1 Key Matrices in Graph Theory

- **Adjacency Matrix (A)**: $A_{ij} = 1$ if edge (i, j) exists; 0 otherwise.
- **Laplacian Matrix (L)**: $L = D - A$, where D is the degree matrix.
- **Signless Laplacian**: $Q = D + A$.
- **Normalized Laplacian**: $L_{\text{sym}} = I - D^{-1/2}AD^{-1/2}$.
- **Probability Transition Matrix (P)**: $P = D^{-1}A$, row-stochastic.

1.1 Signless Laplacian Matrix

Definition 1. *The signless Laplacian $Q = D + A$ where D is the degree matrix and A is the adjacency matrix.*

Theorem 1. *For Q matrix:*

- *symmetric, eigenvalues are real, sum of eigenvalues are still* $2|E|$
- $x^T Qx = \sum_{(i,j)\in E}(x_i + x_j)^2$*, this formula is still neat and what changes is the sign between two entries of the vector*
- *Eigenvalues are non-negative*
- *positive definite, diagonally dominant*

Theorem 2. *The least eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite.*

Proof. **Constructing an Eigenvector for Q with Eigenvalue 0:** Define a vector x as follows:

$$x_i = \begin{cases} 1, & \text{if } v_i \in V_1, \\ -1, & \text{if } v_i \in V_2. \end{cases}$$

Now, compute Qx:

$$(Qx)_i = \sum_{j\sim i}(x_i + x_j).$$

Since G is bipartite, every edge (i, j) connects a vertex from V_1 to a vertex from V_2, meaning $x_i + x_j = 1 + (-1) = 0$. Thus,

$$Qx = 0$$

which means that 0 is an eigenvalue of Q.

Sufficiency (If Q has 0 as an eigenvalue, then G is bipartite) Suppose Q has a 0 eigenvalue, i.e., there exists a nonzero vector x such that

$$Qx = 0.$$

This implies:

$$\sum_{j\sim i}(x_i + x_j) = 0, \quad \forall v_i \in V(G).$$

which means that for every edge (i, j),

$$x_i + x_j = 0 \quad \Rightarrow \quad x_j = -x_i.$$

Thus, we can assign all vertices to two sets:

- One set where $x_i > 0$
- Another set where $x_i < 0$

Since all edges connect vertices in different sets, G is bipartite. □

By eigenvalues of Q, can you tell whether the graph is connected or not? No! You can tell whether it is bipartite or not, or you can tell the number of bipartiteness.

And there is an example here:

if you take a triangle with a single vertex, and you take a star on four vertices, Both graphs have Q-eigenvalues $0, 1, 1, 4$. they are Q-copsetral, but one is disconnected, the other is connected. And it does not contradict anything we have done.

The first has 0 but is not bipartite, so you have to be careful that in the above theorem, it works for connected graphs. But it tells the number of bipartite components. The sole vertex itself is a bipartite graph.

Multiplicity of eigenvalue 0 In a general graph, multiplicity of the 0 eigenvalue equals the number of bipartite components.

1.2 Q-spectral

So Q-eigenvalue 0 is a bipartite detector, but there is something that is stronger than the bipartite and signless Laplacian and Laplacian matrix.

Theorem 3. *If G is bipartite, then the Q-spectral is the same as the L-spectral.*

Think about when you have a bipartite graph, what does the L look like? Bipartite naturally produces a block matrix.

By comparison, if we look at the Q, The difference is just some positive ones on these blocks.

Ok, how can we argue that they have the same spectrum? In linear algebra class, two matrices have the same eigenvalues; they are similar matrices. That is a good way to think about it.

How do you say two matrices are similar? If I can find a matrix S such that $S^{-1}LS = Q$, then they would be similar matrices and hence have the same eigenvalues.

Let me recall some properties of bipartite graphs. A graph is bipartite if its vertex set can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V. This means the adjacency matrix A of a bipartite graph has a special block form:

$$A = [[0, B], [B^T, 0]]$$

Signless Laplacian matrix and Adjacency matrix of Line graph

There is a straightforward relation between the eigenvalues of the signless Laplacian matrix of a graph and the adjacency eigenvalues of its line graph.

Check that $Q = SS^T$ where S is $V \times E$ incidence matrix. Let's examine the entries of $S^T S$. This is an edge-by-edge matrix where the diagonal elements are 2. For the off-diagonal entries, they are 1 if the corresponding edges share a mutual neighbor; otherwise, they are 0. This configuration yields the line graph.

1.3 Line Graph Connection

The **line graph** $L(G)$ of a given graph G is a graph that represents the adjacency relationship between the edges of G. More formally:

Given a graph $G = (V, E)$, the **line graph** $L(G)$ is a graph whose:

- **Vertex set:** Each vertex in $L(G)$ corresponds to an edge in G.
- **Edge set:** Two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G share a common endpoint.
- Consider a simple graph G with vertex set and edge set:

$$V = \{A, B, C, D\}, \quad E = \{(A, B), (B, C), (C, D), (A, C)\}.$$

  The corresponding line graph $L(G)$ will have:

  - Vertices labeled as e_1, e_2, e_3, e_4, representing the edges of G.
  - Edges between these vertices where their corresponding edges in G share a common endpoint.

- Properties of Line Graphs

  1. **Degree Relationship:** The degree of a vertex in $L(G)$ is the number of edges in G that are adjacent to the edge it represents.
     $e = uv \in E(G)$, degree of $e = d_u(G) + d_v(G) - 2$
  2. **Line Graph of a Complete Graph:**
  3. **Spectral Properties:** The eigenvalues of the adjacency matrix of $L(G)$ provide information about the structure of G.
     example
     Interest in the topic began with an elementary observation that line graphs have the least eigenvalue greater than or equal to 2 (see the second equality in (1)). A natural problem arose to characterize the graphs with such a remarkable property.

let us go back to $S^T S$, the $|E| \times |E|$ matrix.

Theorem 4. *The edge adjacency matrix satisfies:*

$$S^\top S = 2I + AL(G)$$

where S is the incidence matrix of G and $AL(G)$ is the adjacency matrix of the line graph.

How small can the eigenvalue of $A_{L(G)}$ What can you say about the eigenvalues of $S^\top S$?

From the above equal relation we have $A_{L(G)} = S^\top S - 2I$, since $S^\top S$ is positive semidefinite, all eigenvalues are non-negative, subtracting 2I shifts all eigenvalues by -2.

Ok, how let us recall a question in linear algebra. How do eigenvalues of MN and NM compare?

The spectra of MN and NM differ only in the number of zero eigenvalues. For non-zero eigenvalues, they are identical! This property is foundational in applications like singular value decomposition (SVD) and graph theory. If the larger product matrix has additional zero eigenvalues.

So if the eigenvalue -2 is achieved in $A_{L(G)}$, and the graph is connected, then it is also bipartite.

Let us take an example, say $G = K_5$. The line graph has 10 vertices and the degree is 8, so 40 edges. The adjacency eigenvalues are $n-1, -1's$, whisch is $4, -1, -1, -1, -1$; the signless Laplacian eigenvalues or Q-eigenvaleus are caluted by 5+ eigenvalue of adjacency matrix, so which is $8, 3, 3, 3, 3$

Note that the eigenvalues of Q and $S^T S$, the non-zero eigenvalues are exactly same!!! up to multiplicities! So the eigenvalue of $S^T S$ are $6, 1, 1, 1, 1$ and but there are 10 eigenvalues, so the other five are just 0!

Finally, for the line graphs of K_5, the adjacency eigenvalues are $8-2, 3-2, 3-2, 3-2, 3-2, -2, , -2, -2, -2, -2$.

SO now let us compare the adjcency eigenvalues of line graph with the eigenvalues of Q-matrix of orignal graph.

If $\lambda \neq 0$ is an adjacency eigenvalue of the line graph, then $\lambda + 2$ will be the non-zero eigenvalues of Q.

Corollary 1. *The characteristic polynomial of the line graph $P_{L(G)}$ is satisfied by*

$$P_{L(G)}(\lambda) = (\lambda + 2)^{m-n} Q_G(\lambda + 2) \tag{1}$$

where Q_G is the characteristic polynomial of the signless Laplacian

So it is equal to $m - n + 1$ if G is bipartite and equal to $m - n$ if G is not bipartite. This together with formula (2) yields the assertion of Theorem 2.

There are a number of papaer study the least eigenvalue -2

2 Random walk, transition probability matrix, Normalized Laplacian matrix

a random walk on a graph is just a process where, at each step, you move to a randomly chosen neighbor. The probabilities usually depend on the degrees of the nodes. Over time, this process reveals deep insights about the structure of the graph, like which nodes are more "central" or how quickly information spreads. You can think of it like taking a random stroll—each time you reach an intersection, you pick a path randomly and move forward, with no memory of where you were before, except for your current position.

3 Review and Motivation

Recall that for a graph $G = (V, E)$ we define the *transition probability matrix* $P(u, v)$ by

$$P(u,v) = \begin{cases} \frac{1}{d_u} & \text{if } u \sim v \\ 0 & \text{otherwise,} \end{cases}$$

where d_u denotes the degree of vertex u. If f is an initial distribution (row vector), then fP^t gives the distribution after t steps. A distribution π is *stationary* if $\pi P = \pi$.

$P = D^{-1}A$

Example 1

For a k-regular graph, $P = \frac{1}{k}A$. The uniform distribution $\overrightarrow{\mathbb{I}}$ is stationary.

Example 2

For two disjoint K_m graphs, stable distributions $\pi_1 = (1, \ldots, 1, 0, \ldots, 0)$ and $\pi_2 = (0, \ldots, 0, 1, \ldots, 1)$ show non-uniqueness.

Example 3

Star graph G on $k + 1$ vertices has stable distribution $\pi = (\underbrace{1, \ldots, 1}_{k}, k)$.

3.1 Finding Stable Distributions

Define *volume* of $S \subseteq V$ as $\operatorname{vol} S = \sum_{v \in S} d_v$. For undirected graphs, the stationary distribution is:

$$\pi = \left(\frac{d_u}{\operatorname{vol} G} \right)_{u \in V}$$

Verified by:

$$(\pi P)(v) = \sum_{u \sim v} \frac{d_u}{\operatorname{vol} G} \cdot \frac{1}{d_u} = \frac{d_v}{\operatorname{vol} G} = \pi(v)$$

3.2 The Issue of Convergence

Using eigenvalue decomposition $P = D^{-1/2} M D^{1/2}$, convergence depends on eigenvalues ρ_i of M. The bound:

$$\|fP^t - \pi\| \leq \max_{i>0} |\rho_i|^t \cdot \left(\frac{\max \sqrt{d_x}}{\min \sqrt{d_y}} \right)$$

shows convergence if $|\rho_i| < 1$ for $i \neq 0$.

4 A Look Ahead to Weighted Edges

Transition matrix for weighted graphs:

$$P(u,v) = \frac{w_{uv}}{\sum_s w_{us}}$$

Stationary distribution:

$$\pi(u) = \frac{\sum_s w_{us}}{\operatorname{vol} G}$$

4.1 A Short Review

Convergence rate depends on $\rho = \max\{|\rho_1|, |\rho_{n-1}|\}$ where ρ_i are eigenvalues of P.

4.2 Introduction to the Laplacian

Normalized Laplacian:

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2}$$

Entrywise definition:

$$\mathcal{L}(u,v) = \begin{cases} 1 & \text{if } u = v, \\ -\frac{1}{\sqrt{d_u d_v}} & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

4.3 Facts About the Laplacian

Rayleigh quotient:

$$\frac{g\mathcal{L}g^*}{gg^*} = \frac{\sum_{x\sim y}(f(x) - f(y))^2}{\sum_x f(x)^2 d_x}$$

Courant-Fischer theorem gives eigenvalues via extremal values of this quotient.

4.4 Examples of Spectra

Complete Graph K_n

Eigenvalues: 0 (multiplicity 1) and $\frac{n}{n-1}$ (multiplicity $n-1$).

Cycle C_n

Eigenvalues: $1 - \cos\left(\frac{2\pi k}{n}\right)$ for $k = 0, 1, \ldots, n-1$.

Exercise 5

Eigenvalues of n-cube Q_n are $\frac{2k}{n}$ with multiplicity $\binom{n}{k}$.