

Affine Weyl Groups and affine Hecke algebras

Root and weight lattices

Let $V_{\mathbb{R}}$ be a finite-dimensional Euclidean vector space with inner product $(-, -)$. Let $R \subset V_{\mathbb{R}}$ be a reduced, irreducible crystallographic root system, i.e., R is of type $A_n, B_n, C_n, D_n, E_{6,7,8}, F_4$, or G_2 . Let W be the Weyl group of R , generated by the orthogonal reflections $s_{\alpha}, \alpha \in R$.

We write $R^{\vee} = \{\alpha^{\vee} = 2\alpha/(\alpha, \alpha)\}$ for the dual root system. Let a_1, \dots, a_n be a fixed basis of simple roots in R , associated with a decomposition $R = R_+ \sqcup R_-$; then $a_1^{\vee}, \dots, a_n^{\vee}$ is a basis of simple roots for R^{\vee} . Introduce fundamental (co)weights b_i, b_i^{\vee} by the conditions $(a_i^{\vee}, b_j) = (a_i, b_j^{\vee}) = \delta_{ij}$.

Here are some standard lattices associated with the pair R, R^{\vee} :

$$Q = \sum_{i=1}^n \mathbb{Z}a_i \text{ (root lattice)} \quad Q^{\vee} = \sum_{i=1}^n \mathbb{Z}a_i^{\vee} \text{ (coroot lattice)}$$

$$P = \sum_{i=1}^n \mathbb{Z}b_i \text{ (weight lattice)} \quad P^{\vee} = \sum_{i=1}^n \mathbb{Z}b_i^{\vee} \text{ (coweight lattice)}$$

The weight and coweight lattices, P and P^\vee , are dual to Q^\vee and Q , respectively: $(P, Q^\vee) \subset \mathbb{Z}$ and $(P^\vee, Q) \subset \mathbb{Z}$. The crystallographic condition $(\alpha^\vee, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in R$ tells us that $Q \subset P$ and $Q^\vee \subset P^\vee$. The finite abelian group P^\vee/Q^\vee will be important below.

Here are some examples.

1. For $R = A_{n-1}$, simple roots are $a_i = \epsilon_i - \epsilon_{i+1}$, $i = 1 \dots, n-1$. We view R in $V_{\mathbb{R}} = \{x \in \mathbb{R}^n : x_1 + \dots + x_n = 0\}$, with Euclidean inner product obtained by restriction from \mathbb{R}^n .

$Q = Q^\vee = \{x \in \mathbb{Z}^n : x_1 + \dots + x_n = 0\}$, $P = P^\vee$ is obtained by projecting $\mathbb{Z}^n \subset \mathbb{R}^n$ orthogonally onto the hyperplane $V_{\mathbb{R}}$. The fundamental (co)weights are

$$b_i = b_i^\vee = \epsilon_1 + \dots + \epsilon_i - \frac{i}{n}(\epsilon_1 + \dots + \epsilon_n), \quad i = 1, \dots, n-1.$$

In this case, $P/Q = \langle b_1 \rangle = \mathbb{Z}_n$.

Examples

2. Let $V_{\mathbb{R}} = \mathbb{R}^n$, and R is of type B_n :

$$R = \{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid i < j\}.$$

The dual of R is R^{\vee} of type C_n :

$$R^{\vee} = \{\pm 2\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid i < j\}.$$

We choose simple roots in R and R^{\vee} as

$$a_i = a_i^{\vee} = \epsilon_i - \epsilon_{i+1}, \quad i = 1, \dots, n-1, \quad a_n = \epsilon_n, \quad a_n^{\vee} = 2\epsilon_n.$$

We have $Q = \mathbb{Z}^n$, $Q^{\vee} = \{x \in \mathbb{Z}^n : x_1 + \dots + x_n \in 2\mathbb{Z}\}$. The (co)weight lattices are $P = \{x \in \frac{1}{2}\mathbb{Z}^n : x_i - x_j \in \mathbb{Z} \forall i, j\}$, $P^{\vee} = \mathbb{Z}^n$. We have $P/Q \cong P^{\vee}/Q^{\vee} = \mathbb{Z}_2$.

Affine Weyl group

As before, we denote by V the complexification of $V_{\mathbb{R}}$ and extend the action of W onto V by \mathbb{C} -linearity. Pick $c \neq 0$; this replaces the Planck constant. For $\lambda \in V$, we denote by $t(\lambda)$ the translation by λ , acting on V by

$$t(\lambda).v = v - c\lambda, \quad \forall v \in V.$$

Translations act naturally on the space of functions by

$$(t(\lambda).f)(v) = f(v + c\lambda), \quad \forall f \in \mathbb{C}(V).$$

The change of sign is because for any transformation φ on V , its action on functions is by $(\varphi.f)(v) := f(\varphi^{-1}v)$. We see that $t(\lambda)$ acts on functions as $t(\lambda) = e^{c\partial\lambda}$, or $t(\lambda) = q^{\partial\lambda}$ where $q = e^c$.

The *affine Weyl group* is defined as $W_a = W \rtimes t(Q^\vee)$, where $t(Q^\vee)$ denotes the group of translations $t(\lambda)$, $\lambda \in Q^\vee$.

The *extended affine Weyl group* is defined as $\widehat{W} = W \rtimes t(P^\vee)$.

We have $\widehat{W}/W_a \cong P^\vee/Q^\vee$. Both groups act naturally on $\mathbb{C}(V)$.

Affine root system

Let \widehat{V} denote the space of affine-linear complex-valued functions on V . We identify \widehat{V} with $V \oplus \mathbb{C}\delta$, where vectors in V are considered as linear functionals on V via the scalar product (\cdot, \cdot) and where $\delta \equiv c$ is a constant function on V (so $e^\delta = e^c = q$).

Let

$$R_a = \{\tilde{\alpha} = \alpha + m\delta, m \in \mathbb{Z}, \alpha \in R\} \subset \widehat{V}$$

be the affine root system associated with R . The action of \widehat{W} on $\widehat{V} \subset \mathbb{C}(V)$ permutes affine roots. For any $\tilde{\alpha} = \alpha + m\delta$ we have the orthogonal reflection with respect to the hyperplane $\{\tilde{\alpha} = 0\}$ in V ,

$$s_{\tilde{\alpha}} \cdot x = x - \tilde{\alpha}(x)\alpha^\vee, \quad x \in V.$$

It is known that the affine Weyl group W_a is generated by reflections $s_{\tilde{\alpha}}$, $\tilde{\alpha} \in R_a$. This is an example of an infinite reflection group. The extended Weyl group in general is not generated by reflections.

Simple reflections

The affine Weyl group W_a admits a description by generators and relations. For this, we extend the set of simple roots a_i to a basis in R_a by adding $a_0 = \delta - \varphi$, where φ is the highest root in R_+ .

(This is a unique root $\varphi = \sum_{i=1}^n n_i a_i$ such that any other root $\alpha = \sum_{i=1}^n l_i a_i$ satisfies $n_i \geq l_i$ for all i .)

The group W_a is generated by the simple reflections $s_i = s_{a_i}$, $i = 0, \dots, n$ subject to relations $s_i^2 = 1$ and the braid relations

$$s_i s_j \cdots = s_j s_i \cdots \quad \text{for } i \neq j, \text{ with } m_{ij} \text{ factors on either side.}$$

Here $m_{ij} \in \{2, 3, 4, 6, \infty\}$ is the order of $s_i s_j$. (The case $m_{ij} = \infty$ happens for W of type A_1 .)

We can encode this information in a Dynkin diagram, by adding an extra vertex for a_0 and by connecting it to other vertices based on the values of m_{ij} . These are called *affine Dynkin* diagrams. We can also put additional labels n_i based on

$$\delta = a_0 + \varphi = \sum_{i=0}^n n_i a_i, \quad n_0 = 1.$$

Affine Dynkin diagrams

\tilde{A}_1	
\tilde{A}_n	
\tilde{B}_n	
\tilde{C}_n	
\tilde{D}_n	
\tilde{E}_6	
\tilde{E}_7	
\tilde{E}_8	
\tilde{F}_4	
\tilde{G}_2	

with $n \geq 2$ for \tilde{A}_n , $n \geq 3$ for \tilde{B}_n , $n \geq 2$ for \tilde{C}_n and $n \geq 4$ for \tilde{D}_n .

Length function

We define the *length* $l(w)$ of $w \in W_a$ as the length l of a reduced (i.e., shortest) decomposition

$$w = s_{i_1} \dots s_{i_l}, \quad \text{with } 0 \leq i_k \leq n.$$

Let Ω be the subgroup of the elements $\omega \in \widehat{W}$ which map the basis a_0, \dots, a_n to itself. It is known that Ω is an abelian group, isomorphic to P^\vee/Q^\vee , and the extended affine Weyl group is isomorphic to $W_a \rtimes \Omega$. Each $\tilde{w} \in \widehat{W}$ admits a unique presentation as $\tilde{w} = w\omega$ with $w \in W_a$ and $\omega \in \Omega$.

We use this to extend the notion of the length from W_a to \widehat{W} by setting $l(w\omega) = l(w)$, so $l(\omega) = 0$ for all $\omega \in \Omega$.

Note the following property of translations in \widehat{W} . Consider the set of *dominant coweights* defined as $P_+^\vee := \sum_{i=1}^n \mathbb{Z}_{\geq 0} b_i^\vee$. Then

$$l(t(\lambda + \mu)) = l(t(\lambda)) + l(t(\mu)) \quad \forall \lambda, \mu \in P_+^\vee.$$

Relations in extended affine Weyl group

The extended affine Weyl group \widehat{W} can also be described by generators and relations, but it's a little bit more complicated as it is not generated by reflections if $\Omega \neq \{1\}$.

Recall that each $\omega \in \Omega$ permutes simple roots, so for each i $\omega s_i \omega^{-1} = s_j$ for suitable j .

Then group $\widehat{W} = W_a \rtimes \Omega$ is generated by s_i , $i = 0, \dots, n$ and $\omega \in \Omega$, subject to the following:

- (1) Relations in W_a , i.e., $s_i^2 = 1$ and braid relations;
- (2) Relations $\omega\omega' = \omega'\omega$ in Ω (commutativity);
- (3) Relations $\omega s_i \omega^{-1} = s_j$ (semi-direct product structure $W_a \rtimes \Omega$)

Affine braid group

By relaxing relations $s_i^2 = 1$, we can define the braid group of type \widehat{W} . Namely, the *braid group* \mathfrak{B} of \widehat{W} is the group with generators $T_i, i = 0, \dots, n$ and $T_\omega, \omega \in \Omega$, subject to the following relations:

$$\begin{aligned}T_i T_j \cdots &= T_j T_i \cdots \quad \text{for } i \neq j, \text{ with } m_{ij} \text{ factors on either side,} \\T_\omega T_{\omega'} &= T_{\omega\omega'} \quad \text{for } \omega, \omega' \in \Omega, \\T_\omega T_i T_\omega^{-1} &= T_j \quad \text{if } \omega s_i \omega^{-1} = s_j.\end{aligned}$$

Here $m_{ij} = 2, 3, 4, 6$ is the order of $s_i s_j \in W_a$.

We can define elements $T_w, w \in \widehat{W}$ by taking a reduced decomposition

$$w = w_0 \omega, \quad \omega \in \Omega, \quad w_0 = s_{i_1} \dots s_{i_l}, \quad \text{with } 0 \leq i_k \leq n$$

and setting $T_w = T_{i_1} \dots T_{i_l} T_\omega$. The fact that this definition does not depend on the choice of a reduced decomposition relies on the property that in a Coxeter group (in our case, the group W_a), any two reduced words can be related using braid relations only.

Commutative subgroup

The following property is then clear from the definitions:

$$T_v T_w = T_{vw} \quad \text{if} \quad l(v) + l(w) = l(vw).$$

This allows us to see an important abelian subgroup $\{Y^\lambda \mid \lambda \in P^\vee\}$ inside \mathfrak{B} . Namely, if coweights $\lambda, \mu \in P_+^\vee$ are dominant, then from above $T_{t(\lambda)} T_{t(\mu)} = T_{t(\mu)} T_{t(\lambda)} = T_{t(\lambda+\mu)}$. Hence, the elements

$$Y^\lambda := T_{t(\lambda)}, \quad \lambda \in P_+^\vee$$

form an abelian semigroup. We extend it to a group by setting $Y^\lambda = Y^\mu (Y^\nu)^{-1}$ whenever $\lambda = \mu - \nu$ with $\mu, \nu \in P_+^\vee$.

Affine Hecke algebra

Choose nonzero parameters τ_i , $i = 0, \dots, n$ such that $\tau_i = \tau_j$ if s_i and s_j are conjugated in \widehat{W} . We use τ to denote the set of these parameters. Note that τ_i 's take on at most two different values. The (extended) *affine Hecke algebra* $\widehat{\mathfrak{H}}$ is the quotient of the group algebra $\mathbb{C}\mathfrak{B}$ of the braid group by relations

$$(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0, \quad i = 0, \dots, n.$$

The image of T_i (resp. T_w , Y^λ) in $\widehat{\mathfrak{H}}$ will be denoted by the same symbol T_i (resp. T_w , Y^λ) as in the braid group.

For $\tau_i = 1$ we have $\widehat{\mathfrak{H}} \cong \mathbb{C}\widehat{W}$. The subalgebra $\mathfrak{H} \subset \widehat{\mathfrak{H}}$ generated by T_1, \dots, T_n is the finite *Hecke algebra* of W . The elements T_w , $w \in \widehat{W}$ ($w \in W$, respectively) form a \mathbb{C} -basis of $\widehat{\mathfrak{H}}$ (\mathfrak{H} , respectively). An alternative basis for $\widehat{\mathfrak{H}}$ is given by the elements $T_w Y^\lambda$ with $w \in W$, $\lambda \in P^\vee$.