Calabi–Yau Geometry, Z-Theory, and the Hydrodynamic Paradigm

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Our map

1. Polyakov string and mixed-signature worldsheets
2. Complex and split structures, Kodaira–Spencer hydrodynamics
3. Z-theory and polynomial Hitchin functionals
4. Relativistic hydrodynamics as an intersection problem

Nambu–Goto string

A map of the worldsheet into spacetime (M, G)

$$X : \Sigma \longrightarrow M$$

induces the metric

$$\gamma_{ab} = G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu$$

and the Lorentzian area functional

$$S_{\mathrm{NG}}[X] = T \int_\Sigma \sqrt{-\det \gamma}$$

Polyakov's reformulation

Independent worldsheet metric g

$$S_{\mathrm{P}}[X,g] = \frac{T}{2}\int_{\Sigma}\sqrt{-\det g}\, g^{ab}\, G_{\mu\nu}(X)\partial_a X^{\mu}\partial_b X^{\nu}$$

The g equation identifies its conformal class

$$g = e^{2\chi}\,\gamma$$

On shell $S_{\mathrm{P}}[X,g]$ returns the Nambu–Goto functional

but the worldsheet geometry is now an independent field

Historically, most of research on string theory

dealt with the Wick-rotated problem

Riemannian target space, Riemannian worldsheet

Conformal invariance of the worldsheet theory $\Longrightarrow$

Constraints on the target space geometry

This is where the Calabi-Yau geometry is most prominent

For Type II superstrings, say

$$M = \mathbb{R}^4 \times X$$

$$G = \eta \oplus g_{\mathrm{CY}}$$

Yau's theorem associates a distinguished metric

to topological, complex, and Kähler data

$$c_1(X) = 0, \qquad [\omega] \quad \longrightarrow \quad g_{\mathrm{CY}}, \; Ric(g_{\mathrm{CY}}) = 0$$

The metric is the solution of a nonlinear geometric equation

Physics significance

X-valued 2d sigma model is (almost) conformally invariant

Two real forms of conformal geometry

Euclidean signature

$$J^2 = -1$$

Lorentzian signature

$$K^2 = +1, \qquad T\Sigma = L_+ \oplus L_-$$

The two eigenline distributions are the null foliations

Mixed-signature worldsheets

The string path integral need not be restricted

to everywhere Euclidean metrics

$$\Sigma = \Sigma_{\mathrm{E}} \cup \Gamma \cup \Sigma_{\mathrm{L}}$$

$$J^2 = -1 \quad \text{on } \Sigma_{\mathrm{E}}, \qquad K^2 = +1 \quad \text{on } \Sigma_{\mathrm{L}}$$

The metric degenerates on the signature-changing locus Γ

Work in progress with David Gross

The target-space analogue

In two dimensions

$$J^2 = -1 \quad \longleftrightarrow \quad K^2 = +1$$

suggests in six dimensions

$$\mathsf{SL}(3, \mathbb{C}) \quad \longleftrightarrow \quad \mathsf{SL}(3, \mathbb{R}) \times \mathsf{SL}(3, \mathbb{R})$$

Complex and split structures are two real branches

of the geometry of stable three-forms

Complex and split Calabi–Yau structures

Complex branch

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X \ , \qquad \Omega \in \Omega^{3,0}(X)$$

Split branch

$$TX = F_+ \oplus F_-, \qquad \rho = \rho_+ + \rho_-$$

F_+ and F_- are transverse three-dimensional foliations

ρ_+ and ρ_- provide transverse volume forms

Kodaira–Spencer gravity

A deformation of complex structure is represented by

$$A \in \Omega^{0,1}(X, T^{1,0}X)$$

Integrability is the Maurer–Cartan equation

$$\bar{\partial} A + \frac{1}{2}[A, A] = 0$$

The field both propagates on a fixed complex geometry

and deforms the complex geometry itself

Kodaira–Spencer theory and hydrodynamics

The holomorphic volume form selects divergence-free vector fields

$$\mathfrak{g}_\Omega = \{v \in \mathrm{Vect}(X) \mid \mathcal{L}_v \Omega = 0\}$$

$$\mathfrak{g}_\Omega = \mathrm{Lie}\big(\mathsf{SDiff}_\Omega(X)\big)$$

Kodaira–Spencer gravity, proposed in 1993 by BCOV

is a Chern–Simons-type theory

based on volume-preserving diffeomorphisms

$$\mathrm{KS} \ \sim \ \mathrm{CS}\big(\mathfrak{g}_\Omega\big)$$

Split Kodaira–Spencer theory

Locally let $X = B_+ \times B_-$

A deformation of one foliation relative to the other is

$$A \in \Omega^1 \left(B_-, \mathrm{Lie}\big(\mathsf{SDiff}_{\rho_+}(B_+)\big)\right)$$

Its integrability equation is

$$d_- A + \frac{1}{2}[A, A] = 0$$

Chern–Simons on B_- with the gauge group

$$\mathsf{SDiff}_{\rho_+}(B_+)$$

Z-theory

A la recherche de la m-théorie perdue

2004

A series of bold conjectures on a missing theory

behind topological strings and topological gauge theories

Z-theory is different from mirror symmetry

Mirror symmetry

$$A(X) \longleftrightarrow B(X^\vee)$$

Z-theory

$$A(X) \quad \text{and} \quad B(X)$$

are different corners of one nonperturbative theory

on the same Calabi–Yau manifold

The nonperturbative A-model

Gromov–Witten theory counts pseudoholomorphic curves

$$Z_A^{\mathrm{GW}}(X;t), \qquad t \in H^2(X,\mathbb{C})$$

The proposed completion also sums over

Lagrangian submanifolds with flat bundles

$$L \subset X\,, \qquad [L] = \ell \in H_3(X,\mathbb{Z})$$

$$Z_A^{?}(X;t,s) \sim Z_A^{\mathrm{GW}}(X;t) \sum_{\ell} \mathcal{N}_\ell(t,\hbar) \exp\left(-\frac{1}{\hbar}\int_\ell s\right)$$

The nonperturbative B-model

The closed B-model is described by Kodaira–Spencer gravity

$$Z_B^{\mathrm{KS}}(X;s), \qquad s \in H^3(X,\mathbb{C})$$

Its nonperturbative objects are stable objects

$$\mathcal{E} \in D^b\mathsf{Coh}(X)$$

Their charges belong to $H^{\mathrm{even}}(X,\mathbb{Z})$

and their stability depends on the Kähler data t

$$Z_B^{?}(X;s,t) \sim Z_B^{\mathrm{KS}}(X;s)\sum_{\gamma}\Omega_{\mathrm{DT}}(\gamma;t)e^{-\langle t,\gamma\rangle/\hbar}$$

Both completions see the whole Calabi–Yau geometry

	perturbative sector	nonperturbative objects
A	pseudoholomorphic curves	Lagrangian branes
B	Kodaira–Spencer fields	stable derived objects

The completed partition function should depend on

$$t \in H^2(X, \mathbb{C}) \qquad \text{and} \qquad s \in H^3(X, \mathbb{C})$$

The normalization of the holomorphic volume form is retained

The MNOP corner of Z-theory

Maulik–Nekrasov–Okounkov–Pandharipande, 2004

Gromov–Witten theory $\longleftrightarrow$ Donaldson–Thomas theory

After the change of variables

$$q = -e^{\mathrm{i}u}$$

$$Z'_{\mathrm{GW}}(X; u, t) = Z'_{\mathrm{DT}}(X; q, t)$$

A-model stable maps and B-model stable sheaves

encode the same curve-counting theory on X

A 2004 conjecture becomes a theorem

John Pardon proved that curve-enumeration theories

on Calabi–Yau threefolds are determined by local curves

$$\text{local-curve GW/DT correspondence} \quad \Longrightarrow \quad \text{MNOP}$$

This proves the correspondence with primary insertions

for Calabi–Yau threefolds

John Pardon, Fields Medal 2026

J. Pardon, Universally counting curves in Calabi–Yau threefolds, 2025

Why seven dimensions?

For $Z = X \times S^1$

$$H^3(Z) \simeq H^3(X) \oplus H^2(X) \otimes H^1(S^1)$$

$$H^3(X \times S^1) \simeq H^3(X) \oplus H^2(X)$$

A seven-dimensional three-form combines

$$\varphi = \rho + k \wedge d\theta + \cdots$$

the calibrated complex and Kähler sectors

Z-theory and Hitchin's six-dimensional functional

The independent fields are

$$\rho \in \Omega^3(X), \qquad J \in \Gamma(\mathsf{End}(TX)), \qquad \epsilon \in \Omega^6(X)$$

$$\operatorname{tr} J = 0, \qquad \rho = \rho_0 + dB$$

J is a would-be complex or split structure

ϵ is an independent volume form

The polynomial six-dimensional functional

$$\boxed{S_6^{\pm}[\rho, J, \epsilon] = \int_X \rho \wedge \iota_J \rho + \int_X \epsilon \left(\mathrm{tr}\, J^2 \pm 6\right)}$$

Variation of ϵ fixes the normalization of J

$$\mathrm{tr}\, J^2 = \mp 6$$

Variation of J relates J algebraically to ρ

Eliminating J and ϵ gives Hitchin's volume functional

$$\mathcal{H}_6[\rho] = \int_X \sqrt{|\lambda(\rho)|}$$

Two branches of the six-dimensional theory

Complex branch

$$J^2 = -1, \qquad \Omega = \rho + \mathrm{i}\iota_J \rho$$

Split branch

$$J^2 = +1, \qquad TX = F_+ \oplus F_-$$

Variation of B gives

$$d\rho = 0, \qquad d\left(\iota_J \rho\right) = 0$$

The complex or split structure emerges dynamically

Z-theory and Hitchin's seven-dimensional functional

On a seven-manifold Z, the independent fields are

$$\varphi \in \Omega^3(Z), \qquad h \in \Gamma\left(\mathsf{Sym}^2 T^* Z\right)$$

$$\varphi = \varphi_0 + dB$$

The polynomial functional is

$$\boxed{S_7[\varphi, h] = \int_Z h^{ij}\, \iota_{\partial_i}\varphi \wedge \iota_{\partial_j}\varphi \wedge \varphi + \int_Z \sqrt{\det h}\, d^7x}$$

In seven dimensions the metric is dynamical

The h equation is algebraic

$$\iota_u\varphi \wedge \iota_v\varphi \wedge \varphi \sim h_\varphi(u,v)\,\mathrm{vol}_{h_\varphi}$$

$$h = h_\varphi$$

Eliminating h gives Hitchin's functional

$$\mathcal{H}_7[\varphi] = \int_Z \mathrm{vol}_\varphi$$

The B equation gives

$$d\varphi = 0, \qquad d *_\varphi \varphi = 0$$

The Polyakov principle

$$
\begin{array}{ccc}
S_{\mathrm{NG}}[X] & \longleftarrow & S_{\mathrm{P}}[X, g] \\
\int_X \sqrt{|\lambda(\rho)|} & \longleftarrow & S_6[\rho, J, \epsilon] \\
\int_Z \mathrm{vol}_\varphi & \longleftarrow & S_7[\varphi, h]
\end{array}
$$

A non-polynomial volume is replaced by a polynomial theory

with an independent geometric field

$2d$	worldsheet conformal structure
$6d$	tangent bundle endomorphism
$7d$	metric

Topological sigma models extend the Calabi–Yau world

The A- and B-models originated in mirror symmetry

but the topological sigma-model idea extends to

symplectic geometry, Poisson geometry, coisotropic branes

The Poisson sigma model produces deformation quantization

$$f \star g = fg + \frac{\hbar}{2}\{f, g\} + O(\hbar^2)$$

Can the same methods quantize hydrodynamics?

Fluid dynamics

as an intersection problem

with Paul Wiegmann

Relativistic hydrodynamics on spacetime itself

rather than Hamiltonian evolution on a Cauchy surface

The symplectic manifold associated with spacetime

For a four-dimensional spacetime M^4

$$\mathcal{P}_{M^4} = \Omega^0(M^4) \oplus \Omega^1(M^4) \oplus \Omega^3(M^4) \oplus \Omega^4(M^4)$$

$$\Phi = (S, p, n, \nu)$$

with symplectic form

$$\omega_{\mathcal{P}} = \int_{M^4} \delta S \wedge \delta\nu + \delta p \wedge \delta n$$

Two geometric ingredients

A coisotropic submanifold

$$\mathcal{C} = \mu^{-1}(0) \subset \mathcal{P}_{M^4}$$

defined by the moment map for

$$\mathsf{Diff}(M^4) \ltimes \mathsf{C}^\infty(M^4)$$

and a Lagrangian submanifold

$$\mathcal{L}_{g,e} \subset \mathcal{P}_{M^4}$$

defined by the spacetime metric and equation of state

Relativistic hydrodynamics is the intersection

$$\boxed{\Phi \in \mathcal{C} \cap \mathcal{L}_{g,e} \subset \mathcal{P}_{M^4}}$$

$\mathcal{C}$	$\mathcal{L}_{g,e}$
differential topology	spacetime metric
conservation laws	equation of state
coisotropic	Lagrangian

Their intersection gives the Lichnerowicz–Carter equations

N. Nekrasov and P. Wiegmann, Fluid dynamics as an intersection problem, 2025

Towards quantization

The classical fluid =

a system of PDEs =

a point in the phase space of histories

classical dynamics = coisotropic constraint $\cap$ Lagrangian state

This suggests a quantum problem

quantize the intersection $\mathcal{C} \cap \mathcal{L}$

Possible languages

AKSZ topological sigma models

deformation quantization

Back to quantization of Calabi-Yau problem

One of the outcomes of BCOV approach to all-genus topological string

Holomorphic anomaly equation (later explored by A.Klemm)

Relating $\mathcal{F}_g$'s at different genera g

Witten interpreted as Knizhnik-Zamolodchikov equation

For a theory of self-dual 3-forms in six dimensions

Intersection theory view on chiral fields

The recent reformulation of hydrodynamics

Can be applied to chiral fields in $4n+2$ dimensions

Intersection theory view on chiral fields

The space of middle-dimensional forms

$$\mathcal{P}_n = \Omega^{2n+1}(M^{4n+2})$$

is symplectic: $\omega = \int_{M^{4n+2}} \delta C \wedge \delta C$

Intersection theory view on chiral fields

Hamiltonian action $C \mapsto C + dB$, $B \in \Omega^{2n}(M)$

Coisotropic submanifold $\mathcal{C} = \mu^{-1}(0) \subset \mathcal{P}_n$, $dC = 0$

Conformal structure on $M \implies \star : \Omega^{2n+1}(M^{4n+2}) \to \Omega^{2n+1}(M^{4n+2})$

Lagrangian submanifold: $\mathcal{L} \subset \mathcal{P}_n$, $C = \star C$

Quantum Kodaira–Spencer theory made complex structures interact

Z-theory sought a common nonperturbative completion

Relating A and B topological model and

connecting Calabi-Yau geometry to intersection theory

Recent developments:

signature changing metric
interplay between complex and split structures
connections to hydrodynamics

Can fluid dynamics be informed by Calabi-Yau geometry?

Thank you