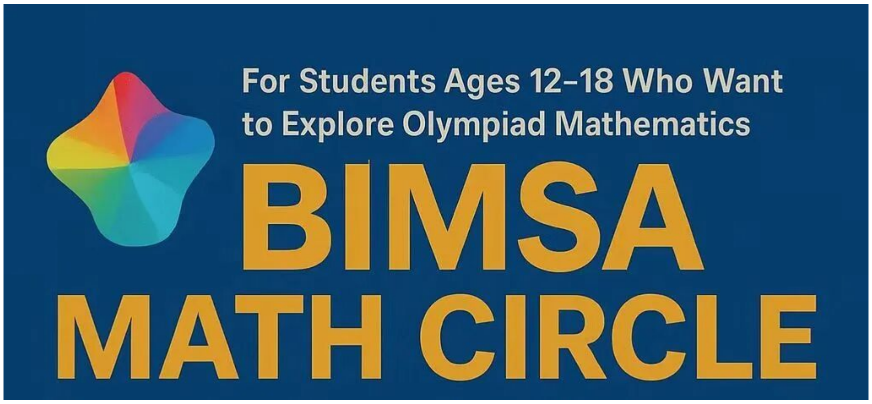

第一讲——代数：基本恒等式与不等式

CEZAR LUPU

1. 基础理论

1.1 两个实变量的恒等式. 设 a 和 b 是两个实数。下面我们将展示最重要的恒等式。

1.1.1 和差平方恒等式.

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a-b)^2 = a^2 - 2ab + b^2.$$

平方差恒等式:

$$a^2 - b^2 = (a-b)(a+b).$$

1.1.2 立方恒等式.

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

1.1.3 对称-反对称分解.

$$a = \frac{(a+b)+(a-b)}{2}, \qquad b = \frac{(a+b)-(a-b)}{2}.$$

1.1.4 平方和分解.

$$a^2 + b^2 = \frac{(a+b)^2 + (a-b)^2}{2}.$$

1.2 三个实变量的恒等式. 设 a, b, c 是三个实数。

1.2.1 和的平方与立方.

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca).$$

完全展开形式:

$$(a+b+c)^3 = a^3+b^3+c^3+3(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)+6abc.$$

对称形式:

$$(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a).$$

1.2.2 对称和恒等式. 令

$$\sigma_1 = a+b+c, \qquad \sigma_2 = ab+bc+ca, \qquad \sigma_3 = abc.$$

则

$$a^2+b^2+c^2 = \sigma_1^2 - 2\sigma_2.$$

1.2.3 基本因式分解.

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc. \tag{1}$$

$$(a+b+c)^3 - (a^3+b^3+c^3) = 3(a+b)(b+c)(c+a). \tag{2}$$

等价地,

$$(a+b+c)^3 - (a^3+b^3+c^3) = 3\big((a+b+c)(ab+bc+ca) - abc\big).$$

或者,

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca). \tag{3}$$

$$(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2-ab-bc-ca).$$

等价形式:

$$a^2+b^2+c^2 = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} + (ab+bc+ca).$$

1.3 关键恒等式总结.

两个变量.

- $(a \pm b)^2$, $a^2-b^2 = (a-b)(a+b)$
- $a^3 \pm b^3$, $(a+b)^3$
- $a^2+b^2 = \frac{(a+b)^2+(a-b)^2}{2}$

三个变量.

- $(a+b+c)^2$, $(a+b+c)^3$
- $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc$
- $(a+b+c)^3 - (a^3+b^3+c^3) = 3(a+b)(b+c)(c+a)$
- $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
- $(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2-ab-bc-ca)$

1.1. 1、2、3 个变量的不等式... 以及更多.

1.1.1. 一个变量的不等式. 设 $x \in \mathbb{R}$。

- 平方的非负性：

$$x^2 \geq 0.$$

- 绝对值不等式：对于 $x \geq 0$：

$$x \geq |x|.$$

- 基本算术-几何平均不等式（*AM–GM*）：对于一个变量

$$x^2 + 1 \geq 2x.$$

- 对数不等式：对于 $x > 0$：

$$\ln x \leq x - 1.$$

- 指数不等式：

$$1 + x \leq e^x.$$

1.1.2. 两个变量的不等式. 设 a 和 b 是两个实数（在需要的地方注明正性假设）。

- *AM–GM* 不等式：如果 $a, b \geq 0$，则

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

- *AM–HM* 不等式：如果 $a, b > 0$，则

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}.$$

- 柯西-施瓦茨不等式（两项）*:*

$$(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2.$$

- 重排不等式（两项）：如果 $a \leq b$ 且 $x \leq y$，则

$$ax + by \geq ay + bx.$$

- 二次平均不等式：

$$\sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a+b}{2}.$$

- 切比雪夫不等式（*2* 个变量）：如果 $a \leq b$ 且 $x \leq y$，则

$$\frac{ax + by}{2} \geq \left(\frac{a+b}{2}\right)\left(\frac{x+y}{2}\right).$$

- 三角不等式：

$$|a + b| \leq |a| + |b|.$$

1.1.3. 三个变量的不等式. 设 $a,b,c>0$，除非另有说明。

- *3 个变量的 AM–GM 不等式*：
$$\frac{a+b+c}{3}\geq\sqrt[3]{abc}.$$
- 柯西-施瓦茨不等式（三项）：
$$(a^2+b^2+c^2)(x^2+y^2+z^2)\geq(ax+by+cz)^2.$$
- 重排不等式（三项）：设 $a_1\leqslant a_2\leqslant a_3$, $x_1\leqslant x_2\leqslant x_3$, 我们有
$$a_1x_1+a_2x_2+a_3x_3\geq a_1x_{\sigma(1)}+a_2x_{\sigma(2)}+a_3x_{\sigma(3)}$$
这里 $x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}$ 是 x_1,x_2,x_3 的一个重新排列。
- 舒尔不等式（一次形式）：对于 $a,b,c\geq 0$：
$$a^3+b^3+c^3+3abc\geq a^2(b+c)+b^2(c+a)+c^2(a+b).$$
等价对称因式分解：
$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$
- 切比雪夫不等式（*3* 个变量）：如果 $a\leq b\leq c$ 且 $x\leq y\leq z$，则
$$\frac{ax+by+cz}{3}\geq\left(\frac{a+b+c}{3}\right)\left(\frac{x+y+z}{3}\right).$$
- 幂平均不等式（*3* 个变量）：如果 $r>s$：
$$\left(\frac{a^r+b^r+c^r}{3}\right)^{1/r}\geq\left(\frac{a^s+b^s+c^s}{3}\right)^{1/s}.$$
- 加权平均不等式：如果 $w_1,w_2,w_3\geq 0$ 且 $w_1+w_2+w_3=1$，则
$$w_1a+w_2b+w_3c\geq a^{w_1}b^{w_2}c^{w_3}.$$

2. 一些简单例子

1. 牛顿不等式. 证明对于任意 $a,b,c>0$，有
$$ab+bc+ca\geq\sqrt{3abc(a+b+c)}.$$

解. 将不等式平方得
$$(ab+bc+ca)^2\geq 3abc(a+b+c).$$

此外，$(ab+bc+ca)^2=a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)$，因此我们的不等式简化为
$$a^2b^2+b^2c^2+c^2a^2\geq abc(a+b+c).$$

现在，令 $bc=x,ca=y$ 和 $ab=z$，我们只需要证明 $x^2+y^2+z^2\geq xy+yz+zx$，这等价于
$$(x-y)^2+(y-z)^2+(z-x)^2\geq 0.$$

□

2. 切萨罗不等式. 证明对于任意 $a, b, c > 0$，有

$$(a+b)(b+c)(c+a) \geq 8abc.$$

解. 由 AM–GM 不等式，我们有

$$a+b \geq 2\sqrt{ab}, \qquad b+c \geq 2\sqrt{bc}, \qquad c+a \geq 2\sqrt{ca}.$$

将这三个不等式相乘得

$$(a+b)(b+c)(c+a) \geq 8\sqrt{a^2b^2c^2} = 8abc.$$

□

3. 内斯比特不等式. 证明对于任意 $a, b, c > 0$，有

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

解. 令

$$b+c = 2A, \qquad c+a = 2B, \qquad a+b = 2C,$$

其中 $A, B, C > 0$。将三个等式相加得

$$a+b+c = A+B+C.$$

解出 a, b, c 得

$$a = B+C-A, \qquad b = C+A-B, \qquad c = A+B-C.$$

将这些代入我们的表达式得

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{B+C-A}{2A} + \frac{C+A-B}{2B} + \frac{A+B-C}{2C}.$$

重写每个分数：

$$\frac{B+C-A}{2A} = \frac{1}{2}\left(\frac{B}{A} + \frac{C}{A} - 1\right),$$

$$\frac{C+A-B}{2B} = \frac{1}{2}\left(\frac{C}{B} + \frac{A}{B} - 1\right),$$

$$\frac{A+B-C}{2C} = \frac{1}{2}\left(\frac{A}{C} + \frac{B}{C} - 1\right).$$

将这三个等式相加得

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{1}{2}\left[\left(\frac{B}{A} + \frac{A}{B}\right) + \left(\frac{C}{A} + \frac{A}{C}\right) + \left(\frac{C}{B} + \frac{B}{C}\right) - 3\right].$$

由于对于每个 $x > 0$ 有 $x + \frac{1}{x} \geq 2$，括号中的每一对至少为 2。因此，

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{1}{2}(6-3) = \frac{3}{2}.$$

□

4. 莱文森不等式. 证明对于任意 $a,b,c>0$，有

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

解. 我们使用简单不等式

$$\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b},\qquad a,b>0,$$

这由 AM–GM 得出。将此不等式乘以 $c>0$ 得

$$c\left(\frac{1}{a}+\frac{1}{b}\right)\geq\frac{4c}{a+b}.$$

循环应用相同不等式并求和:

$$c\left(\frac{1}{a}+\frac{1}{b}\right)+a\left(\frac{1}{b}+\frac{1}{c}\right)+b\left(\frac{1}{c}+\frac{1}{a}\right)\geq 4\left(\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a}\right).$$

但左边简化为

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}.$$

因此

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 4\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right),$$

这正是我们想要证明的。 □

3. 一些较难的例子

1. 1995 年 IMO 第 2 题. 设 a,b,c 为正实数。证明

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

解. 设 $a=\frac{1}{x}$，$b=\frac{1}{y}$，$c=\frac{1}{z}$，其中 $x,y,z>0$。则 $b+c=\frac{1}{y}+\frac{1}{z}=\frac{y+z}{yz}$，且

$$\frac{1}{a^3(b+c)}=\frac{1}{(1/x)^3(y+z)/(yz)}=\frac{x^3yz}{y+z}.$$

类似地，其他项变为 $\frac{y^3xz}{x+z}$ 和 $\frac{z^3xy}{x+y}$。因此不等式等价于

$$\frac{x^3yz}{y+z}+\frac{y^3xz}{x+z}+\frac{z^3xy}{x+y}\geq\frac{3}{2}.$$

提取公因子 xyz:

$$xyz\left(\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\right)\geq\frac{3}{2}.$$

这将问题简化为证明经典不等式

$$\frac{x^2}{y+z}+\frac{y^2}{x+z}+\frac{z^2}{x+y}\geq\frac{3}{2}$$

对于正数 x,y,z，这可以通过使用 AM-GM 不等式的标准方法或通过求和平方来证明：

$$\sum_{cyc}\frac{x^2}{y+z}-\frac{3}{2}=\frac{1}{2}\sum_{cyc}\frac{(x-y)^2}{x+y}\geq 0.$$

当 $x=y=z$，即 $a=b=c$ 时取等号。 □

2. 2000 年 IMO 第 2 题. 设 a,b,c 为正实数且 $abc=1$。证明

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1.$$

解. 由于 $abc=1$，我们可以代入 $a=x/y$，$b=y/z$，$c=z/x$，其中 $x,y,z>0$。则

$$a-1+\frac{1}{b}=\frac{x}{y}-1+\frac{z}{y}=\frac{x+z-y}{y}.$$

类似地，其他因子变为

$$b-1+\frac{1}{c}=\frac{y+x-z}{z},\quad c-1+\frac{1}{a}=\frac{z+y-x}{x}.$$

因此乘积为

$$\frac{(x+z-y)(y+x-z)(z+y-x)}{xyz}.$$

因此，我们只需证明

$$(x+z-y)(y+x-z)(z+y-x)\leq xyz,$$

通过代入 $y+z-x=a, z+x-y=b$ 和 $x+y-z=c$，我们的不等式简化为

$$(a+b)(b+c)(c+a)\geq 8abc$$

这已在上一节中证明。 □

3. 1984 年 IMO 第 1 题. 设 $x,y,z\geq 0$ 且 $x+y+z=1$。证明

$$0\leq xy+yz+zx-2xyz\leq\frac{7}{27}.$$

解. 对于下界，我们有

$$xy+zx+zx-2xyz=(xy+yz+zx)(x+y+z)-2xyz=x^2y+x^2z+xy^2+xz^2+yz^2+y^2z+xyz\geq 0$$

因为 x,y,z 非负。另一方面，对于上界，我们进行如下操作。由于 $x+y+z=1$，注意

$$yz+zx+xy-2xyz=\left(\frac{1}{4}\right)(1-2x)(1-2y)(1-2z)+\left(\frac{1}{4}\right).$$

由 AM–GM

$$(1-2x)(1-2y)(1-2z)\leq\frac{1}{27}$$

因此

$$yz+zx+xy-2xyz\leq(\frac{1}{4})(\frac{1}{27})+\frac{1}{4}=\frac{7}{27}.$$

□

4. 习题

习题 1. [BMO, 2001] 设 $a,b,c>0$ 且 $a+b+c\geq abc$。证明

$$a^2+b^2+c^2\geq abc\sqrt{3}.$$

习题 2. 设 $a,b,c>0$ 且 $a+b+c=1$。证明

$$6(a^3+b^3+c^3)+1\geq 5(a^2+b^2+c^2).$$

习题 3. 设 $x,y,z>0$ 且 $x+y+z=3$。证明

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\geq xy+yz+zx.$$

习题 4. [IMO 预选题 1987] 设 x,y,z 为实数且 $x^2+y^2+z^2=2$。证明

$$x+y+z\leq xyz+2.$$

习题 5. 设 $a,b,c>0$ 且 $(a+b)(b+c)(c+a)=1$。证明

$$ab+bc+ca\leq \frac{3}{4}.$$

习题 6. [IMO 预选题 1997] 设 $x,y,z>0$ 且 $xyz=1$。证明

$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)}\geq \frac{3}{4}.$$

习题 7. 设 $x,y,z>0$ 且 $abc=1$。证明

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a}\leq 1.$$

习题 8. 设 $x,y,z>0$ 且 $xyz=1$。证明

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}\leq 1.$$

习题 9. 证明对于任意正实数 a,b,c，有

$$\frac{1}{a(b+c)}+\frac{1}{b(c+a)}+\frac{1}{c(a+b)}\geq \frac{27}{2(a+b+c)^2}.$$

习题 10. 设 $x,y,z>0$ 且 $x+y+z+xyz=4$。证明

$$x+y+z\geq xy+yz+zx.$$

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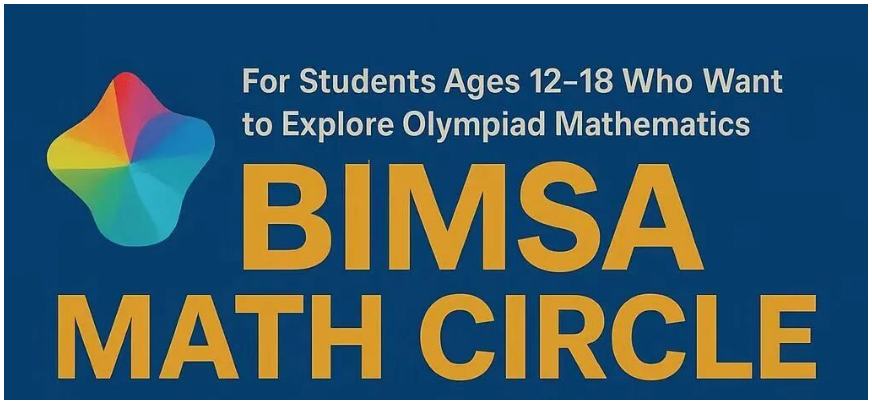

LECTURE 1-ALGEBRA: BASIC IDENTITIES AND INEQUALITIES

CEZAR LUPU

1. A little bit of theory

1.1 Identities for two real variables. Let $a, b \in \mathbb{R}$. In what follows we will exhibit the most important identities.

1.1.1 Square and Difference Identities.

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a-b)^2 = a^2 - 2ab + b^2.$$

Difference of squares:

$$a^2 - b^2 = (a-b)(a+b).$$

1.1.2. Cubic Identities.

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2),$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

1.1.3. Symmetric–Antisymmetric Decomposition.

$$a = \frac{(a+b)+(a-b)}{2}, \qquad b = \frac{(a+b)-(a-b)}{2}.$$

1.1.4. Sum of squares decomposition.

$$a^2 + b^2 = \frac{(a+b)^2 + (a-b)^2}{2}.$$

1.2. Identities in Three Real Variables. Let $a, b, c \in \mathbb{R}$.

1.2.1. Squares and Cubes of Sums.

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca).$$

Fully expanded form:

$$(a+b+c)^3 = a^3+b^3+c^3+3(a^2b+a^2c+b^2c+b^2a+c^2a+c^2b)+6abc.$$

Symmetric form:

$$(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a).$$

1.2.2. Symmetric Sum Identities. Let

$$\sigma_1 = a+b+c, \qquad \sigma_2 = ab+bc+ca, \qquad \sigma_3 = abc.$$

Then

$$a^2+b^2+c^2 = \sigma_1^2 - 2\sigma_2.$$

1.2.3. Fundamental Factorizations.

$$(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca) - abc. \tag{1}$$

$$(a+b+c)^3 - (a^3+b^3+c^3) = 3(a+b)(b+c)(c+a). \tag{2}$$

Equivalently,

$$(a+b+c)^3 - (a^3+b^3+c^3) = 3\big((a+b+c)(ab+bc+ca) - abc\big).$$

or

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca). \tag{3}$$

$$(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2-ab-bc-ca). \tag{4}$$

Equivalently,

$$a^2+b^2+c^2 = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2} + (ab+bc+ca).$$

1.3. Summary of Key Identities.

Two variables.

- $(a\pm b)^2$, $a^2-b^2=(a-b)(a+b)$
- $a^3\pm b^3$, $(a+b)^3$
- $a^2+b^2 = \frac{(a+b)^2+(a-b)^2}{2}$

Three variables.

- $(a+b+c)^2$, $(a+b+c)^3$
- $(a+b)(b+c)(c+a) = (a+b+c)(ab+bc+ca)-abc$
- $(a+b+c)^3-(a^3+b^3+c^3) = 3(a+b)(b+c)(c+a)$
- $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
- $(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2-ab-bc-ca)$

1.1. **Inequalities for one, two and three variables.**

1.1.1. *Inequalities in one variable.* Let $x \in \mathbb{R}$.

- Nonnegativity of squares:
$$x^2 \geq 0.$$
- Absolute value inequality For $x \geq 0$:
$$x \geq |x|.$$
- Basic Arithmetic-Geometric mean inequality (AM–GM) for one variable
$$x^2 + 1 \geq 2x.$$
- Logarithmic Inequality For $x > 0$:
$$\ln x \leq x - 1.$$
- Exponential Inequality
$$1 + x \leq e^x.$$

1.1.2. *Inequalities in two variables.* Let $a, b \in \mathbb{R}$ (positivity assumptions noted where needed).

- AM–GM Inequality If $a, b \geq 0$, then
$$\frac{a+b}{2} \geq \sqrt{ab}.$$
- AM–HM Inequality If $a, b > 0$, then
$$\frac{a+b}{2} \geq \frac{2ab}{a+b}.$$
- Cauchy–Schwarz Inequality (2-term)
$$(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2.$$
- Rearrangement Inequality (2-term) If $a \leq b$ and $x \leq y$, then
$$ax + by \geq ay + bx.$$
- Inequality of Quadratic Means
$$\sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a+b}{2}.$$
- Chebyshev's Inequality (2 variables) If $a \leq b$ and $x \leq y$, then
$$\frac{ax + by}{2} \geq \left(\frac{a+b}{2}\right)\left(\frac{x+y}{2}\right).$$
- Triangle Inequality
$$|a + b| \leq |a| + |b|.$$

1.1.3. *Inequalities in three variables.* Let $a, b, c > 0$ unless otherwise stated.

- AM–GM for 3 variables
$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}.$$
- Cauchy–Schwarz (3-term)
$$(a^2+b^2+c^2)(x^2+y^2+z^2) \geq (ax+by+cz)^2.$$
- Rearrangement Inequality (3-term) For $a_1 \leqslant a_2 \leqslant a_3$ and $x_1 \leqslant x_2 \leqslant x_3$,
$$a_1x_1 + a_2x_2 + a_3x_3 \geq a_1x_{\sigma(1)} + a_2x_{\sigma(2)} + a_3x_{\sigma(3)},$$
where $x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}$ is a reorder of x_1, x_2, x_3.
- Schur's Inequality (Degree 1 Form) For $a, b, c \geq 0$:
$$a^3+b^3+c^3+3abc \geq a^2(b+c)+b^2(c+a)+c^2(a+b).$$
Equivalent symmetric factorization:
$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$
- Chebyshev's Inequality (3 variables) If $a \leq b \leq c$ and $x \leq y \leq z$, then
$$\frac{ax+by+cz}{3} \geq \left(\frac{a+b+c}{3}\right)\left(\frac{x+y+z}{3}\right).$$
- Power Mean Inequality (3 variables) If $r > s$:
$$\left(\frac{a^r+b^r+c^r}{3}\right)^{1/r} \geq \left(\frac{a^s+b^s+c^s}{3}\right)^{1/s}.$$
- Weighted Mean Inequality If $w_1, w_2, w_3 \geq 0$ and $w_1+w_2+w_3 = 1$, then
$$w_1a + w_2b + w_3c \geq a^{w_1}b^{w_2}c^{w_3}.$$

2. Some easy examples

1. Newton's Inequality. Show that for any $a, b, c > 0$, we have
$$ab+bc+ca \geq \sqrt{3abc(a+b+c)}.$$

Solution. By squaring the inequality we have
$$(ab+bc+ca)^2 \geq 3abc(a+b+c).$$
Moreover, $(ab+bc+ca)^2 = a^2b^2+b^2c^2+c^2a^2+2abc(a+b+c)$ and thus our inequality reduces to
$$a^2b^2+b^2c^2+c^2a^2 \geq abc(a+b+c).$$
Now, denote $bc = x, ca = y$ and $ab = z$ and we only need to prove that $x^2+y^2+z^2 \geq xy+yz+zx$ is equivalent to
$$(x-y)^2+(y-z)^2+(z-x)^2 \geq 0.$$
$\square$

2. Cesaro's Inequality. Show that for any $a, b, c > 0$ we have

$$(a+b)(b+c)(c+a) \geq 8abc.$$

Solution. By the AM–GM inequality we have

$$a+b \geq 2\sqrt{ab}, \qquad b+c \geq 2\sqrt{bc}, \qquad c+a \geq 2\sqrt{ca}.$$

Multiplying these three inequalities gives

$$(a+b)(b+c)(c+a) \geq 8\sqrt{a^2b^2c^2} = 8abc.$$

□

3. Nesbitt's Inequality. Show that for any $a, b, c > 0$ we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Solution. Let

$$b+c = 2A, \qquad c+a = 2B, \qquad a+b = 2C,$$

where $A, B, C > 0$. Summing the three equalities gives

$$a+b+c = A+B+C.$$

Solving for a, b, c yields

$$a = B+C-A, \qquad b = C+A-B, \qquad c = A+B-C.$$

Substituting these into our expression gives

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{B+C-A}{2A} + \frac{C+A-B}{2B} + \frac{A+B-C}{2C}.$$

Rewrite each fraction:

$$\frac{B+C-A}{2A} = \frac{1}{2}\left(\frac{B}{A} + \frac{C}{A} - 1\right),$$

$$\frac{C+A-B}{2B} = \frac{1}{2}\left(\frac{C}{B} + \frac{A}{B} - 1\right),$$

$$\frac{A+B-C}{2C} = \frac{1}{2}\left(\frac{A}{C} + \frac{B}{C} - 1\right).$$

Adding these three equalities gives

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{1}{2}\left[\left(\frac{B}{A} + \frac{A}{B}\right) + \left(\frac{C}{A} + \frac{A}{C}\right) + \left(\frac{C}{B} + \frac{B}{C}\right) - 3\right].$$

Since $x + \frac{1}{x} \geq 2$ for every $x > 0$, each pair in parentheses is at least 2. Hence,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{1}{2}(6-3) = \frac{3}{2}.$$

□

4. Levinson's inequality. Show that for any $a, b, c > 0$ we have

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 4\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right).$$

Solution. We use the simple inequality

$$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{a+b}, \qquad a, b > 0,$$

which follows from AM–GM. Multiplying this inequality by $c > 0$ yields

$$c\left(\frac{1}{a} + \frac{1}{b}\right) \geq \frac{4c}{a+b}.$$

Apply the same inequality cyclically and sum:

$$c\left(\frac{1}{a} + \frac{1}{b}\right) + a\left(\frac{1}{b} + \frac{1}{c}\right) + b\left(\frac{1}{c} + \frac{1}{a}\right) \geq 4\left(\frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a}\right).$$

But the left-hand side simplifies to

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}.$$

Thus

$$\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 4\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right),$$

which is exactly what we wanted to prove. □

3. Some harder examples

1. Problem 2, IMO 1995. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution. Set $a = \frac{1}{x},\ b = \frac{1}{y},\ c = \frac{1}{z}$ with $x, y, z > 0$. Then $b + c = \frac{1}{y} + \frac{1}{z} = \frac{y+z}{yz}$, and

$$\frac{1}{a^3(b+c)} = \frac{1}{(1/x)^3(y+z)/(yz)} = \frac{x^3yz}{y+z}.$$

Similarly, the other terms become $\frac{y^3xz}{x+z}$ and $\frac{z^3xy}{x+y}$. Thus the inequality is equivalent to

$$\frac{x^3yz}{y+z} + \frac{y^3xz}{x+z} + \frac{z^3xy}{x+y} \geq \frac{3}{2}.$$

Factor xyz out:

$$xyz\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right) \geq \frac{3}{2}.$$

This reduces the problem to proving the classical inequality

$$\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \geq \frac{3}{2}$$

for positive x, y, z, which can be shown by the standard method using the AM-GM inequality or by summing squares:

$$\sum_{cyc} \frac{x^2}{y+z} - \frac{3}{2} = \frac{1}{2} \sum_{cyc} \frac{(x-y)^2}{x+y} \geq 0.$$

Equality holds when $x = y = z$, i.e., $a = b = c$. □

2. Problem 2, IMO 2000. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Solution. Since $abc = 1$, we can substitute $a = x/y$, $b = y/z$, $c = z/x$ for positive x, y, z. Then

$$a - 1 + \frac{1}{b} = \frac{x}{y} - 1 + \frac{z}{y} = \frac{x+z-y}{y}.$$

Similarly, the other factors become

$$b - 1 + \frac{1}{c} = \frac{y+x-z}{z}, \quad c - 1 + \frac{1}{a} = \frac{z+y-x}{x}.$$

Hence the product is

$$\frac{(x+z-y)(y+x-z)(z+y-x)}{xyz}.$$

Therefore, we are left to prove that

$$(x+z-y)(y+x-z)(z+y-x) \leq xyz,$$

By using the substituions $y+z-x = a, z+x-y = b$ and $x+y-z = c$ our inequality reduces to

$$(a+b)(b+c)(c+a) \geq 8abc$$

which was already proven in the previous section. □

3. Problem 1, IMO 1984. Let $x, y, z \geq 0$ with $x + y + z = 1$. Prove that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27}.$$

Solution. For the lower bound, we have

$$xy+zx+zx-2xyz = (xy+yz+zx)(x+y+z)-2xyz = x^2y+x^2z+xy^2+xz^2+yz^2+y^2z+xyz \geq 0$$

because x, y, z are nonnegative. On the other hand, for the upper bound, we proceed as follows. Since $x + y + z = 1$ notice that

$$yz + zx + xy - 2xyz = \left(\frac{1}{4}\right)(1-2x)(1-2y)(1-2z) + \left(\frac{1}{4}\right).$$

By the AM–GM

$$(1-2x)(1-2y)(1-2z) \leq \frac{1}{27}$$

Therefore

$$yz + zx + xy - 2xyz \leq (\frac{1}{4})(\frac{1}{27}) + \frac{1}{4} = \frac{7}{27}.$$

$\square$

4. Proposed Problems

Problem 1. [BMO, 2001] Let $a, b, c > 0$ such that $a + b + c \geq abc$. Show that

$$a^2 + b^2 + c^2 \geq abc\sqrt{3}.$$

Problem 2. Let $a, b, c > 0$ such that $a + b + c = 1$. Show that

$$6(a^3 + b^3 + c^3) + 1 \geq 5(a^2 + b^2 + c^2).$$

Problem 3. Let $x, y, z > 0$ such that $x + y + z = 3$. Show that

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \geq xy + yz + zx.$$

Problem 4. [IMO Shortlist 1987] Let x, y, z be real numbers such that $x^2 + y^2 + z^2 = 2$. Show that

$$x + y + z \leq xyz + 2.$$

Problem 5. Let $a, b, c > 0$ such that $(a + b)(b + c)(c + a) = 1$. Show that

$$ab + bc + ca \leq \frac{3}{4}.$$

Problem 6. [IMO Shortlist 1997] Let $x, y, z > 0$ such that such that $xyz = 1$. Show that

$$\frac{x^3}{(1 + y)(1 + z)} + \frac{y^3}{(1 + z)(1 + x)} + \frac{z^3}{(1 + x)(1 + y)} \geq \frac{3}{4}.$$

Problem 7. Let $x, y, z > 0$ such that $abc = 1$. Show that

$$\frac{1}{1 + a + b} + \frac{1}{1 + b + c} + \frac{1}{1 + c + a} \leq 1.$$

Problem 8. Let $x, y, z > 0$ such that $xyz = 1$. Show that

$$\frac{1}{2 + a} + \frac{1}{2 + b} + \frac{1}{2 + c} \leq 1.$$

Problem 9. Show that for any positive real numbers a, b, c we have

$$\frac{1}{a(b + c)} + \frac{1}{b(c + a)} + \frac{1}{c(a + b)} \geq \frac{27}{2(a + b + c)^2}.$$

Problem 10. Let $x, y, z > 0$ such that $x + y + z + xyz = 4$. Show that

$$x + y + z \geq xy + yz + zx.$$

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