Calabi-Yau 50 years Conference
Tsinghua University and BIMSA, September 6-11, 2026

Symmetry Theories, Anomalies and η Invariants

(Geometric Engineering and Quiver Approaches)

Mirjam Cvetič

Penn
University of Pennsylvania

CAMTP
University of Maribor

Univerza v Ljubljani
Fakulteta za matematiko in fiziko

Based on:

M.C., R. Donagi, J. Heckman, M. Hübner, 2512.17906 (geometric engineering)

&

V. Chakrabhavi, M.C., J. Heckman, S. Meynet, 2605.30354 (quivers)

[Also, earlier works (geometric engineering):

M.C., R. Donagi, J. Heckman, M. Hübner, E. Torres, 2408.12600

&

M.C., Heckman, Hübner, Torres, 2203.10102;

M.C., Heckman, Hübner, Torres, 2305.09665;

M.C., Heckman, Hübner, Torres, 2307.13027]

I. Motivation – Geometric Engineering Approach

Well-established study of Higher Symmetry Structures in Quantum Field Theory (QFT) via:

String/M theory on non-compact, singular spaces X

a) Typical approach: singular geometry X is resolved locally to smooth X' to great computational success

[Heckman, Park, Rudelius; 2015], [Morrison, Schafer-Nameki, Willett; 2020], [Bhardwaj, Schafer-Nameki; 2020], [Apruzzi, Bonetti, Garcıa-Etxebarria, Hosseini, Schafer-Nameki; 2021], [Tian, Wang; 2021], [M.C., Dierigl, Lin, Zhang; 2021], [Del Zotto, Garcıa-Etxebarria; 2022], [Lawrie, Yu, Zhang; 2023] & many more

b) However, key quantities often topological: calculations expected to be possible in singular spaces X via algebraic/differential topology

It may also be necessary to work with singular X due to the complexity of the singular geometry of X or inability to resolve it.

Examples

- Nested/non-Isolated Singularities
  New results in singular geometry:
  [M.C., Heckman, Hübner, Torres; 2022], [Del Zotto,Garcıa-Etxebarria, Schafer-Nameki; 2022],
  [Acharya, Del Zotto, Heckman, Hübner, Torres; 2023],…,
  [M.C., Donagi, Heckman, Hübner, Torres; 2024], [M.C., Donagi, Heckman, Hübner; 2025]

- Frozen Singularities
  [de Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi; 2001], [Atiyah, Witten; 2001], ...,
  New results in singular geometry:
  [M.C., Dierigl, Lin, Torres, Zhang; 2025], [Torres; 2026]

- Terminal Singularities
  [Denef, Douglas, Florea, Grassi, Kachru; 2005], ...,
  New results in singular geometry:
  [Arras, Grassi, Weigand; 2016],...,
  [Sangiovanni, Valandro; 2024]

- Non-SUSY String Constructions
  [Adams, Polchinski, Silverstein; 2001], [Morrison, Narayan,Plesser; 2004], ...
  New results in singular geometry:
  [Braeger, Chakrabhavi, Heckman, Hübner; 2025]

Recent research focused on the analysis in singular X; contributions to the geometric engineering toolbox from differential cohomology, Chen-Ruan orbifold cohomology, equivariant cohomology, K-theory,...

II. Motivation – Quiver Approach

Less explored for studies of Higher Symmetry Structures

Anticipate that this data can be extracted from the path algebra of branes probing singular X.

Defect Group [Del Zotto, Heckman, Meynet, Moscrop, Zhang; 2022], [Del Zotto and Garcia-Etxebarria; 2022]… [Dramburg, Meynet, Sangiovanni; 2025]

Complementary algebraic approach to geometric engineering w/ direct QFT interpretation

Outline

I. Geometric Engineering Approach

i) Higher symmetry defects/operators

ii) Symmetry Field Theories (SymTFTs) and Nested Ones
(Isolated and non-isolated singularities examples)

iii) Anomaly terms in SymTFTs

iv) Computation of Anomaly Coefficients
(Isolated and non-isolated singularities examples)

Focus on refined data

Outline

II. Quiver Approach

- Present a new procedure for extracting higher symmetries & SymTFT anomaly coefficients form Quiver Data of quantum theory of branes, probing the singular geometry of X.

  [Focus on Quiver Data which directly tracks BPS defects in engineered QFT and its KK reduction.]

In the rest of the talk

M-theory on X – non-compact conical Calabi-Yau three-folds
prototypes: orbifolds $X = \mathbb{C}^3/\Gamma$ with finite $\Gamma \subset SU(3)$

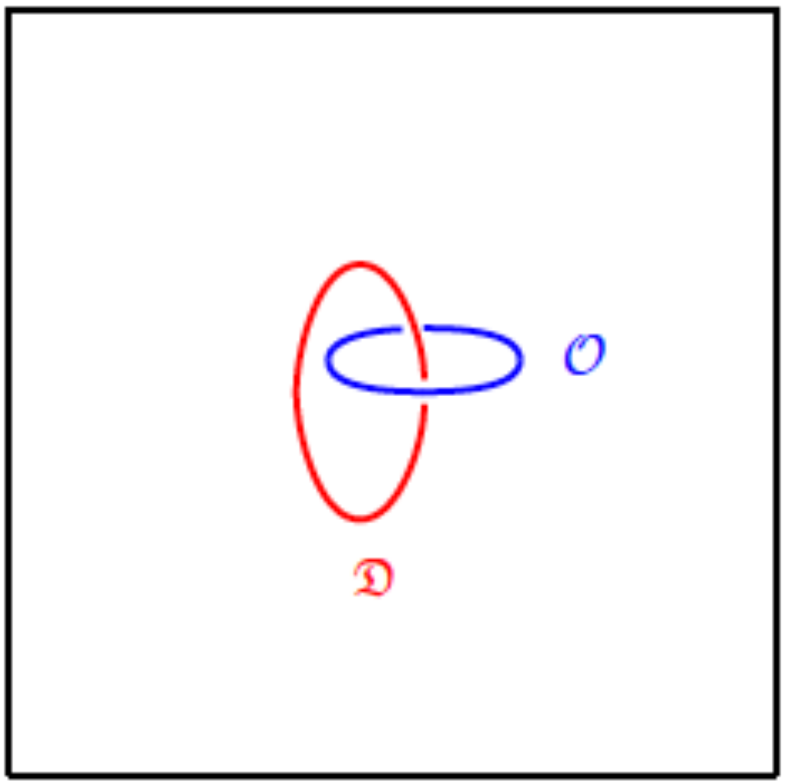
$\Downarrow$

Supersymmetric QFT in 5D

[Gaiotto, Kapustin, Seiberg, Willet; 2014]

1) Higher Global Symmetries in QFT

Introduction of an extended defects $\mathfrak{D}$ and a symmetry operators $\mathcal{O}$ acting on them



Space-time

One-form symmetry generated by
a string defect $\mathfrak{D}$ (1-dim) (Wilson line) and
a symmetry operator $\mathcal{O}$ acting on it.

[Standard zero-form symmetry generated
by a point defect $\mathfrak{D}$ (0-dim) &
a symmetry operator $\mathcal{O}$ acting on it.]

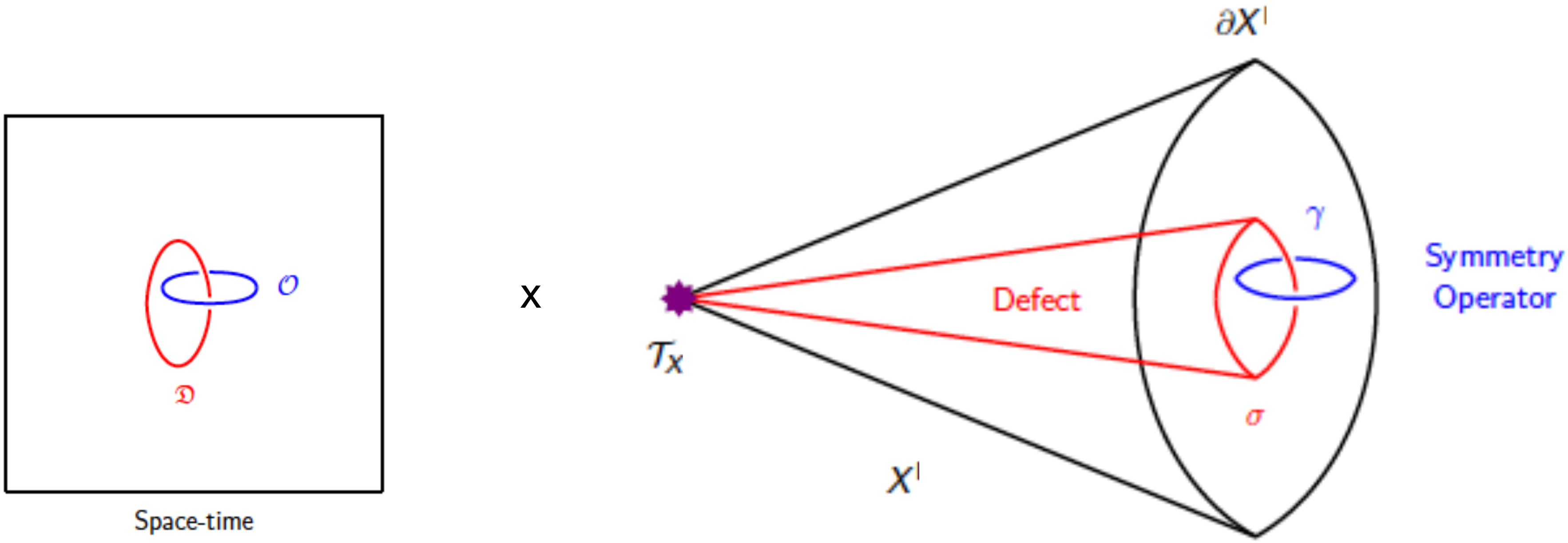
Intertwining of one-form and zero-form symmetries → leads to two-group symmetry

[Benini, Cordova, Hsin; 2019],
[Lee, Ohmori, Tachikawa; 2021],…

Higher Symmetries from String Theory (Geometric Engineering)

Example: Isolated Singularity

QFT $\mathcal{T}_X$ with string theory construction on conical non-compact X



p-dim defects $\mathfrak{D}$ associated w/ branes wrapping non-compact cycles σ on X

Symmetry operators $\mathcal{O}$ w/ branes wrapping cycles γ on the boundary ∂X

In M-theory, defects due to M2 and M5 branes

El. $\mathcal{D}_p^{\mathrm{M2}} = \dfrac{H_{3-p}(X,\partial X)}{H_{3-p}(X)} \cong H_{3-p-1}(\partial X)|_{\mathrm{triv}}$

Mag. $\mathcal{D}_p^{\mathrm{M5}} = \dfrac{H_{6-p}(X,\partial X)}{H_{6-p}(X)} \cong H_{6-p-1}(\partial X)|_{\mathrm{triv}}$

[Del Zotto, Heckman, Park, Rudelius; '16]
[Morrison, Schafer-Nameki, Willett; '20]
[Albertini, Del Zotto, Etxebarria, Hosseini; '20]
…
[M.C., Heckman, Hübner, Torres; '22
[Del Zotto, Garcia Etxebarria, Schäfer-Nameki; 2022],
…
[Heckman, Hübner, Torres, Zhang; 2023],…
[M.C.,Heckman, Hübner, Torres; 2023],...

Focus on electric (M2-branes) p=1 defects (Wilson lines)

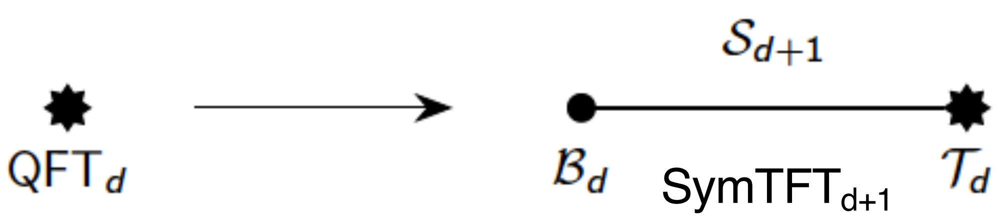
Symmetry operators: $H_\ell(\partial X)$

2a) Symmetry Theories

Symmetry Structures: How symmetries are represented (spontaneously broken or not?), including constraints on representations, anomalies, etc.

[Witten; 1998], [Gaiotto, Kulp; 2020];
[Apruzzi, Bonetti, Garcıa-Etxebarria, Hosseini, Schäfer-Nameki; 2021],
[Freed, Moore, Teleman; 2022], [Kaidi, Ohmori, Zheng; 2023];
[Brennan, Sun; 2024], [Bonetti, Del Zotto, Minasian; 2024], ...

In QFT in d-dim one can often isolate such symmetry structures into a topological field theory (TFT) in (d+1)-dim → Symmetry Topological Field Theory (SymTFT) $\mathcal{S}_{d+1}$



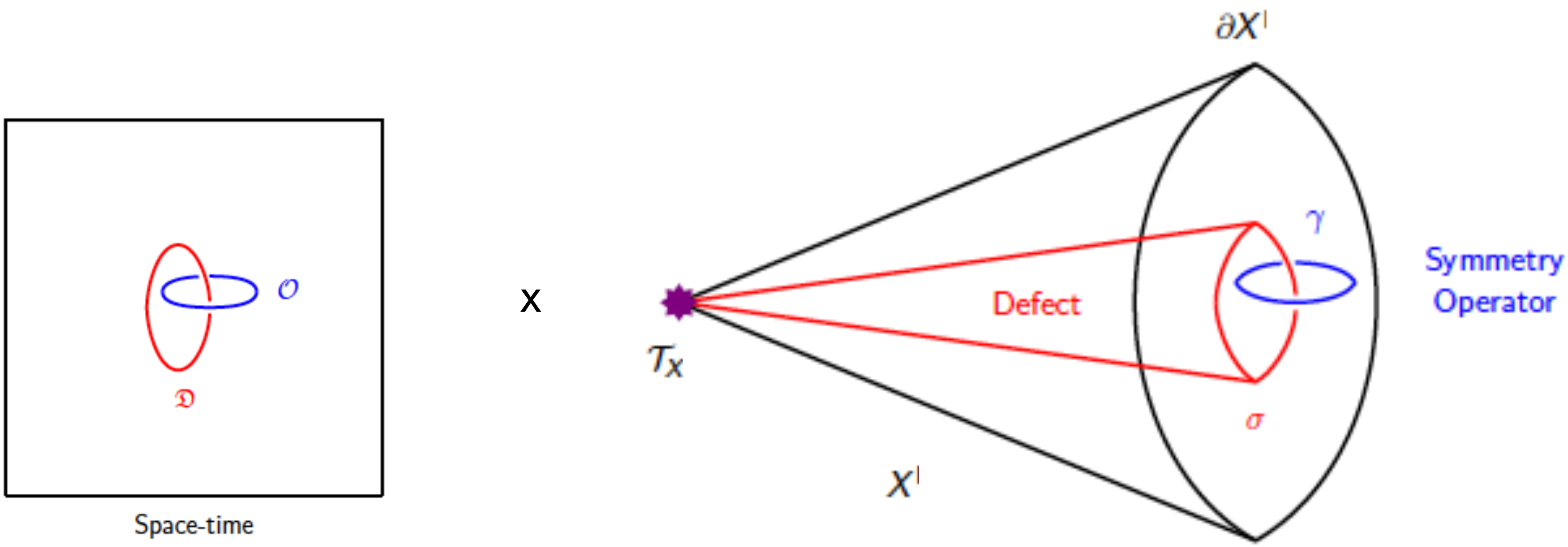
Indicating physical boundary conditions (b.c.) at $\mathcal{T}$ and topological b.c. at $\mathcal{B}$ for the SymTFT.

Many symmetry structures depend only on $\mathcal{B}$ and the TFT.

SymTFT from String Theory (Geometric Engineering)

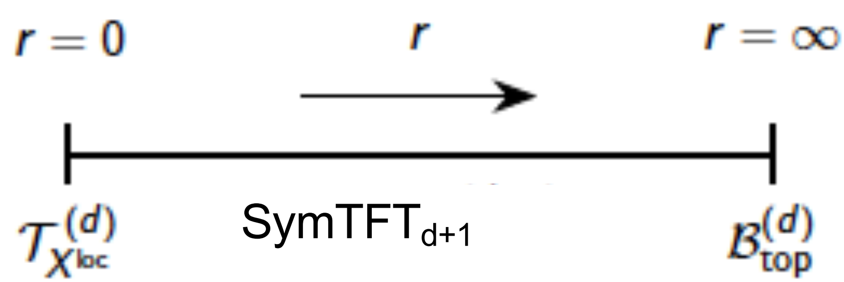
Prototype: Isolated Singularity

QFT $\mathcal{T}_X$ with string theory construction on conical non-compact X



Its SymTFT follows via compactification of ∂X over radial slices:

Physical b.c. from singularities,
topological b.c. from SUGRA b.c.



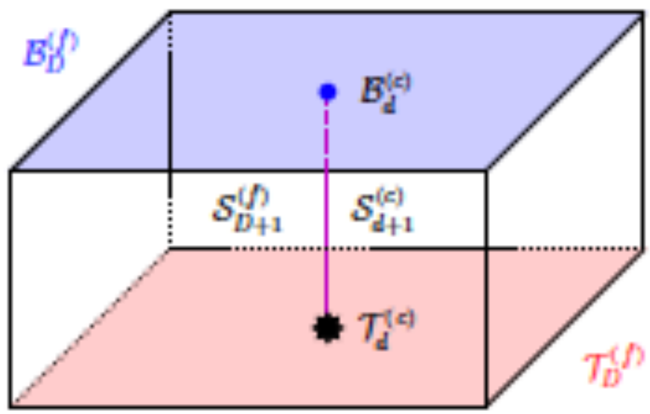
[Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schäfer-Nameki; 2021]
[Heckman, Hübner, Torres, Yu, Zhang;2022][van Beest, Gould, Schäfer-Nameki, Wang; 2022]
[Yu;2023] [Apruzzi, Bonetti, Gould, Schäfer-Nameki; 2023]
[Lawrie, Yu, Zhangl 2023] [Del Zotto, Meynet, Moscrop; 2024]...[Franco, Yu; 2024]…

2b) Generalizations to Nested SymTFTs

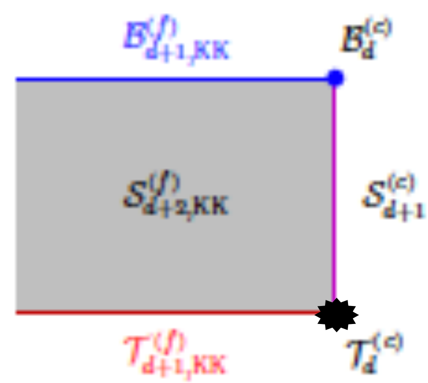
[M.C., Donagi, Heckman, Hübner, Torres; 2024]

Prototype

Consider a QFT in D dim and 'insert' into it a defect QFT in d<D dim.
What is the symmetry theory of the combined system?



(i)



(ii)

The sandwich of the bulk QFT in D-dim with defect QFT in d-dim inserted. → KK reduce (i) to an open nested SymTFTs.

Nested SymTFTs via Geometric Engineering Natural

They are associated with nested/non-isolated singularities

[M.C., Donagi, Heckman, Hübner, Torres; 2024]

Subsequent work

[Garcia-Etxebarria, Huertes, Uranga; 2024],
[Huertes, Uranga; 2024],
[Bhardway, Copetti, Pajer, Schäfer-Nameki; 2024]
…

Conical Prototypes of non-Isolated Singularities

Orbifolds $\mathbb{C}^3/\Gamma$ = Cone(S^5/Γ) w/ finite $\Gamma \subset SU(3)$

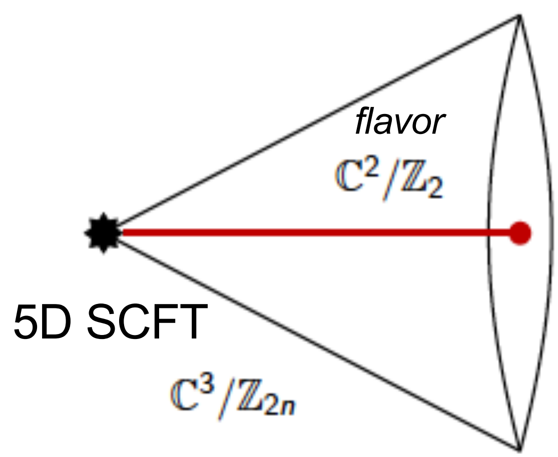
w/flavor branes at co-dim 4 loci $\mathbb{C}^2/\Gamma_{fix}$= Cone($S^3/\Gamma_{fix}$)

$\mathbb{C}^3 = \{z_1, z_2, z_3\}$

$\Gamma = \mathbb{Z}_{2n}$, $\Gamma_{fix} = \mathbb{Z}_2$

Example:

- Setup: $\boxed{\text{M-theory on } X = \mathbb{C}^3/\mathbb{Z}_{2n}(1,1,2n-2)}$



(i) Cone

- SU(n) 5D SCFT at $\{z_1, z_2, z_3\} = 0$ → isolated co-dim 6 singularity
  SU(2) flavor brane at $\{z_1, z_2\} = 0$ → red cone - co-dim 4 singularity
  (flavor brane $\mathbb{C}^2/\mathbb{Z}_2$)
- SU(n) 5D SCFT as defect nested within a 7D SU(2) SYM theory

Past

[M.C., Heckman, Hübner, Torres; 2022]

Such non-compact orbifold examples were chosen to identify geometric origin of higher-form symmetries (0-form, 1-form & 2-group) by studying the symmetry defects via algebraic topology (via cutting & gluing of singular boundary ∂X of the non-compact space X *and* employing Mayer-Vietoris exact sequences).

Subsequent

[M.C., Heckman, Hübner, Torres; 2024]

Concrete construction of the nested SymTFTs of the joint 7D/5D system. Allows for studies of how any topological operator behaves when pushed from the symmetry bulk SymTFT onto the symmetry boundary SymTFT.

Further works [Garcia-Etxebarria, Huertes, Uranga; 2024], [Huertes, Uranga; 2024], [Bhardway, Copetti, Pajer, Schäfer-Nameki; 2024],…

Recent

[M.C., Donagi, Heckman, Hübner, Torres; 2024],
[M.C., Donagi, Heckman, Hübner; 2025]

3) Anomaly Couplings in SymTFTs

Recall: M-theory on $R^{1,d}$ x X with X = Cone(∂X)   d+1=5, dim(X) = 6
The SymTFT on $R^{1,d}$ *x I* follows by compactification on ∂X.

Starting point for anomaly couplings:11D SG topological terms

[Witten; 2001],…

$$\frac{2\pi}{6}\int_{11\mathrm{D}}\frac{C_3}{2\pi}\left[\left(\frac{G_4}{2\pi}\right)^2+\frac{p_1^2-4p_2}{32}\right] \Longrightarrow \int_{12\mathrm{D}} L_{CS}=\frac{2\pi}{6}\int_{12\mathrm{D}}\frac{G_4}{2\pi}\left[\left(\frac{G_4}{2\pi}\right)^2+\frac{p_1^2-4p_2}{32}\right]$$

Globally defined

Chern-Simons action in 12D

$$\Longrightarrow \quad S_{\text{anomaly}}=\int_{\mathbb{R}^{1,d}\times I}\left(\int_{X'}L_{CS}\right)$$

w/ ∂X′ = ∂X, but possibly X≠ X′
(topological coupling)

SymTFT action

G_4- field strength of C_3; $p_{1,2}$ - Pontryagin classes of TM_{11}, specifying X_8 -8-form characteristic class

Anomaly couplings: Isolated Singularity

Recall: M-theory on $X = \mathbb{C}^3/\Gamma$ with finite $\Gamma \subset SU(3)$.
1-form symmetry sourced by background (torsional) fields B_2, which arise by expanding

$G_4 = G^0_4 + B_2 \wedge F_2 + \ldots$
w/ free class $F_2 \in H^2(X') = \mathbb{Z}^{\#}$ projecting to a generator $H^2(X')/H^2(X', \partial X') \simeq Ab(\Gamma) \simeq Z_N$

[Apruzzi, Bonetti, Garcıa-Etxebarria, Hosseini, Schäfer-Nameki; 2021]

There is 1-form self-anomaly & mixed 1-form gravitational anomaly

$$S_{\text{anomaly}} = \int \frac{|\Gamma|}{2\pi} B_2 dB_3 \boxed{+ \beta B_2^3 + \gamma B_2 (p_1/4)} + \ldots$$

$$\beta = \int_{X'} F_2^3$$

$$\gamma = \int_{X'} -\frac{1}{24} F_2 p_1(TX')$$

Calculation of Anomaly Coefficients β and γ
(determined up to shifts that change S_{anomaly} by an integer)

For $B_2 dB_3 \sim B \wedge F$-type terms see [Garcia-Etxebarria, Hosseini; 2024]

Calculation of Coefficient $\alpha = \beta + \gamma$

Employ Dirac operator on X′ coupled to a line bundle L with first Chern class $c_1(L) = F_2$. Its index, due to Atiyah-Patodi-Singer (APS):

[Atiyah, Patodi, Singer; 1975]

bulk - denote $\alpha = \beta + \gamma$ boundary

$$\text{Index } D_L = \int_{X'} \left(\frac{1}{6} c_1(L)^3 - \frac{1}{24} p_1(TX') c_1(L) \right) - \frac{h_L(\partial X') + \eta_L^{\not{D}}(\partial X')}{2}$$

$\eta_L^{\not{D}}(\partial X')$ - spectral invariant of Dirac op. (twisted by L) on ∂X'

[$h_L(\partial X')$ = 0 – dim of kernel of Dirac op. on ∂X']

Integer & $\partial X' = \partial X$

Bulk coeff. $\alpha = \beta + \gamma$ (modulo integers) is $\boxed{\alpha = \frac{1}{2}\eta_L^{\not{D}}(\partial X)}$

For $\mathbb{C}^3/\Gamma$ w/$\Gamma = Z_N(p, q, r)$ w/isolated singularities & $\partial X = S^5/\mathbb{Z}_N$:

$$\alpha = \frac{1}{N}\sum_{k=1}^{N-1} \frac{(-1)^k \omega^k}{(\omega^{-pk/2} - \omega^{pk/2})(\omega^{-kq/2} - \omega^{kq/2})(\omega^{-kr/2} - \omega^{kr/2})}$$

w/ $\omega = \exp(2\pi i/N)$ & L w/ Q=1

[Donnelly; 1978], [Gilkey; 1984], [Bär; 1996]

The Status of α Coefficient Calculations

Old: Resolution of X → X' & intersection theory (in the bulk)

[Jefferson, Katz, Kim, Vafa; 2018], [Bhardwaj, Jefferson; 2019], [Tian, Wang; 2021]…

→ applied to the calculation of α in the bulk

[Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schäfer-Nameki; 2021]

[M.C., Donagi, Heckman, Hübner; 2025]

Agree

New: Calculation of $\eta_L^{\not{D}}(\partial X)$ invariant for singular X (on the boundary)

& APS index $\boxed{\alpha = \frac{1}{2}\eta_L^{\not{D}}(\partial X)}$ [M.C., Donagi, Heckman, Hübner; 2025]

Concrete example: $\mathbb{C}^3/Z_{2n+1}$ (1, 1, 2n−1):

$$\alpha = \frac{1}{2}\eta_L^{\not{D}}(S^5/\mathbb{Z}_{2n+1}) = \frac{n(n+1)}{6(2n+1)}$$

Results match for all $\mathbb{C}^3/\Gamma$, w/$\Gamma = Z_N(p, q, r)$ & isolated singularities

Anomaly Coefficients β & γ

Anomaly coefficients β & γ depend on a choice of generator for Z_N.
For another generator, they are changed by rescaling:

$$B_2 \to Q\, B_2 ,$$

for any charge Q with gcd(N,Q) = 1.
Then he anomaly coefficients β & γ scale as

$$(\beta , \gamma) \to (Q^3 \beta, Q\, \gamma) ,$$

Thus: $\alpha_Q = Q^3 \beta + Q\, \gamma$ w/

$$\alpha_Q = \frac{1}{2}\eta^{\not D}_{L_Q} = \frac{1}{N}\sum_{k=1}^{N-1}(-1)^k \omega^{kQ}\prod_{i=1}^{3}\frac{1}{\omega^{-km_i/2}-\omega^{km_i/2}} \mod 1$$

$L_Q = (R_Q \times S^5)/Z_N$ for the charge Q representation R_Q of Z_N

[Q = q is a preferred generator → determined à la Kawasaki (no time)]

Can solve for β & γ :

$$\beta_{L_Q} = +\frac{1}{12}\eta^{\not D}_{L_{2Q}}(\partial X) - \frac{1}{6}\eta^{\not D}_{L_Q}(\partial X) \qquad \mod 1$$

$$\gamma_{L_Q} = -\frac{1}{12}\eta^{\not D}_{L_{2Q}}(\partial X) + \frac{2}{3}\eta^{\not D}_{L_Q}(\partial X) \qquad \mod 1$$

Anomaly Coefficients α, β & γ

One parameter Examples of Anomaly Coefficients for $C^3/Z_N(m_1,m_2,m_3)$, twisted by a line bundle L_q , w/ Q= q – Kawasaki preferred generator

$\mathbb{Z}_N(m_1,m_2,m_3)$	$\frac{1}{2}\eta^{D\!\!\!/}_{L_q}(S^5/\mathbb{Z}_N)$	q	β_{L_q}	γ_{L_q}
$\frac{1}{2n+3}(1,1,2n+1)$	$\frac{(n+1)(n-1)}{3(2n+3)}$	-2	$-\frac{2n-1}{6(2n+3)}$	$\frac{2n^2+2n-3}{6(2n+3)}$
$\frac{1}{12n+5}(1,3,12n+1)$	$\frac{(2n-1)(6n-10)}{12n+5}$	-12	$\frac{24}{12n+5}$	$\frac{12n^2-26n-14}{12n+5}$
$\frac{1}{30n+7}(1,5,30n+1)$	$\frac{(5n-9)(15n-11)}{30n+7}$	-30	$-\frac{5(30n-53)}{2(30n+7)}$	$\frac{150n^2-230n-67}{2(30n+7)}$
$\frac{1}{56n+9}(1,7,56n+1)$	$\frac{4(7n-13)(28n-23)}{3(56n+9)}$	-56	$-\frac{14(56n-103)}{3(56n+9)}$	$\frac{2(392n^2-658n-123)}{3(56n+9)}$

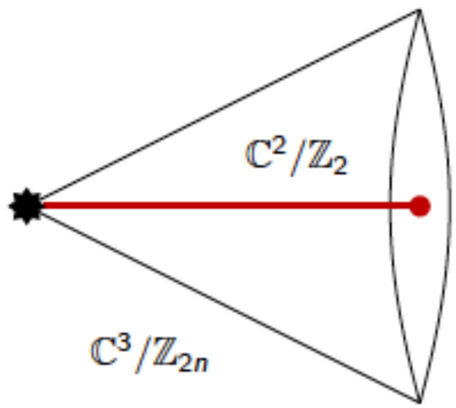
Explicit form also for a broader class of two parameter examples $C^3/Z_N(m_1,m_2,m_3)$, w/ isolated singularities → in the paper

All in agreement w/resolved calculations

Coefficients α for Non-Isolated Singularities

Recall: conical orbifolds $\mathbb{C}^3/\Gamma$ = Cone(S^5/Γ) w/ finite $\Gamma \subset SU(3)$
w/flavor branes at co-dim 4 loci $\mathbb{C}^2/\Gamma_{fix}$ = Cone(S^3/Γ_{fix})

Again, Concrete Example $\mathbb{Z}_{2n}(1,1,2n\text{-}2)$ orbifold: $\Gamma = \mathbb{Z}_{2n}$, $\Gamma_{fix} = \mathbb{Z}_2$



- SU(n) 5D SCFT at ✸ isolated co-dim 6 singularity
- SU(2) 7D SYM at red cone co-dim 4 singularity (flavor brane at $\mathbb{C}^2/\mathbb{Z}_2$)

Coefficients α for Non-Isolated Singularities

Example $\mathbb{Z}_{2n}(1,1,2n-2)$ orbifold:

$\Gamma = \mathbb{Z}_{2n}$, $\Gamma_{\text{fix}} = \mathbb{Z}_2$

- Calculation directly in field theory

[Gukov, Hsin, Pei; 2020]

Agree

- Resolution and intersection computation

[Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schäfer-Nameki; 2021]

$$\alpha = \frac{(n-1)(n-2)}{6n}$$

Agree

- Computation directly in singular geometry:

[M.C. Donagi, Heckman, Hübner; 2025]

$$\boxed{\alpha = \frac{1}{2}\eta_L^{\not D}(S^5/\Gamma)}$$

$$\frac{1}{2}\eta_L^{\not D}(S^5/\Gamma) = \frac{1}{2n}\sum_{k=1,k\neq n}^{2n-1}\frac{(-1)^k\omega^{k(-2)}}{(\omega^{-k/2}-\omega^{k/2})(\omega^{-k/2}-\omega^{k/2})(\omega^{-k(2n-2)/2}-\omega^{k(2n-2)/2})}$$

[Degaratu; 2008]

$\omega = \exp(2\pi i/N)$ & L_Q w/ $Q = -2$

Anomaly Coefficients : α, β & γ for $\Gamma = \mathbb{Z}_N$, $\Gamma_{fix} = \mathbb{Z}_g$

Eta Invariance for & L_Q

[Degaratu; 2008]

$$\frac{1}{2}\eta_L^{D\!\!\!/} = \frac{1}{N}\sum_{\gamma\in \mathrm{FF}(\mathbb{Z}_N)} (-1)^k \omega^{kQ_L} \prod_{i=1}^{3} \frac{1}{\omega^{-km_i/2} - \omega^{km_i/2}}$$

$$= \frac{1}{N}\sum_{\ell=0}^{g-1}\left(\sum_{k=\ell N/g+1}^{(\ell+1)N/g-1} (-1)^k \omega^{kQ_L} \prod_{i=1}^{3} \frac{1}{\omega^{-km_i/2} - \omega^{km_i/2}}\right)$$

$$\alpha_L = \frac{1}{2}\eta_L^{D\!\!\!/} \mod 1$$

$$\alpha_L = \beta_L + \gamma_L \qquad \beta_{L_Q} = +\frac{1}{12}\eta^{D\!\!\!/}_{L_{2Q}}(\partial X) - \frac{1}{6}\eta^{D\!\!\!/}_{L_Q}(\partial X) \mod 1$$

$$\gamma_{L_Q} = -\frac{1}{12}\eta^{D\!\!\!/}_{L_{2Q}}(\partial X) + \frac{2}{3}\eta^{D\!\!\!/}_{L_Q}(\partial X) \mod 1$$

One-parameter examples:

$\mathbb{Z}_N(m_1, m_2, m_3)$	$\frac{1}{2}\eta^{D\!\!\!/}_{L_q}(S^5/\mathbb{Z}_N)$	q	β_{L_q}	γ_{L_q}
$\frac{1}{2n+2}(1,1,2n)$	$\frac{n(n-1)}{6(n+1)}$	-2	$-\frac{n-1}{6(n+1)}$	$\frac{n^2-1}{6(n+1)}$
$\frac{1}{6n+3}(1,2,6n)$	$\frac{n(n-1)}{2(2n+1)}$	-3	$-\frac{n-1}{6(2n+1)}$	$\frac{3n^2-2n-1}{6(2n+1)}$
$\frac{1}{12n+4}(1,3,12n)$	$\frac{n(n-1)}{3n+1}$	-4	$-\frac{n-1}{6(3n+1)}$	$\frac{6n^2-5n-1}{6(3n+1)}$
$\frac{1}{20n+5}(1,4,20n)$	$\frac{5n(n-1)}{3(4n+1)}$	-5	$-\frac{n-1}{6(4n+1)}$	$\frac{10n^2-9n-1}{6(4n+1)}$

In agreement w/ explicit intersection calculations for resolved geom.

Also examples with non-cyclic/non-Abelian Γ & Z_N x Z_M → in the paper

For non-isolated (flavor) singularities

Non-Abelian flavor symmetry for $\mathbb{C}^3/\mathbb{Z}_N(m_1, m_2, m_3)$ w/ $g_k \equiv \gcd(N, m_k) > 1$

$$\rightarrow \quad \mathfrak{f} \equiv \mathfrak{su}(g_1) \oplus \mathfrak{su}(g_2) \oplus \mathfrak{su}(g_3)$$

Additional (Mixed) Flavor Brane Anomalies

[Via Wess-Zumino Action/7D SYM effective action reduction]

$$\frac{2\pi}{2(2\pi)^2} \sum_{k=1}^{3} \delta_k \int_{6\mathrm{D}} B_2^{(N/g)} \mathrm{Tr}(F_2^{(k)} F_2^{(k)}) \quad \text{and} \quad \frac{2\pi}{6(2\pi)^3} \sum_{k=1}^{3} \epsilon_k \int_{6\mathrm{D}} \mathrm{Tr}(F_2^{(k)} F_2^{(k)} F_2^{(k)})$$

$$\delta_k = \ell(t_4^{(ij)}, t_2) = q g_i g_j / N\,, \qquad \{i, j, k\} = \{1, 2, 3\} \qquad\qquad \epsilon_k = \frac{m_k}{N}$$

[M.C., R. Donagi, J. Heckman, M. Hübner, E. Torres; 2024]

II. Quiver Approach – Focus on Anomaly Coefficients

[V. Chakrabhavi, M.C., J. Heckman, S. Meynet; 2605.3035]

Concretely: Study 4D Type IIA (by reducing M-theory of X on S^1) w/ D0 probing X, realizing a quiver quantum mechanics or T^3-dual to Type IIB w/ D3 probing X, realizing D=4, N=1 quiver QFT.

For $X = \mathbb{C}^n/\Gamma$ with boundary $\partial X = S^{2n-1}/\Gamma$: (n=3)

$$\frac{1}{2}\eta_\chi^{\bar\partial}\left(S^{2n-1}/\Gamma\right) = \frac{(-1)^n}{|\Gamma|}\sum_{\gamma\in\Gamma_{\mathbf{FF}}}\frac{\chi(\gamma)}{\det(1-\gamma)}$$

[Degaratu; 2008]

w/ canonical lift $\Gamma \to SU(n)$, representation of Γ w/ character χ &

$\Gamma_{\mathbf{FF}} \subset \Gamma$ the subset acting fixed-point freely on the sphere S^{2n-1}

Degaratu's Theorem:

$$\frac{1}{2}\eta_\chi^{\bar\partial}(S^{2n-1}/\Gamma) = (-1)^n \operatorname*{Res}_{t=1}\frac{M_\chi(t)}{1-t}$$

w/ $M_\chi(t)$ denotes the twisted (by character χ) Molien series:

$$M_\chi(t) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi(\gamma)}{\det(1-t\gamma)}$$

Untwisted Molien series ($\chi=1$) – generating function,
counting the number of Γ- invariant holomorphic functions
(independent polynomial invariants at each polynomial degree) for $X = \mathbb{C}^n/\Gamma$.

Twisted Molien series – generating function, counting symmetric polynomials of specific degrees, "twisted" by a finite group action.

Note, anomaly takes the form:

$$\alpha = \frac{1}{2}\eta_L^{\not{D}}(Y) = \frac{1}{2}\left(\eta_L^{\bar\partial}(Y) - \eta_\varnothing^{\bar\partial}(Y)\right)$$

$Y = \partial X = S^{2n-1}/\Gamma$ (n=3)

w/ $\eta_L^{\bar\partial}$ & $\eta_\varnothing^{\bar\partial}$ expressed in terms of twisted and untwisted Molien series as:

$$\boxed{\alpha = \operatorname*{Res}_{t=1} \frac{M_{\mathrm{tw}}(t) - M_{\mathrm{untw}}(t)}{1-t}}$$

From Geometric Engineering to Representation Theory!

Example: X= $\mathbb{C}^3/\Gamma$ w/$\Gamma=\mathbb{Z}_3$:

$$M_{\mathrm{tw}}(t) = \frac{1}{3}\sum_{k=0}^{2}\frac{\omega^{-k}}{(1-\omega^k t)^3} = \frac{3(t+2t^4)}{(1-t^3)^3}$$

$$M_{\mathrm{untw}}(t) = \frac{1}{3}\sum_{k=0}^{2}\frac{1}{(1-\omega^k t)^3} = \frac{1+7t^3+t^6}{(1-t^3)^3}$$

$\omega = e^{2\pi i/3}$.

$\Rightarrow$ $\alpha = -1/9$

agrees w/geometric engineering

Quiver Hilbert Series and η Invariants

- Introduce matrix-valued Hilbert series H(t), derived from the brane-probe of X (quiver-theoretic refinement of the Molien series).

[Eager; 2011],
also [Ginzburg; 2006], [Bocklandt; 2006],[Martelli, Sparks, Yau; 2006]

- w/ matrix H(t)'s (i,j)- entry counts (F-term equivalence) classes of paths from quiver node i to node j (weighted by total R-charge).

Diagonal entries(i=j) count loops at a given quiver node i;
off-diagonal ones (i≠j) count open paths between i and j,
interpolating between distinct fractional branes.

- For orbifolds $\mathbb{C}^3/\Gamma$, w/{R_i} – irreducible representations of Γ,
  R – representation by which Γ acts on $\mathbb{C}^3$
  A_{ij} - the McKay adjacency matrix defined as

$$R \otimes R_i = \bigoplus_j A_{ij} \; R_j$$

[McKay;1980],
[Kronheimer, Nakajima;1990]…
[Douglas, Moore; 1996],
[Lawrence, Nekrasov, Vafa;1998]..

matrix-valued Hilbert series greatly simplifies:

$$\boxed{H(t) = \left(\mathbf{1} - tA + t^2 A^T - t^3 \mathbf{1}\right)^{-1}}$$

Connections to Molien series

$H_{00}(t)$ (trivial rep. node) correspond to untwisted Molien series:

$$\boxed{H_{00}(t)} = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\frac{1}{\det(1-t\rho(g))}$$

$\rho(g) = \boldsymbol{\gamma}$ from before

$H_{ij}(t)$ correspond to twisted Molien series:

$$H_{ij}(t) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\frac{\chi_i(g)\chi_j(g^{-1})}{\det(1-t\rho(g))}$$

w/ χ_i (g) is the character of the irreducible representation R_i.

$$M_{\mathrm{untw}}(t) = H_{00}(t), \qquad M_{\mathrm{tw}}(t) = H_{0i}(t) \qquad (i \neq 0)$$

Anomaly coefficient from quiver data:

$$\boxed{\alpha_i = \operatorname*{Res}_{t=1}\frac{H_{0i}(t) - H_{00}(t)}{1-t} \mod 1,}$$

[i =1 for isolated sing]

$H_{00}(t)$ counts gauge-invariant operators, $H_{0i}(t)$ counts operators in the ith twisted sector

= $H_{\mathrm{untw}}(t)$

= $H_{\mathrm{tw}}(t)$

Quiver probing $\mathbb{C}^3/\mathbb{Z}_3$

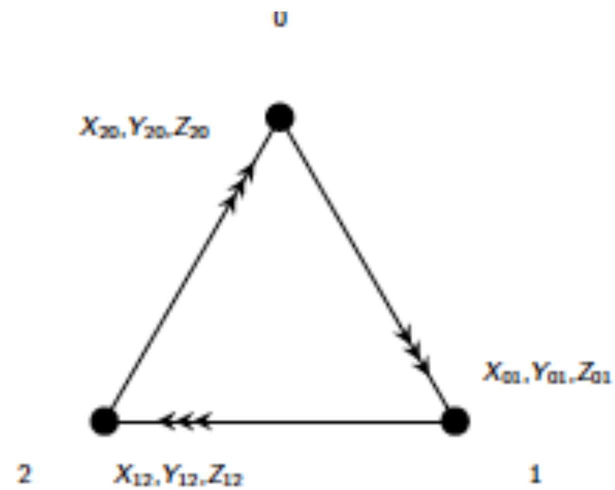


Figure: Quiver for D-brane probe of $\mathbb{C}^3/\mathbb{Z}_3$.

Adjacency matrix A:

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{pmatrix}$$

Matrix valued Hilbert series H(t):

$$H(t) = \begin{pmatrix} -\frac{1+7t^3+t^6}{(-1+t^3)^3} & -\frac{3(t+2t^4)}{(-1+t^3)^3} & -\frac{3t^2(2+t^3)}{(-1+t^3)^3} \\ -\frac{3t^2(2+t^3)}{(-1+t^3)^3} & -\frac{1+7t^3+t^6}{(-1+t^3)^3} & -\frac{3(t+2t^4)}{(-1+t^3)^3} \\ -\frac{3(t+2t^4)}{(-1+t^3)^3} & -\frac{3t^2(2+t^3)}{(-1+t^3)^3} & -\frac{1+7t^3+t^6}{(-1+t^3)^3} \end{pmatrix}$$

$$\boxed{\alpha = \operatorname*{Res}_{t=1} \frac{H_{01}(t) - H_{00}(t)}{1-t} = -1/9}$$

Agrees with geometric engineering

Final Remarks

Details in [Chakrabhavi, M.C., Heckman, Meynet; 2026]

- The η-invariant anomaly coefficient extracted from a matrix-valued quiver Hilbert series H(t) also for non-isolated (flavor singularities); α for all Abelian orbifolds via H(t). [i≠1]
- Approach valid for quivers probing other toric CYs,
  e.g., cones over $Y^{p,q}$, [Martelli, Gauntlett, Sparks, Waldram;2004]
  $L^{p,q,r}$ [M.C., Lü, Page, Pope; 2005] ,…

  → obtained new results for η invariants for families of cones over $Y^{p,q}$

- For non-isolated singularities, additional flavor sectors are supported on lower-dimensional strata. (These flavor symmetries can mix with the one-form symmetry through a 2-group structure.) This refined data is captured by tracking the higher-order poles:

$$\alpha^{\mathrm{ref}} = \frac{1}{2}\eta^{\mathrm{ref}} = \mathrm{Res}^{(2)}_{t=1}\left[\frac{H_{\mathrm{tw}}(t) - H_{\mathrm{untw}}(t)}{1-t}\right] + \mathrm{Res}_{t=1}\left[\frac{H_{\mathrm{tw}}(t) - H_{\mathrm{untw}}(t)}{1-t}\right] \mod 1$$

$\mathrm{Res}^{(2)}$ captures data from the $(1 - t)^{-2}$ term

w/ $$\alpha^{\mathrm{ref}} \equiv \beta + \gamma + \sum_k \delta_k + \sum_k \epsilon_k \mod 1$$

Thank you!