KP integrability in topological recursion through the x-y swap relation

Alexander Alexandrov

IBS Center for Geometry and Physics, Pohang, Korea

April 11, 2025

BIMSA

Based on a joint work with Boris Bychkov, Petr Dunin-Barkowski, Maxim Kazarian, and Sergey Shadrin

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The theory of integrable hierarchies was actively developed by the Kyoto school of Date, Jimbo, Kashiwara, Miwa, and Sato in the 80s of the last century. They found a fundamental relation between integrable hierarchies, representation theory of the infinite dimensional Lie algebras, and free field formalism.

The **Kadomtsev–Petviashvili (KP) hierarchy** was introduced by **[Sato '81]**. It can be represented in terms of **tau-functions**

$$\tau(\mathbf{t}) \in \mathbb{C}[\![t_1, t_2, t_3, \dots]\!],$$

which are nothing but the vacuum expectation values of some group elements in the free field formalism, by the **Hirota bilinear identity**

$$\oint_{\infty} e^{\sum_{k>0}(t_k - t'_k)z^k} \tau(\mathbf{t} - [z^{-1}])\tau(\mathbf{t}' + [z^{-1}])dz = 0,$$

where

$$\mathbf{t} \pm [z^{-1}] := \{t_1 \pm z^{-1}, t_2 \pm \frac{1}{2}z^{-2}, t_3 \pm \frac{1}{3}z^{-3}, \dots\}.$$

The first nontrivial term in the expansion of the left hand side of the Hirota bilinear identity gives the **KP equation**

$$\tau\tau_{1111} - 4\tau_1\tau_{111} + 3\left(\tau_{11}\right)^2 + 3\tau\tau_{22} - 3\left(\tau_2\right)^2 - 4\tau\tau_{13} + 4\tau_1\tau_3 = 0,$$

where

$$\tau_{i_1 i_2 \dots} = \frac{\partial}{\partial t_{i_1}} \frac{\partial}{\partial t_{i_2}} \dots \tau.$$

The second derivative of this equation with respect to t_1 gives the standard KP equation

$$3u_{22} = \left(4u_3 - 12uu_1 - u_{111}\right)_1,$$

where $u = \frac{\partial^2}{\partial {t_1}^2} \log(\tau)$.

Sato Grassmannian and KdV reduction

The KP hierarchy can be described in several different ways. In particular, it has an interpretation as Plücker relations for the semi-infinite **Sato Grassmannian** which is equivalent to the **free charged fermion description**. A point of the Sato Grassmannian can be described (non-uniquely!) by the **basis vectors**

$$\Phi_i = z^{-i}(1 + O(z)), \qquad i \in \mathbb{Z}_{>0}.$$

This description is equivalent to the Hirota bilinear identity, and the tau-function in the **Miwa parametrization** is given by

$$\tau\Big|_{t_k = \frac{1}{k}(z_1^k + \cdots + z_n^k)} = \frac{\det_{i,j \le n} z_j \Phi_i(z_j)}{\Delta(z^{-1})}.$$

If a tau-function of the KP hierarchy does not depend on the even time variables,

$$\frac{\partial}{\partial t_{2k}} \tau_{\mathrm{KdV}}(\mathbf{t}) = 0 \qquad \forall\, k > 0,$$

than it is a tau-function of the **Korteweg–De Vries (KdV) hierarchy**.

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Topological recursion (TR) **[Chekhov, Eynard, Orantin, '06-'08]** associates to an input data that consists of two meromorphic functions x and y defined on a compact Riemann surface Σ a system of symmetric meromorphic n-differentials $\omega_n^{(g)}$ on Σ^n, $n \geq 1$, via an explicitly given recursive procedure

$$(\Sigma, x, y) \quad \xrightarrow{TR} \quad \{\omega_n^{(g)}\}_{\substack{g\geq 0, n\geq 1\\ 2g-2+n>0}}$$

The triple (Σ, x, y) is traditionally called the **spectral curve**.

Topological recursion originates from the Virasoro constraints and spectral curve description of correlation functions in matrix models developed in '80s-'00s with the roots in Wigner's semicircular distribution **[Ambjørn, Chekhov, Kristjansen, Makeenko, Jurkiewicz, Marshakov, Mironov, Morozov,. . .]**.

Many examples from geometry and physics are related tau-functions of the KP hierarchy.

Question: What is the general relation between topological recursion and KP?

Topological recursion have several forms and can be defined on a spectral curve of any genus. I will focus on the genus zero case, $\Sigma = \mathbb{C}P^1$.

Let x and y be rational functions. Assume that all zeros $\{p_1, \dots, p_N\}$ of dx are simple and dy is regular and non-zero at these points. Let σ_i be the deck transformation with respect to x near p_i. We set

$$\omega_1^{(0)}(z_1) := -y(z_1)dx(z_1), \quad \omega_2^{(0)}(z_1, z_2) := \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

The **correlation differentials** $\omega_n^{(g)}$, $g \geq 0$, $n \geq 1$, $2g - 2 + n > 0$, are computed recursively as

$$\omega_n^{(g)}(z_{[\![n]\!]}) = \frac{1}{2}\sum_{i=1}^{N} \operatorname*{res}_{z=p_i} \frac{\int_z^{\sigma_i(z)} \omega_2^{(0)}(z_1, \cdot)}{\omega_1^{(0)}(\sigma_i(z)) - \omega_1^{(0)}(z)} \Bigg(\omega_{n+1}^{(g-1)}(z, \sigma_i(z), z_{[\![n]\!]\setminus\{1\}}) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = [\![n]\!]\setminus\{1\} \\ (g_i, |I_i|) \neq (0,0)}} \omega_{|I_1|+1}^{(g_1)}(z, z_{I_1}) \omega_{|I_2|+1}^{(g_2)}(\sigma_i(z), z_{I_2}) \Bigg).$$

The role of functions x and y in topological recursion is not symmetric.

Let us have completely symmetric assumptions for x and y: both are meromorphic, the zeros of dx and dy are simple and disjoint, dy is regular at the zero locus of dx and vice versa. Then we have two instances of topological recursion, for (x,y) and $(x^\vee, y^\vee) := (y,x)$

$$(\Sigma, x, y) \xrightarrow{TR} \{\omega_n^{(g)}\} \xleftrightarrow{x-y} \{\omega_n^{\vee,(g)}\} \xleftarrow{TR} (\Sigma, y, x) = (\Sigma, x^\vee, y^\vee)$$

A universal differential-algebraic relation that captures the exchange of x and y in the topological recursion procedure was conjectured in **[Borot, Charbonnier, Garcia-Failde, Leid, Shadrin '21]**, and proved in genus zero in **[Hock '22]**.

THEOREM (ABDKS '22)

$$\frac{\omega_n^{\vee,(g)}(z_{[\![n]\!]})}{\prod_{i=1}^n dx_i^\vee} = (-1)^n [\hbar^{2g}] \sum_\Gamma \frac{\hbar^{2g(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \prod_{i=1}^n \sum_{k_i=0}^\infty \partial_{y_i}^{k_i} [u_i^{k_i}] \frac{dx_i}{dy_i}$$

$$\frac{1}{u_i} e^{u_i \mathcal{S}(\hbar u_i \partial_{x_i}) \sum_{\tilde g=0}^\infty \hbar^{2\tilde g} \frac{\omega_1^{(\tilde g)}(z_i)}{dx_i} - u_i \frac{\omega_1^{(0)}(z_i)}{dx_i}}$$

$$\prod_{e\in E(\Gamma)} \prod_{j=1}^{|e|\geq 2} \Big\lfloor_{(\tilde u_j, \tilde x_j)\to(u_{e(j)}, x_{e(j)})} \tilde u_j \mathcal{S}(\hbar \tilde u_j \partial_{\tilde x_j}) \sum_{\tilde g=0}^\infty \hbar^{2\tilde g} \frac{\tilde\omega_{|e|}^{(\tilde g)}(\tilde z_{[\![|e|]\!]})}{\prod_{j=1}^{|e|} d\tilde x_j} + \delta_{(g,n),(0,1)}(-x_1).$$

Description of the $x - y$ swap formula

- The sum is taken over all connected graphs Γ with n labeled vertices and multiedges of index ≥ 2, where the index of a multiedge is the number of its legs and we denote it by $|e|$. By $g(\Gamma)$ we denote the first Betti number of Γ.
- For a multiedge e with index $|e|$ we control its attachment to the vertices by the associated map $e\colon [\![|e|]\!] \to [\![n]\!]$ that we denote also by e, abusing notation (so $e(j)$ is the label of the vertex to which the j-th leg of the multiedge e is attached). Do note that this map can be an arbitrary map from $[\![|e|]\!]$ to $[\![n]\!]$; in particular, it might not be injective, i.e. we allow a given multiedge to connect to a given vertex with several of its legs.
- $\tilde{\omega}_2^{(0)}(\tilde{x}_1, \tilde{x}_2) = \omega_2^{(0)}(\tilde{x}_1, \tilde{x}_2) - \frac{d\tilde{x}_1 d\tilde{x}_2}{(\tilde{x}_1 - \tilde{x}_2)^2}$ if $e(1) = e(2)$, and $\tilde{\omega}_2^{(0)}(\tilde{x}_1, \tilde{x}_2) = \omega_2^{(0)}(\tilde{x}_1, \tilde{x}_2)$ otherwise. For all $(g, n) \neq (0, 2)$ we simply have $\tilde{\omega}_n^{(g)} = \omega_n^{(g)}$.
- $[x^m] \sum_{i=-\infty}^{\infty} a_i x^i := a_m$.
- By $\lfloor_{a \to b}$ we denote the operator of substitution $a \to b$, that is, $\lfloor_{a \to b} f(a) = f(b)$ for any function f.
- The function $\mathcal{S}(z)$ is defined as $\mathcal{S}(z) := \frac{e^{z/2} - e^{-z/2}}{z}$.

For each $g \geq 0$, $n \geq 1$, equation is manifestly a finite sum of finite products of differential operators applied to $\omega_m^{(\tilde{g})}$ for $2\tilde{g} - 2 + m \geq 0$

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Denote by $\overline{\mathcal{M}}_{g,n}$ the **Deligne–Mumford compactification of the moduli space** $\mathcal{M}_{g,n}$ of all compact Riemann surfaces S of genus g with n distinct marked points. It is a non-singular complex orbifold of dimension $3g-3+n$, which is empty unless the **stability condition**

$$2g-2+n>0$$

is satisfied. Intersection theory on $\overline{\mathcal{M}}_{g,n}$ " $=$ " Two-dimensional topological gravity **[Witten '91]**.

For each marking index i consider the cotangent line bundle $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$, whose fiber over a point $[S, z_1, \ldots, z_n] \in \overline{\mathcal{M}}_{g,n}$ is the complex cotangent space $T^*_{z_i}S$ of S at z_i. Let $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ denote the first Chern class of $\mathbb{L}_i$. We consider the intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n} \in \mathbb{Q}.$$

The integral on the right-hand side vanishes unless the stability condition is satisfied, all k_i are non-negative integers, and the **dimension constraint** holds true,

$$3g-3+n=\sum_{i=1}^{n} k_i.$$

Let T_i, $i \geq 0$, be formal variables. We introduce the **Kontsevich–Witten tau-function**

$$\tau_{\mathrm{KW}} := \exp\left(\sum_{g,n} \hbar^{2g-2+n} \sum_{k_1,\dots,k_n\geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n} \frac{\prod T_{k_i}}{n!}\right) \in \mathbb{Q}[\mathbf{T}][\![\hbar]\!],$$

that can be considered as a generating function of the Gromov–Witten invariants of a point. A parameter $\hbar$ here is introduced to trace the Euler characteristic of the punctured curve Σ (**topological expansion**).

Witten's conjecture, proved by Kontsevich, states that the partition function τ_{KW} becomes a tau-function of the KdV hierarchy after the change of variables $T_k = (2k+1)!!t_{2k+1}$.

THEOREM (KONTSEVICH '92)

The generating function τ_{KW} is a tau-function of the KdV hierarchy in the variables t_k.

Brézin–Gross–Witten tau-function

The **Brézin–Gross–Witten (BGW)** model was introduced in lattice gauge theory 40 years ago. It can be defined in terms of the asymptotic expansion of a matrix integral.

This tau-function has a natural enumerative geometry interpretation given by the intersection theory of Norbury's Θ-classes, also related to super Riemann surfaces. Norbury's Θ-classes are the cohomology classes, $\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$. Consider the generating function of the intersection numbers

$$\tau_\Theta := \exp\left(\sum_{g,n} \hbar^{2g-2+n} \sum_{k_1,\ldots,k_n \geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{k_1} \psi_2^{k_2} \cdots \psi_n^{k_n} \frac{\prod T_{k_i}}{n!}\right).$$

Genus zero part is trivial! For this generating function **[Norbury '17]** has suggested a direct analog of Witten's conjecture.

Theorem (Chidambaram, Garcia-Failde, Giacchetto '22)

The generating function τ_Θ is the BGW tau-function of the KdV hierarchy in the variables $t_{2k+1} = \frac{T_k}{(2k+1)!!}$,

$$\tau_\Theta = \tau_{\mathrm{BGW}}.$$

Series expansions

$$\log \tau_{\mathrm{KW}} = \left(\frac{{t_1}^3}{6} + \frac{t_3}{8} \right) \hbar + \left(\frac{t_3 {t_1}^3}{2} + \frac{5\, t_5 t_1}{8} + \frac{3\, {t_3}^2}{16} \right) \hbar^2 + \left(\frac{15\, t_1 t_3 t_5}{4} + \frac{3\, {t_3}^2 {t_1}^3}{2} + \frac{5\, t_5 {t_1}^4}{8} + \frac{35\, t_7 {t_1}^2}{16} + \frac{3\, {t_3}^3}{8} + \frac{105\, t_9}{128} \right) \hbar^3 + O(\hbar^4),$$

$$\log \tau_{\mathrm{BGW}} = \frac{t_1}{8} \hbar + \frac{{t_1}^2}{16} \hbar^2 + \left(\frac{{t_1}^3}{24} + \frac{9\, t_3}{128} \right) \hbar^3 + \left(\frac{{t_1}^4}{32} + \frac{27\, t_3 t_1}{128} \right) \hbar^4 + \left(\frac{{t_1}^5}{40} + \frac{27\, t_3 {t_1}^2}{64} + \frac{225\, t_5}{1024} \right) \hbar^5 + \left(\frac{{t_1}^6}{48} + \frac{45\, t_3 {t_1}^3}{64} + \frac{1125\, t_5 t_1}{1024} + \frac{567\, {t_3}^2}{1024} \right) \hbar^6 + O(\hbar^7).$$

The asymptotic expansion of the **Kontsevich matrix model** **[Kontsevich '92]** describes the Kontsevich–Witten tau-function in the **Miwa parametrization**

$$\tau_{\mathrm{KW}}\Big|_{t_k=\frac{1}{k}\operatorname{Tr}\Lambda^{-k}} = \mathcal{C}^{-1}\int_{\mathcal{H}_N}[d\Phi]\exp\left(\frac{1}{\hbar}\operatorname{Tr}\left(\frac{\Phi^3}{3!}-\frac{\Lambda\Phi^2}{2}\right)\right),$$

where $\mathcal{H}_N$ is the space $N\times N$ Hermitian matrices and $[d\Phi]$ is the standard measure on it. KdV integrability of the BGW model follows from the relation to the generalized Kontsevich-type model **[Mironov, Morozov, and Semenoff '96]**:

$$\tau_{\mathrm{BGW}}\Big|_{t_k=\frac{1}{k}\operatorname{Tr}\Lambda^{-k}} = \tilde{\mathcal{C}}^{-1}\int_{\mathcal{H}_N}[d\Phi]\exp\left(\frac{1}{2\hbar}\operatorname{Tr}\left(\Lambda^2\Phi+\Phi^{-1}-2\hbar N\log\Phi\right)\right).$$

Asymptotic expansion near the critical point $\Phi=\Lambda$.

TR FOR THE KW AND BGW TAU-FUNCTIONS

The Kontsevich–Witten tau-function can be described by TR on the **Airy spectral curve** **[Eynard, Orantin ’07; Zhou ’13]**

$$x = -\frac{z^2}{2}, \quad y = z.$$

The correlation differentials generate the intersection numbers

$$\omega_n^{(g)}(z_{[\![n]\!]}) = \sum_{k_1+\cdots+k_n=3g-3+n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\cdots\psi_n^{k_n} \frac{(2k_1+1)!!dz_1}{z_1^{2k_1+2}} \cdots \frac{(2k_n+1)!!dz_n}{z_n^{2k_n+2}}.$$

The Brézin–Gross–Witten tau-function corresponds to the **Bessel spectral curve** **[A. ’18; Do, Norbury ’18]**

$$x = -\frac{z^2}{2}, \quad y = z^{-1}.$$

One critical point of $dx = -zdz$, $z = 0$. The deck transformation is $\sigma(z) = -z$.

For both curve the dual TRs are trivial, because $dy = dz$ and $dy = dz^{-1}$ have no zeroes on Σ, therefore for $2g-2+n>0$ we have $\omega_n^{\vee,(g)} = 0$.

More examples: Hypergeometric tau-functions

Hypergeometric tau-function of 2-component KP hierarchy

$$\tau^{\mathrm{OS}}(\mathbf{t},\mathbf{s}) = \sum_{\lambda} s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s}/\hbar) \exp\left(\sum_{(i,j)\in\lambda} \hat{\psi}(\hbar(i-j),\hbar)\right),$$

where s_λ are the Schur functions. With this tau-function one can associate two functions

$$x = \log z - \hat{\psi}(y(z),0), \qquad y = \sum_{k\geq 1} k s_k(0) z^k,$$

where $\psi(z) = \hat{\psi}(z,0)$. Under some natural analyticity assumptions, which, in particular, imply that dx and dy are rational, these functions define a genus zero spectral curve.

There are several families of $\psi(z,\hbar)$ and $s(z,\hbar)$ such that topological recursion works.

- Hurwitz numbers (usual, monotone, strictly monotone, spin, orbifold, Bousquet-Mélou–Schaeffer, weighted,...)
- Matrix models (Two-matrix model, Gaussian complex matrix model, normal matrix models, Gaussian Hermitian model, HCIZ matrix model, unitary matrix model,...)
- Maps and hypermaps, dessins d'enfants, HOMFLY polynomials of torus knots, $c = 1$ string, Mariño–Vafa numbers, Hodge integrals, constellations,...

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Let Σ be a smooth complex curve, we call it the **spectral curve**. Consider a collection of n-differentials $\omega_n^{(g)}$ on Σ^n defined for all $g \geq 0$, $n \geq 1$.

We do not assume that these differentials satisfy TR relations!

We assume that all $\omega_n^{(g)}$'s are symmetric and meromorphic with no poles on the diagonals for $(g,n) \neq (0,2)$, and $\omega_2^{(0)}$ is also symmetric and meromorphic but it has a second order pole on the diagonal with biresidue 1. It will be convenient to arrange the n-differentials $\omega_n^{(g)}$, $g \geq 0$, for each fixed n into generating series:

$$\omega_n := \sum_{g=0}^{\infty} \hbar^{2g-2+n} \omega_n^{(g)}.$$

Expansion at regular points

Definition

A point $o \in \Sigma$ is called **regular** for the system of differentials $\{\omega_n^{(g)}\}$ if $\omega_n^{(g)} - \delta_{(g,n),(0,2)} \frac{dx_1 dx_2}{(x_1-x_2)^2}$ is regular at $(o,\dots,o) \in \Sigma^n$ for all $(g,n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$, $(g,n) \neq (0,1)$, where x is a local coordinate on Σ at o.

Note that the regularity condition for $\{\omega_n^{(g)}\}$ is independent of a choice of local coordinate and that we have no condition for $\omega_1^{(0)}$.

For a regular point o, and an arbitrary local coordinate x, the forms $\omega_n^{(g)}$ can be expanded

$$\omega_n = \sum_{g=0}^{\infty} \hbar^{2g-2+n} \omega_n^{(g)} = \delta_{n,2} \frac{dx_1 dx_2}{(x_1-x_2)^2} + \sum_{k_1,\dots,k_n=1}^{\infty} f_{k_1,\dots,k_n} \prod_{i=1}^{n} x_i^{k_i-1} dx_i,$$

where the coefficients $f_{k_1,\dots,k_n}$ expand as $\sum_{g=0}^{\infty} \hbar^{2g-2+n} f^{(g)}_{k_1,\dots,k_n}$. Introduce the associated partition function $F = F_{o,x}$ as

$$F(t_1,t_2,\dots) := \sum_{g,n} \hbar^{2g-2+n} F_n^{(g)} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1,\dots,k_n=1}^{\infty} f_{k_1,\dots,k_n} t_{k_1} \dots t_{k_n},$$

where $F_n^{(g)}$ are homogeneous formal series of degree n.

KP integrability as a property of a system of differentials

This way we associate to a given collection of differentials $\omega_n^{(g)}$ a large family of potentials $F_{o,x}$: the freedom in its definition consists of the choice of a regular point o and the choice of a local coordinate x at o. We would like to study conditions assuring that the partition function

$$Z_{o,x} \coloneqq \exp(F_{o,x})$$

is a tau-function of the KP hierarchy.

Theorem (ABDKS '23)

The KP integrability is an internal property of the collection of differentials: if $Z_{o,x}$ is a tau-function of the KP hierarchy for some choice of a regular point o and a coordinate x at this point, then it is a tau-function of the KP hierarchy for any other choice of a regular point and a local coordinate at that point.

A change of the local spectral parameter and the regular point provide a symmetry of the KP hierarchy, given by an element of the Virasoro subgroup of $\widehat{GL_\infty}$.

For a given formal power series $\tau(\mathbf{t})$ the kernel $K(x_1,x_2)$ is defined as

$$K(x_1,x_2) := \frac{1}{x_1-x_2}\tau\Big|_{t_k=\frac{1}{k}(x_1^k-x_2^k)}.$$

If τ is a tau-function of the KP hierarchy, then K is called the **Baker–Akhiezer kernel**. It is a sum of one specific term $(x_1-x_2)^{-1}$ and some formal power series in x_1,x_2. For the KP tau-function these coefficients are known as **affine coordinates**, they characterizes tau-function uniquely. Let

$$W_n(x_1,\dots,x_n) := \left(\prod_{i=1}^{n}\sum_{k=1}^{\infty} x_i^{k-1}\partial_{t_k}\right)\log\tau\Big|_{\mathbf{t}=\mathbf{0}},$$

then we have the **determinantal formulas** which are implied by the Wick formula:

$$W_1(x_1) = \lim_{x_1'\to x_1}\left(K(x_1,x_1') - \frac{1}{x_1-x_1'}\right);$$

$$W_2(x_1,x_2) = -K(x_1,x_2)K(x_2,x_1) - \frac{1}{(x_1-x_2)^2};$$

$$W_n(x_{[\![n]\!]}) = (-1)^{n-1}\sum_{\sigma\in C_n}\prod_{i=1}^{n} K(x_i,x_{\sigma(i)}), \qquad n\geq 3,$$

THEOREM (ZHOU '15)

A formal power series τ is a KP tau-function if and only if the determinantal formulas hold.

In the context of KP integrability as a property of a system of differentials, we consider a given system of differentials $\{\omega_n^{(g)}\}$, and we assume that the tau-function and the corresponding kernel K are associated with a particular choice of the regular point o of the spectral curve and the local coordinate x at this point. We introduce an invariant version of the Baker–Akhiezer kernel as the bi-half-differential, defined globally on the spectral curve

$$\mathbb{K}(z_1, z_2) := K(z_1, z_2)\sqrt{dz_1 dz_2}$$

$$= \frac{\sqrt{dx_1\, dx_2}}{x_1 - x_2} \exp\Bigg(\sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} \int_{z_2}^{z_1} \cdots \int_{z_2}^{z_1} \omega_n^{(g)} + \frac{1}{2} \int_{z_2}^{z_1}\int_{z_2}^{z_1} (\omega_2^{(0)} - \omega_2^{(0),\mathrm{sing}}) \Bigg).$$

then the determinantal formulas take the form

$$\omega_n(x_{[\![n]\!]}) = (-1)^{n-1} \sum_{\sigma \in C_n} \prod_{i=1}^{n} \mathbb{K}(x_i, x_{\sigma(i)}), \qquad n \geq 2.$$

THEOREM (ABDKS '23)

The $x - y$ swap relation preserves KP integrability: the original system of differentials $\omega_n^{(g)}$ is KP integrable if and only if the dual system of differentials $\omega_n^{\vee(g)}$ is KP integrable.

KP integrability restricts the topology of the spectral curve:

THEOREM (ABDKS '23)

If $\omega_2^{(0)}$ has no other poles than on the diagonal and $\{\omega_n^{(g)}\}$ satisfy the KP integrability property, then the spectral curve is rational.

This condition on $\omega_2^{(0)}$ is always satisfied, by definition, in the set-up of topological recursion:

THEOREM (ABDKS '23)

If the differentials $\{\omega_n^{(g)}\}$ produced by topological recursion are KP integrable, then the spectral curve Σ is rational.

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Integral transform

For any two functions x and y let us consider the integral transform of a function $f^\vee(z)$,

$$f(z) = \frac{\mathrm{i}}{\sqrt{2\pi\,\hbar}} \int f^\vee(\chi)\, y'(\chi)\, e^{\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi))+\int_z^\chi x\,dy\right)}\, d\chi.$$

Applying the Taylor expansion of the exponent at the point $\chi = z$ we have

$$\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi)) + \int_z^\chi x\,y'\,dz\right) = \frac{1}{2}x'(z)y'(z)\frac{(\chi-z)^2}{\hbar} + O((\chi-z)^3).$$

After a change of variable $\chi = z+\sqrt{\hbar}\xi/\sqrt{x'(z)y'(z)}$ we can rewrite (and, actually, define) this integral as

$$f(z) = \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{f^\vee\left(z+\frac{\sqrt{\hbar}\,\xi}{\sqrt{x'(z)y'(z)}}\right)\; y'\left(z+\frac{\sqrt{\hbar}\,\xi}{\sqrt{x'(z)y'(z)}}\right)}{\sqrt{x'(z)y'(z)}}\; e^{\frac{\xi^2}{2}+\sum_{k\geq 3}\hbar^{\frac{k}{2}-1}a_k(z)\frac{\xi^k}{k!}}\, d\xi,$$

where $a_k(z) \coloneqq (\partial_z^{k-1}(xy') - xy^{(k)})/(x'(z)y'(z))^{k/2}$.

$f(z)$ is a series in nonnegative integer powers of $\hbar$ whose coefficients are certain rational combinations of derivatives of different order of the functions $f^\vee$, x, and y.

$x - y$ swap for Baker–Akhiezer kernel

If the systems of differentials $\{\omega_n^{(g)}\}$ and hence $\{\omega_n^{\vee,(g)}\}$ are KP integrable, in the complicated combinatorial expression of $x - y$ duality they both can be replaced by a simpler determinantal formulas. This observation raises a natural question: what happens with the Baker–Akhiezer kernel under the $x - y$ swap? It proves out that there are explicit formulas relating $x - y$ dual kernels.

Let us consider two integral transforms which are inverse to each other. Let for some function $f^\vee(z)$

$$f(z) = \frac{\mathrm{i}}{\sqrt{2\pi\,\hbar}} \int f^\vee(\chi)\, y'(\chi)\, e^{\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi))+\int_z^\chi x\,dy\right)}\, d\chi.$$

Then

$$f^\vee(z) = \frac{1}{\sqrt{2\pi\,\hbar}} \int f(\chi)\, x'(\chi)\, e^{-\frac{1}{\hbar}\left(y(z)(x(z)-x(\chi))+\int_z^\chi y\,dx\right)}\, d\chi.$$

All integrals of this type are understood purely formally in the sense of asymptotic expansions for small absolute value of $\hbar$ near the critical points $\chi = z$.

$x - y$ swap for Baker–Akhiezer kernel

This type of transformation connects the Baker–Akhiezer kernels on the two sides of the $x - y$ duality.

Theorem (ABDKS '23)

The Baker–Akhiezer kernels $\mathbb{K}$ and $\mathbb{K}^\vee$ related by the $x - y$ swap are expressed in terms of one another by the following double integrals:

$$K(z_1,z_2) = \frac{-\mathrm{i}}{2\pi\hbar}\iint K^\vee(\chi_1,\chi_2)\, y'(\chi_1)y'(\chi_2)\, d\chi_1 d\chi_2 \times$$
$$e^{-\frac{1}{\hbar}\left(x(z_2)(y(z_2)-y(\chi_1))+\int_{z_2}^{\chi_1} x dy\right)} e^{\frac{1}{\hbar}\left(x(z_1)(y(z_1)-y(\chi_2))+\int_{z_1}^{\chi_2} x dy\right)};$$

$$K^\vee(z_1,z_2) = \frac{-\mathrm{i}}{2\pi\hbar}\iint K(\chi_1,\chi_2)\, x'(\chi_1)x'(\chi_2) d\chi_1 d\chi_2 \times$$
$$e^{-\frac{1}{\hbar}\left(y(z_2)(x(z_2)-x(\chi_1))+\int_{z_2}^{\chi_1} y dx\right)} e^{\frac{1}{\hbar}\left(y(z_1)(x(z_1)-x(\chi_2))+\int_{z_1}^{\chi_2} y dx\right)}.$$

Here $\mathbb{K}(z_1,z_2) = K(z_1,z_2)\sqrt{dx_1 dx_2}$, $\mathbb{K}^\vee(z_1,z_2) = K^\vee(z_1,z_2)\sqrt{dy_1 dy_2}$, $x' = \partial_z x$, $y' = \partial_z y$.

These integral transforms are inspired by and closely related to the Kontsevich matrix model and its generalizations.

$x-y$ DUAL OF A TRIVIAL CASE

An important special case concerns the situation when the $x-y$ dual side is known to be trivial.

We assume that $\omega_0^{\vee,(2)} = \frac{dz_1 dz_2}{(z_1-z_2)^2}$ and for all $(g,n) \neq (0,2)$

$$\omega_n^{\vee,(g)} = 0.$$

Here z is a meromorphic function on Σ that can serve as a coordinate at the point $o \in \Sigma$. In this case

$$Z^{\vee}_{o,z} = 1$$

is a tau-function of the KP hierarchy, and under the $x-y$ duality relation the Baker–Akhiezer kernel is given by the double integral:

$$\frac{\mathbb{K}(z_1,z_2)}{\sqrt{dz_1 dz_2}} = \frac{-\mathrm{i}\sqrt{x'(z_1)x'(z_2)}}{2\pi\hbar} \iint \frac{\sqrt{y'(\chi_1)y'(\chi_2)}}{\chi_1-\chi_2} d\chi_1 d\chi_2 \times$$
$$e^{-\frac{1}{\hbar}\left(x(z_2)(y(z_2)-y(\chi_1))+\int_{z_2}^{\chi_1} x dy\right)} e^{\frac{1}{\hbar}\left(x(z_1)(y(z_1)-y(\chi_2))+\int_{z_1}^{\chi_2} x dy\right)}.$$

Note that in the KP integrable case the kernel $\mathbb{K}$ carries the complete information on the corresponding KP tau-functions. The semi-infinite plane corresponding to the tau-function $Z = Z_{o,z}$ is spanned by the Taylor coefficients of the expansion of $\mathbb{K}(z_1, z_2)/\sqrt{dz_1 dz_2}$ in z_2.

THEOREM (ABDKS ’23)

Assume $x'y'$ has a pole of degree ≥ 3 at $z = 0$. Let $\Phi_i^{*,\vee}(z)$, $i \geq 1$ have the form $z^{-i}(1 + O(z))$ and generate the semi-infinite plane corresponding to the tau-function $Z^{*,\vee}(\mathbf{t}) = Z^{\vee}(-\mathbf{t})$, where $Z^{\vee}$ is $x - y$ dual to the tau-function $Z = Z_{o,z}$. Then the following vectors Φ_i, $i \geq 1$, have the form $\Phi_i(z) = z^{-1}(1 + O(z))$ and generate the semi-infinite plane corresponding to the tau-function Z:

$$\Phi_i(z) = \sqrt{\frac{-x'(z)}{2\pi\hbar}} \int \Phi_i^{*,\vee}(\chi) \sqrt{y'(\chi)}\, e^{\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi)) + \int_z^{\chi} x\,dy\right)} d\chi, \qquad i \geq 1.$$

The so-called $p - q$ duality describes a relation between two minimal models coupled to two-dimensional topological gravity. In our language it should correspond to the swap of x and y defined to be polynomials of fixed finite degrees p and q, respectively. We claim that in these special cases the our statements provide the $\mathbf{p} - \mathbf{q}$ **duality** proposed in **[Kharchev, Marshakov ’95]**

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$x - y$ swap relation and KP integrability

A typical source of situations with trivial dual differentials comes from the theory of topological recursion. Assume z is a global coordinate on $\mathbb{C}P^1$, and $y(z) = (az+b)/(cz+d)$ with $ad - bc \neq 0$, and $x(z)$ is rational. Then the theory of topological recursion assigns to this input a system of differentials $\{\omega_n^{(g)}\}$. These differentials are regular at the points $o \in \mathbb{C}P^1$ where $dx \neq 0$.

Theorem (ABDKS '23)

Let $\{\omega_n^{(g)}\}$ be a system of differentials constructed by topological recursion for the input consisting of $\Sigma = \mathbb{C}P^1$ with a global coordinate z and $y(z) = (az+b)/(cz+d)$. Let $z = 0$ be a regular point for the system of differentials. Then $Z_{0,z}$ is a KP tau-function. Moreover, if $x'y'$ has a pole of degree at least 3 at $z = 0$, then the tau-function $Z_{0,z}$ is described by the point of the semi-infinite Grassmannian spanned by

$$\Phi_i(z) = \sqrt{\frac{-x'(z)}{2\pi\hbar}} \int \chi^{-i} \sqrt{y'(\chi)}\, e^{\frac{1}{\hbar}\left(x(z)(y(z)-y(\chi)) + \int_z^\chi x\,dy\right)} d\chi, \qquad i \geq 1.$$

Many Kontsevich-type matrix models can be obtained using this approach.

As a corollary of this theorem, we obtain the following explicit formulas in two important families of examples of topological recursion.

- Let $x = z^{-r}/r$, $y = -z^{-1}$, $r \geq 2$. In this case the associated partition function $Z_{0,z}$ is known to be the string solution of the r-th Gelfand–Dickey hierarchy, governed the intersection theory of **Witten's class**. The corresponding point on the Sato Grassmannian is given by

$$\Phi_i(z) = \frac{1}{\sqrt{2\pi\hbar z^{r+1}}} \int \chi^{-i-1} e^{\frac{1}{r\,(r+1)\,\hbar}\left(z^{-r} - rz^{-r-1}(\chi - z) - \chi^{-r}\right)} d\chi, \qquad i \geq 1.$$

- Let $x = z^{-r}/r$, $y = -z$, $r \geq 2$. The corresponding point on the Sato Grassmannian is given by

$$\Phi_i(z) = \frac{\mathrm{i}}{\sqrt{2\pi\hbar z^{r+1}}} \int \chi^{-i} e^{\frac{1}{r\,(r-1)\,\hbar}\left(-z^{1-r} + (r-1)z^{-r}(\chi - z) + \chi^{1-r}\right)} d\chi, \qquad i \geq 1.$$

Note that TR is degenerate.

For $x = z^r/r$ and $y = z^{-1}$ the point $z = \infty$ is regular, so we consider the partition function $Z_{\infty,z^{-1}}$.

THEOREM (ABDKS '23)

The partition function $Z_{\infty,z^{-1}}$ is given by the following matrix integral:

$$Z_{\infty,z^{-1}}\Big|_{t_k=\frac{1}{k}\operatorname{Tr}\Lambda^{-k}} = \frac{\int_{\mathcal{H}_N}[dM]\exp\left(\frac{1}{\hbar}\operatorname{Tr}\left(\frac{\Lambda^r M}{r}+\frac{M^{1-r}}{r(r-1)}-\hbar N\ln M\right)\right)}{\int_{\mathcal{H}_N}[dM]\exp\left(\frac{1}{\hbar}\operatorname{Tr}\left(\frac{M^{1-r}}{r(r-1)}-\hbar N\ln M\right)\right)},$$

where $\mathcal{H}_N$ is the space of $N\times N$ Hermitian matrices and $[dM]$ is the flat measure on it.

This matrix integral was introduced in **[Mironov, Morozov, Semenoff '96]** and is knows as a higher Brézin–Gross–Witten tau-function. The statement of this theorem was conjectured in **[Alexandrov, Dhara '22; Chidambaram, Garcia-Failde, Giacchetto '22]**.

Partition function $Z_{\infty,z^{-1}}$ has a geometric meaning. The formal power series expansion of its logarithm gives the intersection numbers of the ψ-classes with the so-called r-theta classes $\Theta^r_{g,n}$ on the moduli spaces of curves $\overline{\mathcal{M}}_{g,n}$. In particular, for $r=2$ it is reduced to a generating series of ψ- and a combination of κ-classes **[Kazarian, Norbury '21]**.

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Any TR on a rational spectral curve is KP integrable

- Σ is of genus zero, that is, $\Sigma = \mathbb{C}P^1$;
- dx and dy are any meromorphic differentials (It is sufficient to assume that dx and dy are defined in a vicinity of the points $q_1, \dots, q_n$).

Theorem (ABDKS '24)

The system of generalized TR differentials for the input data as above possesses KP integrability property.

Idea: Deformation to the trivial case.

The upshot of this Theorem and the restriction of the KP integrability to genus zero spectral curve can be then formulated as follows:

Corollary

A system of differentials $\{\omega_n^{(g)}\}$ produced by generalized topological recursion is KP integrable if and only if the spectral curve is rational.

The non-perturbative disconnected $n|n$ half-differentials $\Omega_n^{\mathrm{np},\bullet}$ (fermionic correlation functions) are defined as

$$\Omega_n^{\mathrm{np},\bullet}(z_{[\![n]\!]}^+, z_{[\![n]\!]}^-) := \prod_{1\le k<l\le n} \frac{E(z_k^+, z_l^+)E(z_k^-, z_l^-)}{E(z_k^+, z_l^-)E(z_k^-, z_l^+)} \prod_{i=1}^n \frac{\exp\left(\frac{1}{\hbar}\int_{z_i^-}^{z_i^+} \omega_1^{(0)}\right)}{E(z_i^+, z_i^-)} \times$$

$$\frac{\exp\left(\sum_{\substack{m_0,\dots,m_n\ge 0\\ \sum_{i=0}^n m_i = m \ge 1}} \frac{1}{\prod_{i=0}^n m_i!}\left(\frac{1}{2\pi \mathrm{i}}\oint_{\mathfrak{B}} \partial_w\right)^{m_0} \prod_{i=1}^n \left(\int_{z_i^-}^{z_i^+}\right)^{m_i} \omega_m\right)\Theta_*\left(w + \mathcal{A}\left(\sum_{i=1}^n (z_i^+ - z_i^-)\right)\middle|\mathcal{T}\right)}{\exp\left(\sum_{m\ge 1}\frac{1}{m!}\left(\frac{1}{2\pi \mathrm{i}}\oint_{\mathfrak{B}}\partial_w\right)^m \omega_m\right)\Theta_*(w|\mathcal{T})}$$

THEOREM (ABDKS '24, CONJECTURED IN EYNARD, ORANTIN '11)

The system of differentials $\{\omega_n^{\mathrm{np}}\}$ is KP integrable, that is, we have:

$$\omega_n^{\mathrm{np}}(z_{[\![n]\!]}) = \det{}^{\circ}(\Omega_1^{\mathrm{np}}(z_i, z_j)), \quad n \ge 2.$$
$$\Omega_n^{\mathrm{np},\bullet}(z_{[\![n]\!]}^+, z_{[\![n]\!]}^-) = \det(\Omega_1^{\mathrm{np}}(z_i^+, z_j^-)).$$

Open questions

- TR and integrability in the non-regular points. Multi-component hierarchies.
- Family of tau-functions generated by topological recursion.
- BKP hierarchy in special points. Other integrable systems.
- Matrix models from TR.
- More general non-perturbative solutions and blobbed TR.

Work in progress with Boris Bychkov, Petr Dunin-Barkowski, Maxim Kazarian, and Sergey Shadrin

Thank you!