on the left of the line and on the right of the line.

Theorem

1) Consider the full subcategory of $D^b(Coh(\mathbb{C}^*/q^{\mathbb{Z}}))$ of objects endowed with two AHN filtrations: with slopes given by the slopes of rays on the left (resp. on the right) of the line. Then this category is abelian, and it is equivalent to the category $Hol_{(l_i),q}((\mathbb{C}^*)^2)$.

2) GRHC holds for $Hol_{(l_i),q}((\mathbb{C}^*)^2)$, and the abelian category $Hol_{(l_i),q}((\mathbb{C}^*)^2)$ is identified with the heart of the partially wrapped Fukaya category of P_{log} consisting of objects having one-dimensional complex support.

Example: elliptic case

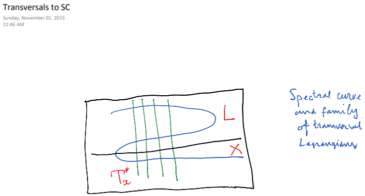
Take $\tau = 2\pi i\hbar, \hbar \in \mathbb{R}_{>0}$, and E be the corresponding elliptic curve. There are two versions of the GRHC with the following de Rham sides:
a) (elliptic difference equations) $f(x+u) = A(x)f(x)$,where $u \in E$ is fixed. Then $M = E \times \mathbb{C}_z^*, \omega^{2,0} = dx \wedge \frac{dz}{z}$, and $P_{log} = E \times \mathbb{P}^1$.

b) (modules over Sklyanin algebras). Here we consider holonomic modules over the algebra corresponding to the quantization of $M = \mathbb{P}^2 - E$ endowed with the symplectic form $\omega^{2,0}$ which has a pole of order 1 on the smooth cubic E. Then $P_{log} = \mathbb{P}^2$.
In both cases irreducible components of D_{log} are elliptic curves.
There are proposals for the RH correspondence in the cases a) and b) (see e.g. a big paper by Eric Rains in arXiv). In order to reformulate them as GRHC one should compare one side with the corresponding Fukaya category.

Rest of my talk is devoted to holonomic modules over quantum tori.

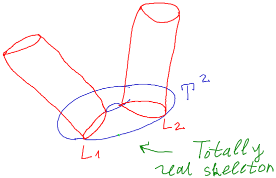
RH functor via family Floer homology and Lagrangian skeleta

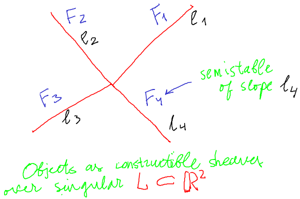
Let $L \subset T^*X$ be a complex Lagrangian. The family of vector spaces $Hom_{\mathcal{F}(T^*X)}(L, T^*_x), x \in X$ outside of ramification points of $pr : L \to X$ is a locally constant sheaf, and by the usual RH correspondence gives an object of $Hol(X)$. For $(\mathbb{C}^*)^2$ (which is not exact symplectic) the same approach gives an object of the category of holonomic $A_{q'}$-modules, where $q' = e^{2\pi i/\hbar}$. One can prove modular invariance of this category with respect to $\hbar$, so we obtain in the end an object of the category of holonomic A_q-modules. Hence $X \subset T^*X$ and $\mathbb{C}^* \subset (\mathbb{C}^*)^2$ are some kind of complex Lagrangian skeleta (cf.: Lagrangian skeleta in real Floer theory, they determine Fukaya categories of exact manifolds).



Totally real skeleton and rotation of complex structure

There is an isomorphism of manifolds $(\mathbb{C}^*)^2 \simeq T^*(T^2)$. Notice that T^2 is totally real w.r.t. $Re(\omega^{2,0})$. It is Lagrangian for $Im(\omega^{2,0}) = Re(-i\omega^{2,0})$. Rational rays (l_i) give rise to closed geodesics on T^2. Their (positive) conormal bundles are conic Lagrangian subvarieties in $T^*(T^2)$. This leads to a microlocal description of the Betti side for $Hol_{(l_i),q}((\mathbb{C}^*)^2)$ in terms of constructible sheaves with prescribed microlocal support. Alternatively, one can rotate the complex structure and identify T^2 with the elliptic fiber E_τ of the Hitchin fibration. In this case the Hitchin map is the "tropical map" $(\mathbb{C}^*)^2 = \mathbb{C} \times E_\tau \to \mathbb{C} = \mathbb{R}^2, (x, y) \mapsto (log|x|, log|y|)$. In physics language: the fiber $\mathbb{C}^* \simeq S^1 \times \mathbb{R}$ is an (A, B, A)-brane of Maslov index zero.





Lagrangian skeleta and $\mathbb{C}^*$-actions

The cotangent bundle T^*X carries a holomorphic $\mathbb{C}^*$-action with the attractor set X. Category of holonomic D_X-modules is described in terms of X. Similar story in the above examples with q-difference and elliptic difference equations.
One can expect that there is a class of complex symplectic manifolds endowed with $\mathbb{C}^*$-action for which there exists "the category of holonomic objects supported on the attractor set (Lagrangian skeleton)". Both sides of GRHC should have a description in terms of this Lagrangian skeleton.
This class of complex symplectic manifolds contains conical symplectic resolutions as well as Coulomb branches of some $3d\,\mathcal{N}=4$ gauge theories (cf. with works of Cautis, Kamnitzer, Proudfoot, Webster and others on Coulomb branches and quantization of symplectic resolutions; in those works one has two $\mathbb{C}^*$-actions).

Higher dimensional quantum tori

De Rham side for higher-dimensional quantum tori can be approached in several ways. I am going to outline an approach which gives the de Rham side of GRHC for some cluster varieties.

Let $\mathbf{k}$ be a field, and $V \simeq \mathbb{Z}^d$ be a lattice. Fix a homomorphism of abelian groups $\mathbf{q} : \wedge^2 V \to \mathbf{k}^*$, where $\mathbf{k}^* = \mathbf{k} - \{0\}$. After a choice of basis of V it gives us a collection of elements $q_{ij} \in \mathbf{k}^*, 1 \leq i \leq d$ such that $q_{ii} = 1, q_{ij} = q_{ji}^{-1}$. The d-dimensional algebraic *quantum torus* $A_{\mathbf{q}} := A_{\mathbf{q}}(d)$ is an associative $\mathbf{k}$-algebra $\mathbf{k}\langle \hat{x}_i^{\pm 1}\rangle_{1\leq i \leq d}/(\hat{x}_i\hat{x}_j = q_{ij}\hat{x}_j\hat{x}_i)$.
As a $\mathbf{k}$-vector space $A_{\mathbf{q}}$ is spanned by $\hat{x}^{\gamma}, \gamma \in V$, where for $\gamma = (\gamma_1, ..., \gamma_d)$ we define $\hat{x}^{\gamma} = \hat{x}_1^{\gamma_1}...\hat{x}_d^{\gamma_d}$.
Let $W = Hom(V, \mathbb{Z})$ be the dual lattice. We have the corresponding vector spaces $V_{\mathbb{R}} = V \otimes_{\mathbb{Z}} \mathbb{R}, W_{\mathbb{R}} = W \otimes_{\mathbb{Z}} \mathbb{R}$. The natural non-degenerate pairing $W \otimes V \to \mathbb{Z}, (r, \gamma) \mapsto \langle r, \gamma \rangle$ extends to an $\mathbb{R}$-linear non-degenerate pairing $W_{\mathbb{R}} \otimes V_{\mathbb{R}} \to \mathbb{R}$.

Let $\Sigma \subset W_{\mathbb{R}}$ be a fan (not necessarily complete). For each closed face $\sigma \in \Sigma, 0 \leq dim\, \sigma \leq d$ we define a $\mathbf{k}$-associative algebra $A_{\sigma,\mathbf{q}}$ as a subalgebra of $A_{\mathbf{q}}$ which is $\mathbf{k}$-linearly spanned by monomials $x^{\gamma}, \gamma \in \sigma^{\vee} \subset V$.
Inclusions of faces give rise to a diagram of algebras $A_{\sigma,\mathbf{q}}$. The naturally defined category $Coh(X_{\Sigma,\mathbf{q}}) := Coh_{\mathbf{q}}(X_{\Sigma})$ of modules of finite type over this diagram is called the category of **coherent sheaves on the quantum toric variety $X_{\Sigma,\mathbf{q}}$.**
Similarly, we have triangulated categories $D(Coh(X_{\Sigma,\mathbf{q}}))$ and $D^b(Coh(X_{\Sigma,\mathbf{q}}))$ (the unbounded and bounded derived categories of coherent sheaves on $X_{\Sigma,\mathbf{q}}$) as well as the category of perfect complexes on $X_{\Sigma,\mathbf{q}}$ denoted by $Perf(X_{\Sigma,\mathbf{q}}) := Perf_{\mathbf{q}}(X_{\Sigma})$.

Assume that $\mathbf{k} = \mathbb{C}$ and $d = 2n$. Fix a constant Poisson structure $\pi = \sum_{1 \leq i,j \leq 2n} \pi_{ij} x_i \partial_{x_i} \wedge x_j \partial_{x_j}$ such that $\pi_{ij} \in \mathbb{C}, \pi_{ij} = -\pi_{ji}$ and $det((\pi_{ij})) \neq 0$. We denote by $\omega^{2,0} = \sum_{1 \leq i,j \leq 2n} \omega_{ij} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j}$ the corresponding constant symplectic structure on $(\mathbb{C}^*)^{2n}$. We have the quantum torus $A_{\mathbf{q}}$, where $\mathbf{q} = (q_{ij})_{1 \leq i,j \leq 2n}$, where $q_{ij} = exp(\pi_{ij})$. Assume $q_{ij} = q^{a_{ij}}$, where $q \in \mathbb{C}^*, a_{ij} \in \mathbb{Z}, a_{ij} = -a_{ji}$ are the matrix elements of the matrix a such that $det(a) \neq 0$. Denote the corresponding quantum torus by $A_q := A_q(2n)$.

Fix a fan $\Sigma_{Lag} \subset \mathbb{R}^{2n}$ such that all its faces are isotropic, while all its maximal faces are Lagrangian. Such a fan is called Lagrangian fan. Let us also assume that Σ_{Lag} is *primitive*. Choose a complete primitive fan $\Sigma \supset \Sigma_{Lag}$.
For $n = 1$ a finite collection of rational rays considered previously gives Σ_{Lag}.

Let $P := P_\Sigma$ be the corresponding projective Poisson toric variety. It contains a smooth open Poisson subvariety $P_{log} := P_{\Sigma_{Lag}}$ corresponding to Σ_{Lag}. Then P_{log} is a log-extension of the open symplectic leaf $(\mathbb{C}^*)^{2n} \subset P_{log}$. The closed subset $P - P_{log} \subset P$ is a union of Poisson toric strata.

The general algebraic construction with diagrams of algebras gives us not only the category $Perf_q(P)$ of perfect complexes on the quantum toric variety $P = P_\Sigma$, but also the categories of compactly supported sheaves, e.g. the category $Perf_{q,c}(\partial P_{log})$ of perfect complexes with compact support on the quantum boundary subvariety ∂P_{log}.

There is a restriction functor $j_q^* : Perf_{q,c}(P_{log}) \to Perf_{q,c}(\partial P_{log})$, which admits a right adjoint embedding functor j_{q*}. Moreover the above categories depend on Σ_{Lag} only, not on Σ.

We will denote the category $Perf_{q,c}(P_{log})$ by $Hol^{\Delta}_{\Sigma_{Lag}}(A_q)$ and call it the triangulated category of holonomic A_q-modules associated with Σ_{Lag}.
With some additional work one can define an abelian subcategory $Hol_{\Sigma_{Lag}}(A_q)$ in the heart of the natural t-structure of $Hol^{\Delta}_{\Sigma_{Lag}}(A_q)$ (it consists of complexes with cohomology in degree zero). For $q = 1$ this subcategory corresponds to coherent sheaves with Lagrangian support whose closure is compact in P_{log}.
This is the de Rham side of GRHC.

GRHC for higher-dimensional quantum tori

To the same data one can assign a partially wrapped Fukaya category $\mathcal{F}_{\Sigma_{Lag}}$ (Betti side). There are many technical details here which I omit in this discussion. If $|q| \neq 1$ generalizing the case $n = 1$ we propose a conjectural description of the Betti side in terms functors from a certain category of constructible sheaves on Σ_{Lag} to a certain explicitly described triangulated subcategory $\mathcal{C} \subset D^b(Coh((\mathbb{C}^*/q^{\mathbb{Z}})^n))$. The GRHC claims equivalence of $Hol^{\Delta}_{\Sigma_{Lag}}(A_q)$ with $\mathcal{C}$ which maps the abelian subcategory $Hol_{\Sigma_{Lag}}(A_q)$ to an explicitly described abelian subcategory of $\mathcal{C}$.

We also propose even a more general conjecture for arbitrary $q \in \mathbb{C}^*$, including roots of 1. Furthermore, we propose a quantum version of Orlov theorem which describes the behavior of these categories under blow-ups. This should give in the end also GRHC for the corresponding cluster varieties.

Subsequent slides are devoted to some applications of GRHC:

- brane quantization and representations of spherical Cherednik algebras,
- generalized non-abelian Hodge theory and periodic monopoles,
- generalized $P = W$ conjecture.

For more: see the video of my talk at the IHES conference "Mathematics inspired by physics" (June 1-2, 2026) at the IHES Youtube channel.

Brane quantization

The work of physicists Gukov-Koroteev-Nawata-Du Pei-Saberi (see arXiv:2206.03565) is devoted to the "brane quantization" of certain Coulomb branches of $3d\,\mathcal{N}=4$ theories. Applications include finite-dimensional representations of the spherical DAHA of rank one (DAHA=double affine Hecke algebra introduced by Cherednik in 90's).

The idea of brane quantization goes back to Kapustin and Witten. Roughly, physicists say that the A-model with the hyperkähler (hence complex symplectic) target $(M,\omega^{2,0})$ should give rise to the "category of A-branes $\mathcal{F}^{ext}$" (extended Fukaya category). The category $\mathcal{F}^{ext}$ contains objects with coisotropic support, including "canonical coisotropic brane $\mathcal{B}_{cc}$" which has support M. Then for any "Lagrangian brane" $\mathcal{B}\in Ob(\mathcal{F}^{ext})$ the space $Hom_{\mathcal{F}^{ext}}(\mathcal{B}_{cc},\mathcal{B})$ ("space of strings between two branes") is a module over $A=End_{\mathcal{F}^{ext}}(\mathcal{B}_{cc})$. Physicists expect that in this way one gets the derived equivalence $A-mod\simeq\mathcal{F}^{ext}$.

GRHC and brane quantization

Mathematical problem: there is no definition of the "extended Fukaya category $\mathcal{F}^{ext}$ " in terms of the Floer theory.

Assuming global GRHC one can propose a temporary solution: consider the category of all modules of finite type over the quantized algebra $\mathcal{O}_q(M)$ (or the sheaf of quantized algebras $\underline{\mathcal{O}}_{M,q}$), not just holonomic modules. If q is a formal parameter then in the quasi-classical limit $q \to 1$ modules become coherent sheaves with complex coisotropic support. Motivated by GRHC one can hope that whatever $\mathcal{F}^{ext}$ is, it can be embedded into the category $\mathcal{O}_q(M)$-mod. Then $\mathcal{B}_{cc}$ should correspond to $\mathcal{O}_q(M)$. "Lagrangian branes" should correspond to objects of the usual Fukaya category $\mathcal{F} \subset \mathcal{F}^{ext}$.

Spherical DAHA as deformation quantization

Let M be the moduli space of flat $SL(2,\mathbb{C})$-connections on the punctured 2-dimensional torus with prescribed semisimple monodromy about the puncture. One can show (see e.g. Oblomkov, arXiv:math/0306393) that the quantized algebra $\mathcal{O}_q(M)$ is isomorphic to the spherical DAHA $S\ddot{H}$ of rank one. The algebra $S\ddot{H}$ is an algebra over a certain localization of $\mathbb{C}[q^{\pm 1/2}, t^{\pm 1}]$. It has generators $\hat{x}, \hat{y}, \hat{z}$ subject to relations:

$$[\hat{x},\hat{y}]_q = (q^{-1}-q)\hat{z}, [\hat{y},\hat{z}]_q = (q^{-1}-q)\hat{x}, [\hat{z},\hat{x}]_q = (q^{-1}-q)\hat{y},$$

$$q^{-1}\hat{x}^2 + q\hat{y}^2 + q^{-1}\hat{z}^2 - q^{-1/2}\hat{x}\hat{y}\hat{z} = (q^{-1/2}t - q^{1/2}t^{-1})^2 + (q^{1/2}+q^{-1/2})^2,$$

where $[a,b]_q = q^{-1/2}ab - q^{1/2}ba$. In the quasiclassical limit $q = e^{2\pi i\hbar} \to 1$ this algebra becomes the commutative Poisson algebra of regular functions on the moduli space $M := \mathcal{M}_{flat}(T^2 - \{pt\}, SL(2,\mathbb{C}), t) \simeq \{(x,y,z) \in \mathbb{C}^3 | f(x,y,z) := x^2 + y^2 + z^2 - xyz - 2 = t^2 - t^{-2}\}$ with x,y,z being monodromies about three loops $\alpha, \beta, \gamma = \alpha\beta^{-1}$ such that $\pi_1(T^2 - \{pt\}) = \langle \alpha, \beta, c | \alpha\beta\alpha^{-1}\beta^{-1} = c\rangle$, and the local monodromy Mon_c about the puncture is conjugate to the given matrix $diag(t^2, t^{-2})$. Here $\omega^{2,0} = \frac{dx\wedge dy\wedge dz}{df}$.

GRHC and representations of spherical DAHA

Gukov with coauthors formulated a conjecture about derived equivalence of the abelian category of compact Lagrangian branes in M and the category of finite-dimensional $S\ddot{H}$-modules. They proposed physics arguments in favor of the conjecture. Mathematically, their conjecture should be a special case of GRHC, since *finite-dimensional* $\mathcal{O}_q(M)$-modules correspond to objects supported on *compact* Lagrangian subsets in M.

Compact Lagrangians and finite-dimensional representations

After rotating the complex structure M becomes the moduli space of parabolic Higgs bundles over elliptic curve with tame singularity at pt. The homology group $H_2(M,\mathbb{Z})$ is generated by classes of compact singular fibers of the corresponding Hitchin fibration. One should also check which compact Lagrangian cycles correspond to finite-dimensional representations of $\mathcal{O}_q(M)=S\ddot{H}$, and what conditions on q and t these geometric restrictions impose. E.g. the dimension of representation corresponding to a compact 2-cycle should be a positive integer, and also it should be equal (up to the factor $const\cdot\hbar$) to the period $\omega^{2,0}$ over the cycle. There is a $PSL(2,\mathbb{Z})$-action on the category of $\mathcal{O}_q(M)-mod^{fd}$, which is an analog of the previously mentioned modular invariance of the category of holonomic $A_q=\mathcal{O}_q((\mathbb{C}^*)^2)$-modules.

Open problem: describe the Betti side of the GRHC for a quantized cluster variety as well as compact Lagrangians corresponding to finite-dimensional representations.

From arXiv:2206.03565: Hitchin fibrations in the unramified and tamely ramified cases

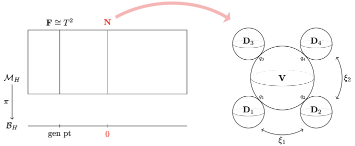


Figure 1: Schematic illustration of the Hitchin fibration $\mathcal{M}_H(C_p, \mathrm{SU}(2)) \to \mathcal{B}_H$ and global nilpotent cone at $\beta_p = 0 = \gamma_p$ and a generic value of α_p.

the Hitchin fibration $\pi : \mathcal{M}_H(C_p, G) \to \mathcal{B}_H$ is the pre-image $\mathbf{N} = \pi^{-1}(0)$ of $0 \in \mathcal{B}_H$ which, in the limit $\alpha_p = \beta_p = \gamma_p = 0$, is the "pillowcase" $\mathbf{V} \cong (S^1 \times S^1)/\mathbb{Z}_2$ with four torsion (orbifold) points.

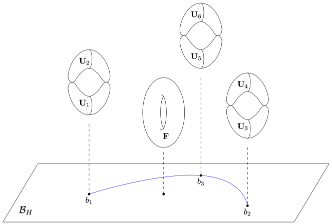


Figure 2: The Hitchin fibration with a generic ramification contains three singular fibers of Kodaira type I_2 at the base points b_i $(i = 1, 2, 3)$.

Generalized non-abelian Hodge theory in dimension 1

In the non-abelian Hodge theory for curves one has an analytic family over $\mathbb{CP}^1_\zeta$ of moduli spaces, such that at $\zeta = 0$ (resp. $\zeta = \infty$) the fiber is the moduli space of semistable parabolic Higgs bundles on the complex curve X (resp. same Higgs moduli space but with opposite complex structure), and at $\zeta \in \mathbb{C}^*$ the fiber is the moduli space of parabolic local systems on X. There is a precise correspondence between parameters of the family (eigenvalues, parabolic weights, etc.). One can ask about generalization of NAHT from $M = T^*X$ to arbitrary symplectic surface $(M, \omega^{2,0})$.

First, with Maxim we introduced the notion of analytic twistor families of (abelian) A_∞-categories $\mathcal{C}_\zeta, \zeta \in \mathbb{CP}^1$, which (roughly) means that:

1) There is an equivalence of A_∞-categories $G_\zeta : \mathcal{C}_\zeta \simeq \mathcal{C}^*_{-1/\overline{\zeta}}$, where $\mathcal{C}^*$ denotes the opposite category $\mathcal{C}^{op}$ endowed with the opposite $\mathbb{C}$-linear structure (we call it the Hermitian conjugate category).

2) *Hom*-spaces of each category are endowed with positive Hermitian forms compatible with the functors G_ζ.

Second, we proposed the twistor family of categories for generalized NAHT in dimension one. We start with symplectic surface $(M, \omega^{2,0})$ and choose a log extension M_{log}. Then at $\zeta \in \mathbb{C}^*$ we place the (partially wrapped) Fukaya category $\mathcal{F}_\zeta$ of $(M, \omega^{2,0})$ corresponding to the symplectic form

$$\omega_\zeta = \frac{\omega^{2,0}}{\zeta} + \zeta\overline{\omega^{2,0}}.$$

This symplectic form splits naturally as $Re(\omega_\zeta) + i\, Im(\omega_\zeta)$. We treat $B_\zeta = Im(\frac{\omega^{2,0}}{\zeta} + \zeta\overline{\omega^{2,0}})$ as B-field. Twistor property 1) follows from the identity $\omega_{-\frac{1}{\overline{\zeta}}} = \overline{\omega}_\zeta$.

We also defined the notion of parabolic structure on $\mathcal{F}_\zeta$ in terms of the behavior of objects near D_{log}. This gives $\mathcal{C}_\zeta, \zeta \in \mathbb{C}^*$. At $\zeta = 0$ we place the category $Coh^{ss} := Coh^{ss}(M)$ of semistable parabolic coherent sheaves on M which have pure 1-dimensional support. For that we defined the notion of parabolic structure and stability on Coh^{ss}. At $\zeta = \infty$ we place the Hermitian conjugate category $(Coh^{ss})^*$. Parabolic structures for both categories can be spelled out in the language of 2-dimensional Calabi-Yau categories with boundary (more generally, with corners).

The category of harmonic bundles which plays a key role in the classical NAHT should be generalized as well. E.g. for $((\mathbb{C}^*)^2, \frac{dx \wedge dy}{xy})$ it can be identified with the category of 2-periodic monopoles in $\mathbb{R}^3$ with prescribed simple (a.k.a.Dirac) singularities. In general, one observes that the geometry of the real blow-up of D_{log} at a point (it controls the parabolic structures) is the one of a real $3d$ manifold with $\mathbb{Z}$-affine structure, endowed with an affine $1d$ foliation which has transversal complex structure. Then e.g. for the category of elliptic difference equations $f(x+u) = A(x)f(x), u \in E_\tau$ the transversal curve is the elliptic curve E_τ, and the corresponding "harmonic" category is the one of 3-periodic monopoles.

$P = W$ conjecture

The original $P = W$ conjecture of Cataldo-Hausel-Migliorini arXiv:1004.1420 says that the NAHT homeomorphism of M_{Dol} (moduli space of Higgs bundles on a complex curve) and M_{Betti} (corresponding moduli space of local systems on the same curve) constructed by Simpson identifies the weight filtration on M_{Betti} with the perverse filtration on M_{Dol}. The latter comes from the decomposition theorem applied to $\mathbf{R}h_*(\underline{\mathbb{Q}}_{M_{Dol}})$, where $h : M_{Dol} \to \mathcal{A}$ is the Hitchin map. The original $P = W$ conjecture was proved by different groups of people (including Maulik, Shen, Yin, Hausel, Mellit, Minets, Schiffman).
There are also parabolic and irregular parabolic versions of $P = W$ conjecture (i.e. for parabolic Higgs bundles with regular and irregular singularities). There are proposals for its generalization to cluster varieties as well as to log-CY manifolds (including Zhang, Harder, Katzarkov, Pantev). Therefore one can speak about $P = W$ phenomenon, or as physicists would probably say, about PW duality.

Generalized $P = W$ conjecture

Motivated by GRHC I suggested in 2016 that there should be a generalized $P = W$ conjecture, for the moduli stacks $\mathcal{M}^{gen}_{Betti}$ and $\mathcal{M}^{gen}_{Dol}$ of objects of the t-structures of the Fukaya category $\mathcal{F}_{\zeta=1}$ and $Coh^{ss}(M)$ respectively. The appropriate cohomology should be the dual to critical cohomology with compact support (i.e. Borel-Moore homology). We used BM homology in the work with Maxim in arXiv:1006.2706, where we introduced the notion of Cohomological Hall algebra (COHA) for a class of 3-dimensional Calabi-Yau categories. The notion of COHA provided an alternative approach to our theory of motivic Donaldson-Thomas invariants developed earlier in arXiv:0811.2435, but it was not related to the $P = W$ phenomenon.

Surprisingly, few years later, physicists Chuang-Diaconsecu-Pan published a short article arXiv:1202.2039 in which they connected motivic DT-invariants of stable pairs in local CY 3-folds to the original $P = W$ conjecture.

Recently $P = W$ conjecture for the stacks $\mathcal{M}_{Dol}$ and $\mathcal{M}_{Betti}$ was studied by Davison, Hennecart, Mejia (see arXiv:2212.07668, 2112.10830, arXiv:2307.09920). They used BM homology of stacks $\mathcal{M}_{Dol}$ and $\mathcal{M}_{Betti}$, and relative versions of COHA and BPS Lie algebra (BPS Lie algebra is related to COHA more or less as the Lie algebra $\mathfrak{g}$ is related to the universal enveloping algebra $U(\mathfrak{g})$).
Their stacky version of $P = W$ conjecture for $\mathcal{M}_{Betti}$ and $\mathcal{M}_{Dol}$ implies the $P = W$ conjecture of Cataldo-Maulik for the intersection cohomology of the corresponding coarse moduli spaces:

$$\Phi(\mathcal{B}^i IH^*(M_{Dol})) = W^{2i} IH^*(M_{Betti}).$$

Here $\mathcal{B}^i$ and W^{2i} are perverse and weight filtrations on the intersection cohomology, and Φ is an isomorphism of intersection cohomology IH^* induced by a certain isomorphism $\Phi : H_*^{BM}(M_{Dol}) \simeq H_*^{BM}(M_{Betti})$.

The techniques developed by Davison-Hennecart-Mejia can be applied to a bigger class of 2-dimensional Calabi-Yau categories (and some results hold for any sufficiently good category of homological dimension ≤ 2). Then it is natural to ask a general question: what does it mean that two categories $\mathcal{A}$ and $\mathcal{B}$ (or their stacks of objects) are *PW* dual? One can also ask about the non-linear version (i.e. for shifted symplectic stacks without underlying categories).

Categories $\mathcal{A}$ and $\mathcal{B}$ are assumed to be abelian (or abelian A_∞), but one can hope for a derived version as well.The stacks of objects $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{B}}$ should be "good" in the sense which has yet to be clarified. In particular for such stack $\mathcal{M}$ there should be a good moduli space in the the sense of Alper (certain properties of the map $p: \mathcal{M} \to M$ to moduli scheme, see arXiv:0804.2242, and for derived version see arXiv:2309.16574). Good stacks should satisfy other properties which in the end guarantee that the cohomology sheaves $\mathcal{H}^i(p_*(\mathcal{D}\underline{\mathbb{Q}}_{\mathcal{M}}[dim_{vir}\mathcal{M}])$ are pure of weight i. Here $\mathcal{D}$ is the Verdier duality functor. By taking derived global sections and using the purity one defines perverse filtration on the BM homology of the stack $\mathcal{M}$. In usual NAHT stacks $\mathcal{M}_{Dol}$ and $\mathcal{M}_{Betti}$ are good.

It is not clear at this time which conditions on the twistor families of categories give rise to PW dual categories. Nevertheless, being optimistic about the future, one can conjecture that:

The categories $Coh^{ss}(M)$ and $\mathcal{F}_{\zeta=1}$ (or the corresponding stacks $\mathcal{M}^{gen}_{Betti}$ and $\mathcal{M}^{gen}_{Dol}$) are PW dual.