Local Fukaya categories

Mohammed Abouzaid
Stanford University

Calabi–Yau at 50: Legacy and Future
Tsinghua University, 10 September 2026

supported by the NSF and the Simons Foundation

The problem

A complex manifold is built out of local pieces, and so are its algebraic invariants like coherent sheaves.

A symplectic manifold is also built out of local pieces, but its algebraic invariants like the Fukaya category are not, or at least not in an obvious way.

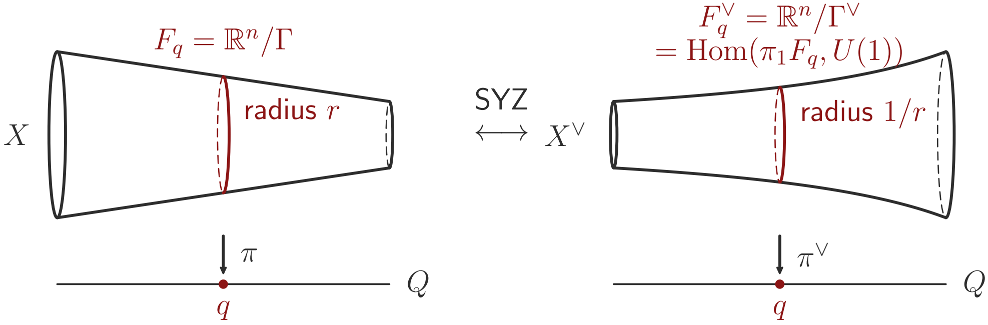
We now have an answer, thanks to an idea of Varolgunes for localising these invariants. I will explain how it gets implemented for Fukaya categories, and what this has to do with Calabi–Yau's.

This is joint work in progress with Yoel Groman and Umut Varolgunes.

SYZ mirror symmetry

The SYZ conjecture

Strominger, Yau and Zaslow conjectured that Calabi–Yau manifolds carry *special Lagrangian fibrations*: families of half-dimensional subspaces on which the Kähler form vanishes and the holomorphic volume form has constant phase. The mirror phenomenon should be induced by duality of tori. There is a long history of results confirming this for non-singular fibrations.



Moduli of A-branes are Calabi–Yau

We think of these moduli in terms of *local systems*: an A-brane is a Lagrangian L carrying a rank-one unitary local system, so the branes supported on one SYZ fibre are the points of the dual torus.

Letting L move as well, flux together with holonomy give the local model

$$H^1(L;\mathbb{R}) \times H^1(L;U(1)) \cong H^1(L;\mathbb{C}^*).$$

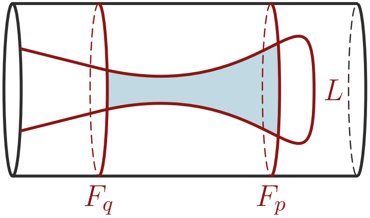
The symplectic notion of flux corresponds to analytic coordinates on this moduli.

For an SYZ torus L^n, the cup product identifies $\bigwedge^n H^1(L;\mathbb{C})$ with $H^n(L;\mathbb{C})$, and the fundamental class identifies that with $\mathbb{C}$. This gives a nowhere-vanishing holomorphic volume form. So the moduli of A-branes is Calabi–Yau.

Family Floer cohomology

Fukaya showed that Floer cohomology varies *analytically* in the flux coordinates. In particular, one can assign to L a sheaf $\mathcal{L}_L$ of coherent analytic complexes on Y, whose fibre is the Floer cohomology of L with a torus fibre carrying a rank-one unitary local system, and the family Floer map is a map from Floer cohomology to sheaf cohomology :

$$HF^*(L, L') \to H^*\big(Y; \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{L}_{L'}, \mathcal{L}_L)\big).$$



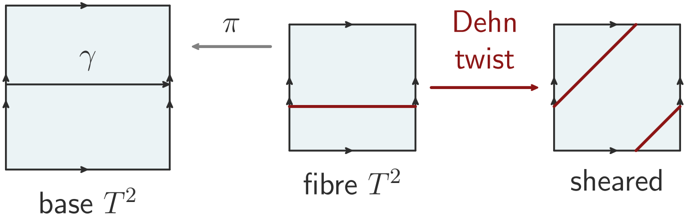
The Floer complexes $CF^*(L, F_q)$ for varying q are of rank 2.

The differential is a function of q, which varies analytically.

Homological mirror symmetry without corrections

Theorem (A., *J. Amer. Math. Soc.* 34 (2021))
Let X be a closed symplectic manifold with a Lagrangian torus fibration over a base Q with $\pi_2(Q) = 0$, equipped with a section. The Fukaya category of X embeds fully faithfully in the derived category of coherent sheaves on the rigid analytic mirror Y.

This result in particular applies to all the abelian varieties for which we expect mirror symmetry to hold, but it applies more generally to twisted examples such as the Kodaira–Thurston surface.

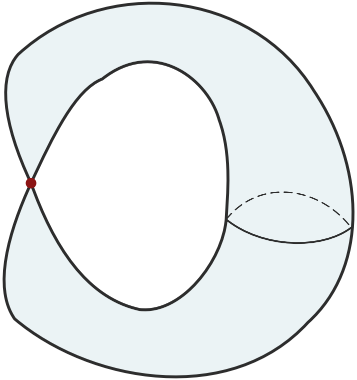


Singularities

Naive T-duality cannot be enough

Strominger, Yau and Zaslow already knew this.
A Lagrangian torus fibration on a compact Calabi–Yau with nonzero Euler characteristic, for example a Calabi–Yau hypersurface in a toric variety, must have singular fibres, and fibrewise duality is a deep problem near the discriminant.

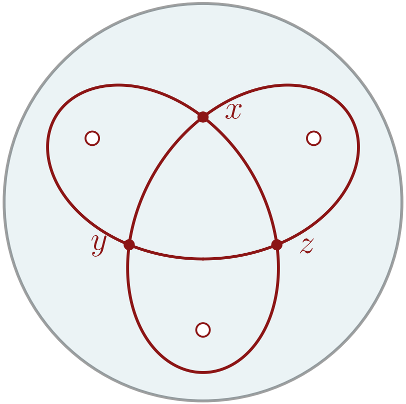
Question
What is the mirror of a singular fibre?



The answer: immersed Lagrangians

In the deformation framework of Fukaya–Oh–Ohta–Ono, and with the immersed Floer theory of Akaho–Joyce, the moduli of A-branes contains more than local systems once the Lagrangian is allowed to be *immersed*: each degree-one self-intersection carries its own deformation parameter.

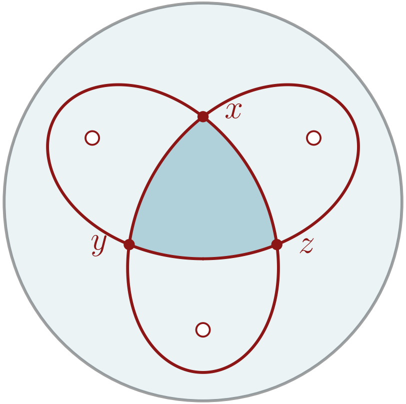
The model case is Seidel's immersed curve in the pair of pants: three self-intersections, so parameters x, y, z, and the discs it bounds give the potential $W = xyz$.



The answer: immersed Lagrangians

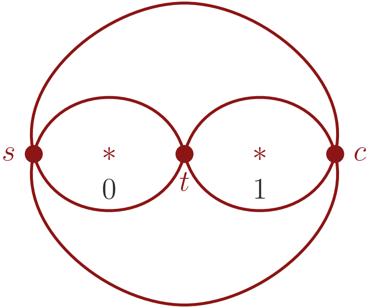
In the deformation framework of Fukaya–Oh–Ohta–Ono, and with the immersed Floer theory of Akaho–Joyce, the moduli of A-branes contains more than local systems once the Lagrangian is allowed to be *immersed*: each degree-one self-intersection carries its own deformation parameter.

The model case is Seidel's immersed curve in the pair of pants: three self-intersections, so parameters x, y, z, and the discs it bounds give the potential $W = xyz$.



Immersed Lagrangians resolve singular Lagrangians

Above dimension 2, we know that SYZ fibrations have more deeply singular fibres. There is no Floer theory for such objects, but by analogy with resolution of singularities, we can replace a singular Lagrangian by a nearby immersed one.



Theorem (A.–Sylvan '21)
Moduli spaces of A-branes supported on "resolutions" of singularities of SYZ fibrations recover the local geometry of the mirror space.

Smallest case beyond $(\mathbb{C}^*)^n$: the complement of $\prod_j x_j = 1$ in $\mathbb{C}^{n+1}$, mirror to the conic bundle over $(\mathbb{C}^*)^n$ with discriminant $\{\sum_j y_j = 1\}$.

Non-commutative Calabi–Yau

In what sense are these Calabi–Yau?

What I have been hiding so far is that, when one starts on the symplectic side, the moduli of A-branes sometimes cannot be directly described as a space. The categories are Calabi–Yau in Kontsevich's sense:

Definition
An A_∞ category $\mathcal{A}$ is Calabi–Yau of dimension n if the diagonal bimodule is perfect over $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ and there is a bimodule quasi-isomorphism

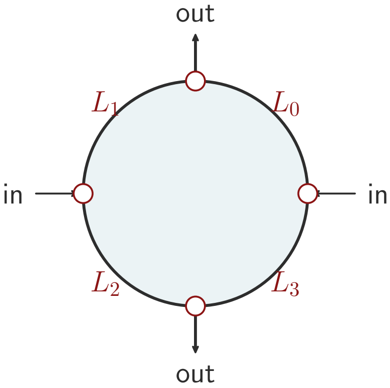
$$\mathbb{R}\,\mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A},\ \mathcal{A} \otimes \mathcal{A}) \ \simeq\ \mathcal{A}[-n],$$

coming from a class in negative cyclic homology.

The basic idea here is that $\mathcal{A}$ is the structure sheaf (holomorphic functions) and $\mathbb{R}\,\mathrm{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A},\ \mathcal{A} \otimes \mathcal{A})$ is the dualising complex (i.e. holomorphic n-forms), so the Calabi–Yau condition is an isomorphism between them.

Fukaya categories are Calabi–Yau

Theorem (A. 2010, Ganatra 2019)
In Floer theory, the volume form is provided by a count of discs with two inputs and two outputs.



The question of non-degeneracy is subtle, because symplectic manifolds do not, in general, have enough Lagrangian branes. The presence of enough Lagrangians is the generation criterion (A., *Publ. IHES* 112 (2010)).

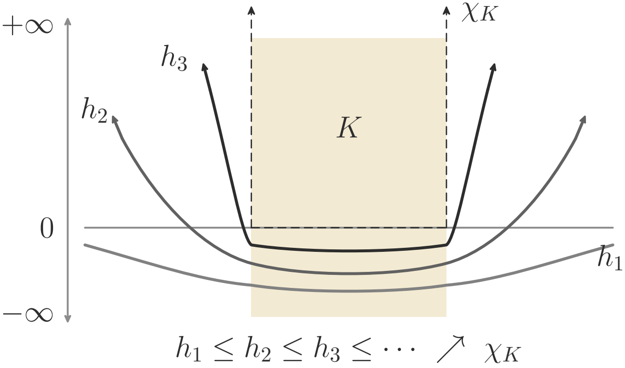
Fukaya category with supports

joint work in progress with Yoel Groman and Umut Varolgunes

Floer cohomology with support

Varolgunes introduced the following construction, which has an older origin in work of Floer and Hofer: given a compact subset K, take Hamiltonians converging to the indicator function which vanishes on K and is $+\infty$ off it, and complete the direct limit of the continuation maps.

The result is Floer cohomology *with support* in K, written $HF^*(L, K)$.

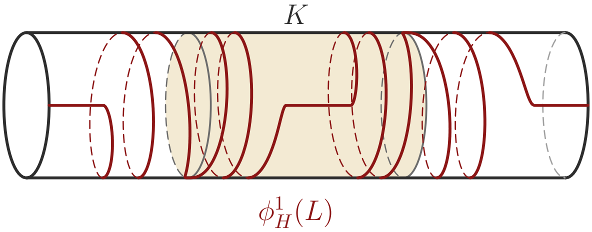


An inclusion $K \subset K'$ gives a restriction map $HF^*(L, K') \to HF^*(L, K)$, and triple inclusions compose. The group $HF^*(L, K)$ is *not* determined by $L \cap K$: it retains global information.

The computation for a Lagrangian section

Let $\mathbb{T} = (\mathbb{R}/\mathbb{Z})^2$ with $\omega = dp \wedge dq$, let $\pi(p, q) = p$, and let $K = \pi^{-1}([a, b])$ with $0 < b - a < 1$. Let L be the section $\mathbb{R}/\mathbb{Z} \times \{0\}$.

The Hamiltonians of the acceleration datum are pulled back from the base, so the flow shears L along the fibres: it winds k times near the boundary of K and unwinds again just outside it.



Inverting T, the answer is a completed sum of one generator in each winding class:

$$HF^*(L, K) \cong \widehat{\bigoplus_{N \in \mathbb{Z}}} \mathbb{K}\, \alpha_N, \qquad |\alpha_N| = 0.$$

The algebra structure

The degree-zero part should be the ring of functions on the analytic annulus of modulus $b - a$:

$$HF^0(L, K) \overset{?}{\cong} \mathbb{K}\langle x, y \rangle / (xy - e^{b-a}).$$

The two generators are the two winding directions, and $b - a$ is the area of the band.

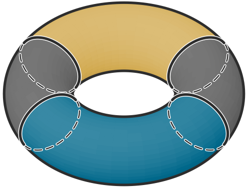
This is the simplest local model. The mirror of a neighbourhood of a fibre is a small piece of the mirror analytic space, and the descent statements that follow are about gluing those pieces together.

Varolgunes descent

Let $\pi\colon X \to Q$ be an *integrable system*: a map locally defined by Poisson commuting functions. An SYZ fibration is the example to keep in mind, and it is the one the last section uses.

Theorem (Varolgunes)
If Q_1 and Q_2 are compact subsets of Q, the restriction maps fit into a long exact sequence

$$\begin{aligned} \cdots \to HF^*\big(L, \pi^{-1}(Q_1 \cup Q_2)\big) \to HF^*\big(L, \pi^{-1}Q_1\big) \oplus HF^*\big(L, \pi^{-1}Q_2\big) \\ \to HF^*\big(L, \pi^{-1}(Q_1 \cap Q_2)\big) \to \cdots \end{aligned}$$

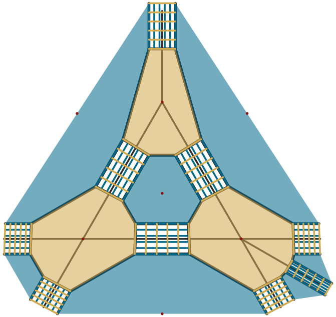


The Fukaya category with supports

Let M be a closed symplectic manifold. We construct a strict presheaf of A_∞ categories over the Novikov ring $\mathbb{K}_{\geq 0}$,

$$K \longmapsto \mathcal{F}_M(K),$$

indexed by the compact subsets of M.



Restriction and descent

Theorem (A.–Groman–Varolgunes, in progress)
For every pair of compact subsets $K \subset K'$ of M there is a strict A_∞ functor $\mathcal{F}_M(K') \to \mathcal{F}_M(K)$, the identity on objects, strictly compatible with triple inclusions. The symplectomorphism group acts: ϕ induces $\mathcal{F}_M(K) \to \mathcal{F}_M(\phi(K))$.

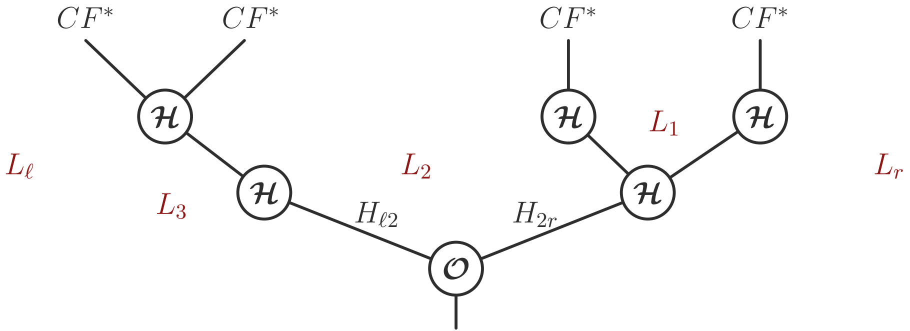
This is a genuine presheaf of A_∞ categories, not a diagram commuting up to coherent homotopy.

Theorem (A.–Groman–Varolgunes, in progress)
If $(K_1, \dots, K_N)$ form a cover of K which is pulled back from Q, then the canonical functor $\mathcal{F}_M(K) \to \check{\mathrm{C}}(\mathcal{F}_M; K_1, \dots, K_N)$ is a quasi-isomorphism of A_∞ categories.

This is what I promised at the beginning: a local-global way of computing Fukaya categories.

All Hamiltonians at once

Admitting essentially all Hamiltonians means recording all of them, all continuation maps and all higher homotopies, in a cubically enriched multicategory. The morphism complex from L_ℓ to L_r is a completed direct sum over labelled planar trees: the morphism space between two Lagrangians takes *every other Lagrangian* into account.



Locality

When does the local category see the ambient space?

Question

When does $\mathcal{F}_M(K)$ depend only on a neighbourhood of K?

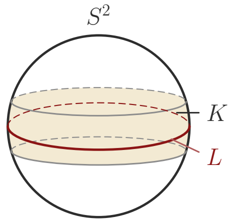
The smallest example

K a band around the equator of S^2: $\mathcal{F}_{S^2}(K)$ is modules over the Clifford algebra $\mathbb{K}\langle x\rangle/(x^2 - T^A)$, the Floer algebra of the equator.

Theorem (A.–Groman–Varolgunes, in progress)

If K is a small neighbourhood of an SYZ fibre, $HF^*(L, K) \cong HF^*(K \cap L)$.

We expect that this property (of locality) is in fact a consequence of a deepter property, related to the Calabi-Yau condition, rather than a feature of SYZ fibrations.



Much more work to be done

1. Prove homological mirror symmetry for a sufficiently fine cover.
2. Enlarge the objects: the microlocal approach of quantization suggestions that it is possible to incorporate singular objects.
3. Settle locality: What is the natural codition under which these categories are not deformed?
4. Beyond discrete rings: Floer homotopy theory suggests a chromatic refinement of many classical mirror phenomena.

Thank you

to Calabi and to Yau for 50 years of mathematics