Emergence of nonlinear Jeans-type instabilities for quasilinear wave equations

Chao Liu

Email: chao_liu@hust.edu.cn

Huazhong University of Science and Technology (HUST)

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§1. Introduction
(Equations, goals, motivations and main results)

Final Goals of this Project

The fully nonlinear Jeans-type instabilities for the Euler–Poisson system and the Einstein–Euler system.

The Recently Completed Work (This talk!)

The fully nonlinear Jeans-type instabilities for a simplified toy model: a quasilinear wave equation with several difficult nonlinear terms which models the Euler–Poisson and Einstein–Euler system in some aspects.

Ongoing Work

The fully nonlinear Jeans-type instabilities for the Euler–Poisson and 2nd order perturbations (Bardeen invariants, by Hwang, Noh) of Einstein–Euler system.

Expecting some opinions from the audiences

The fully nonlinear Jeans-type instabilities for the Einstein–Euler system:
1. The definition of density contrast is ill-defined due to gauge dependent.
2. The covariant fractional density gradient by Ellis and Bruni (1989)? same mechanism for 1st order, different for 2nd order?

Equations and goals

Mathematical version

$$\partial_t^2 \varrho - \left(\frac{\mathrm{m}^2 (\partial_t \varrho)^2}{(1+\varrho)^2} + 4(\mathrm{k} - \mathrm{m}^2)(1+\varrho) \right) \Delta \varrho = F(t, \varrho, \partial_\mu \varrho)$$

where the nonlinear source terms are

$$F(t, \varrho, \partial_\mu \varrho) := \underbrace{\frac{2}{3}\varrho(1+\varrho)}_{\text{(i) self-increasing}} \underbrace{-\frac{1}{3}\partial_t \varrho}_{\text{(ii) damping}} + \underbrace{\frac{4}{3}\frac{(\partial_t \varrho)^2}{1+\varrho}}_{\text{(iii) Riccati}}$$

$$+ \underbrace{\left(\mathrm{m}^2 \frac{(\partial_t \varrho)^2}{(1+\varrho)^2} + 4(\mathrm{k} - \mathrm{m}^2)(1+\varrho) \right) q^i \partial_i \varrho}_{\text{(iv) convection}} - \mathrm{K}^{ij}(t, \varrho, \partial_\mu \varrho) \partial_i \varrho \partial_j \varrho.$$

related to shock formation problem for quasilinear wave by J. Speck

$$\varrho|_{t=\mathbf{t}_0} = \underbrace{\beta}_{\text{const.}} + \underbrace{\psi(x^k)}_{\text{inhomogeneity}} \quad \text{and} \quad \partial_t \varrho|_{t=\mathbf{t}_0} = \underbrace{\beta_0}_{\text{const.}} + \underbrace{\psi_0(x^k)}_{\text{inhomogeneity}}, \quad \text{in } \{\mathbf{t}_0\} \times \mathbb{R}^n,$$

- (Goal) Find self-increasing blowup solutions (formations of nonlinear cosmological structures).
- (Result) The solutions can attain arbitrarily large values over time, leading to self-increasing singularities at some future endpoints of null geodesics, provided the inhomogeneous perturbations of data are sufficiently small (long wave feature!).

Physics version

seen in physics literatures

After time transform $t \to \ln t$, the equation becomes:

$$\begin{aligned}\partial_t^2 \varrho - \mathrm{g}^{ij}\partial_i\partial_j\varrho = & \frac{2}{3t^2}\varrho(1+\varrho) - \frac{4}{3t}\partial_t\varrho + \frac{4}{3}\frac{(\partial_t\varrho)^2}{1+\varrho} + \mathrm{g}q^i\partial_i\varrho \\ & - \frac{1}{t^2}\mathrm{K}^{ij}(t,\varrho,\partial_\mu\varrho)\partial_i\varrho\partial_j\varrho, \quad \text{in } [t_0,t^\star)\times\mathbb{R}^n,\end{aligned}$$

$$\varrho|_{t=t_0} = \beta + \psi(x^k) \quad \text{and} \quad \partial_t\varrho|_{t=t_0} = \beta_0 + \psi_0(x^k), \quad \text{in } \{t_0\}\times\mathbb{R}^n,$$

where

$$\mathrm{g}^{ij} = \mathrm{g}^{ij}(t,\varrho,\partial_t\varrho) := \mathrm{g}(t,\varrho,\partial_t\varrho)\delta^{ij} = \left(\mathrm{m}^2\frac{(\partial_t\varrho)^2}{(1+\varrho)^2} + 4(\mathrm{k}-\mathrm{m}^2)\frac{1+\varrho}{t^2}\right)\delta^{ij}.$$

- Now focus on this equation!! A time transform $t \to e^t$ leads back to the previous equation.

A short version for the motivation

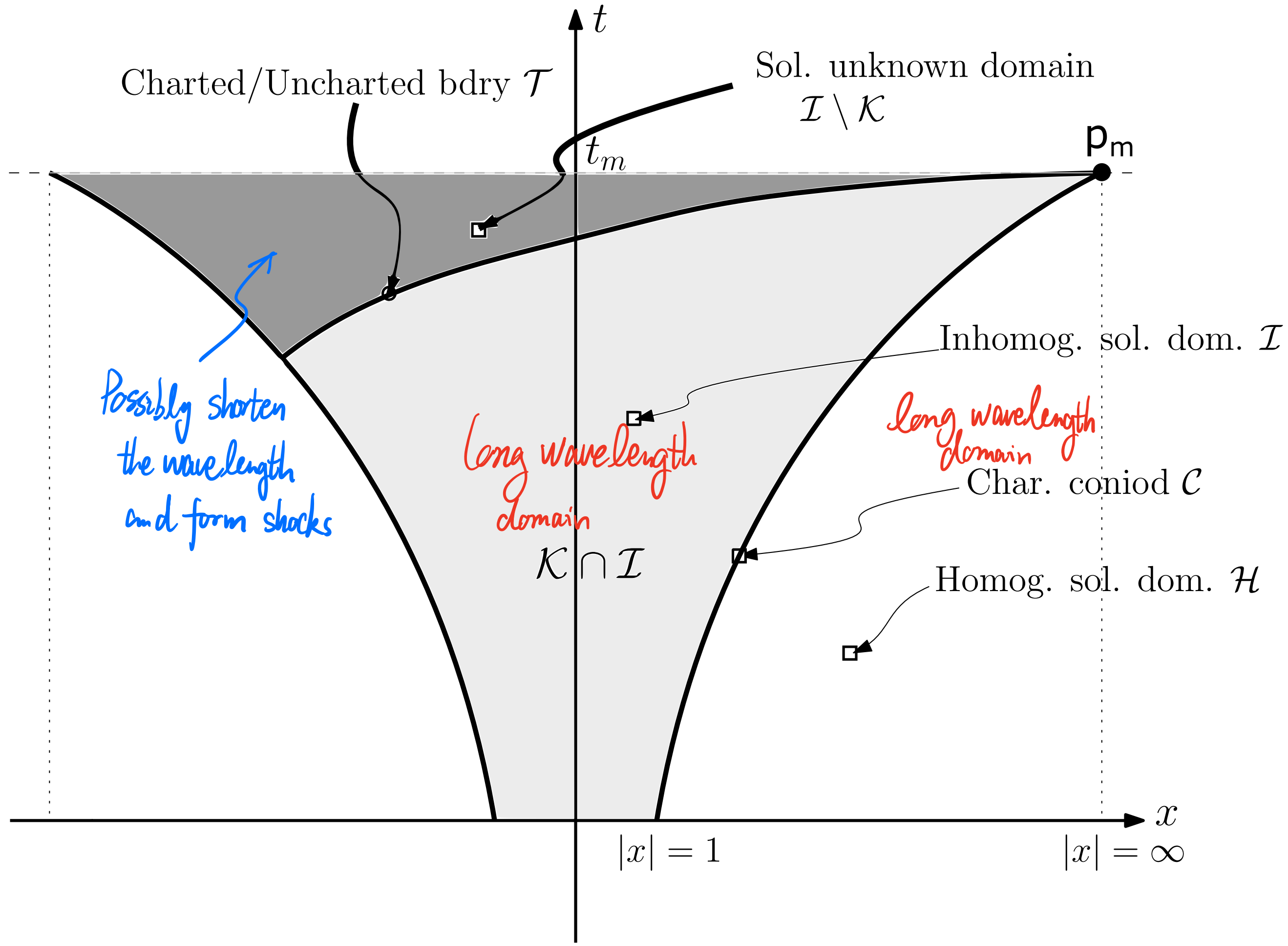
Relate to the famous problem in astrophysics:

Important astrophysical question

- Cosmological perturbation theories (linear, higher order approximations) by e.g., Jeans, Bonner, Lifshitz, Bardeen, Hawking, Ellis, Bruni, Zel'dovich, Peebles, Brandenberger, Mukhanov, Hwang, Noh... (long lists in astrophy.). Stability/instability;
- (Jeans instability) The formations of the nonlinear cosmological structures;
- (Core mathematical mechanism) Nonlinear Jeans-type instability. Rare mathematical studies currently.

- A toy model for above system. Neglecting rotations, shears of the fluids, and tidal forces, Euler–Poisson (or Einstein–Euler) leads to this type of QNLW.

Main Theorem shown by one picture



Rough expression of the main theorem

If the initial inhomogeneities $\|(\psi, \psi_0)\|_{\mathtt{X}}$ are sufficiently small, then $\mathfrak{t} = \mathfrak{t}(t, x) \to t$, and $\varrho(t, x^k) \to f(\mathfrak{t})$ and their derivatives as well of the reference ODEs, at least within a sufficiently large domain $\mathcal{D}$ close to $[t_0, t_m) \times \mathbb{R}^n$.

Rough expression of the main theorem

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$$\boxed{\partial_{\mathfrak{t}}^2 f(\mathfrak{t}) + \frac{\mathrm{a}}{\mathfrak{t}} \partial_{\mathfrak{t}} f(\mathfrak{t}) - \frac{\mathrm{b}}{\mathfrak{t}^2} f(\mathfrak{t})(1 + f(\mathfrak{t})) - \frac{\mathrm{c}}{1 + f(\mathfrak{t})} (\partial_{\mathfrak{t}} f(\mathfrak{t}))^2 = 0,}$$

$$f(t_0) = \beta > 0 \quad \text{and} \quad \partial_t f(t_0) = \beta_0 > 0.$$

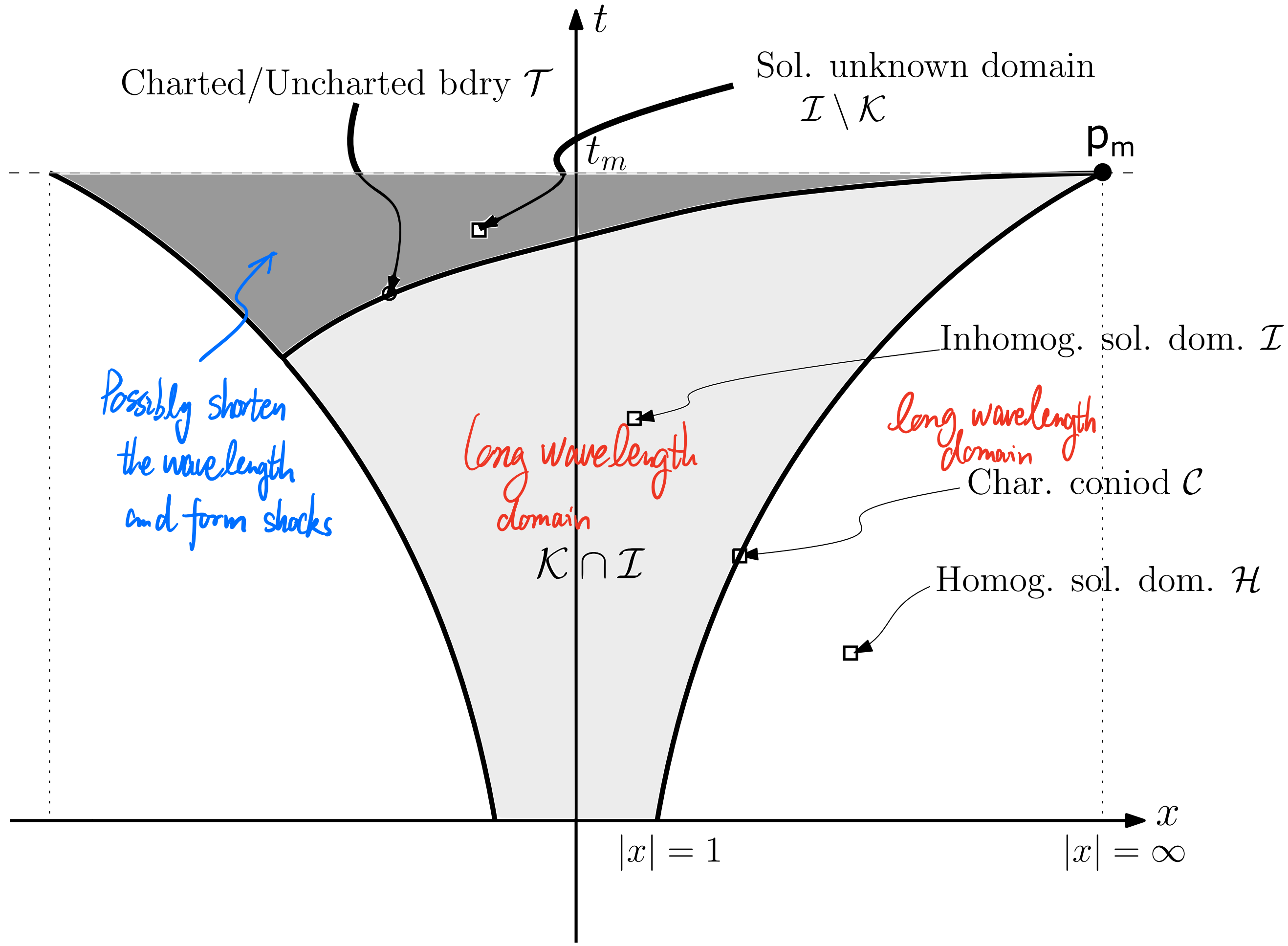
Moreover,

$$e^{\mathrm{Const.} \times t^{\frac{2}{3}}} - 1 \lesssim \varrho(t, x^k) \sim f(\mathfrak{t}) \lesssim \frac{1}{\mathrm{Const.} - t^{\frac{2}{3}}} - 1,$$

and if the initial data $\beta_0 \gtrsim \beta$, then

$$\frac{1}{(t^{-\frac{1}{3}} - \mathrm{Const.})^3} - 1 \lesssim \varrho(t, x^k) \sim f(\mathfrak{t}) \lesssim \frac{1}{\mathrm{Const.} - t^{\frac{2}{3}}} - 1,$$

Main Theorem



$$\mathcal{C} := \left\{ (t, x) \in [t_0, t_m) \times \mathbb{R}^n \;\middle|\; |x| = 1 + \int_{t_0}^{t} \sqrt{\mathsf{g}(y, f(y), f_0(y))} dy \right\}.$$

Remark (directional bias)

The direction of convection is assumed to be constant and can be normalized $q^i = |q|\delta_1^i$.

Generations

remark

The convection term introduces a directional bias in gq^i, causing the wave to propagate more strongly in a particular direction depending on the sign and magnitude of gq^i.

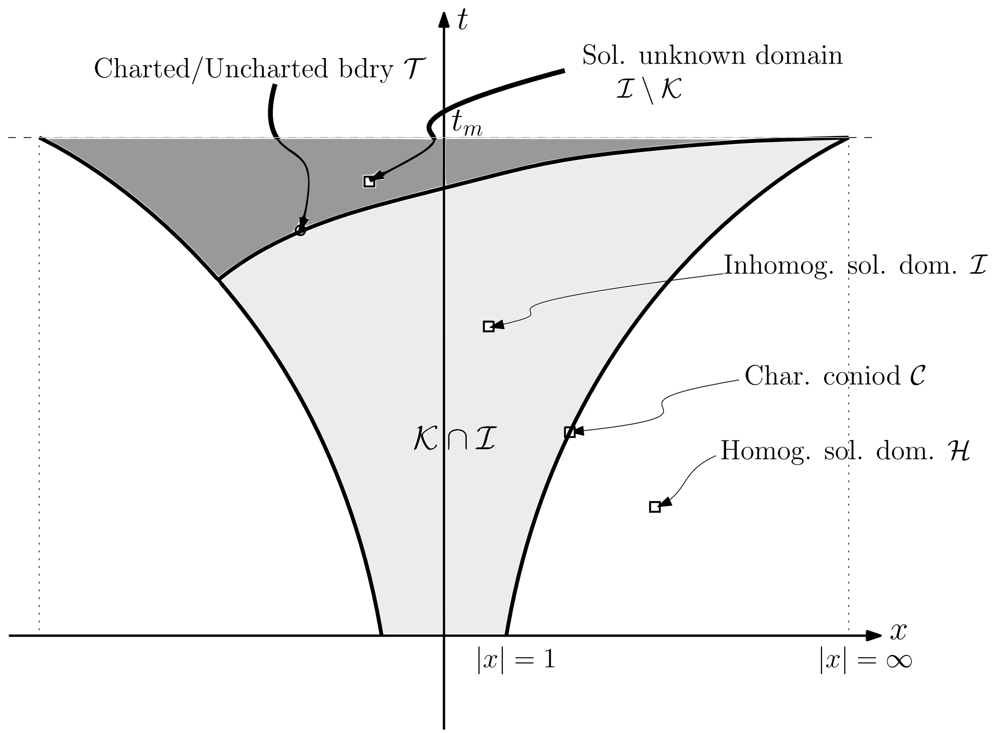


Figure: $q^i = |q|\delta_1^i$

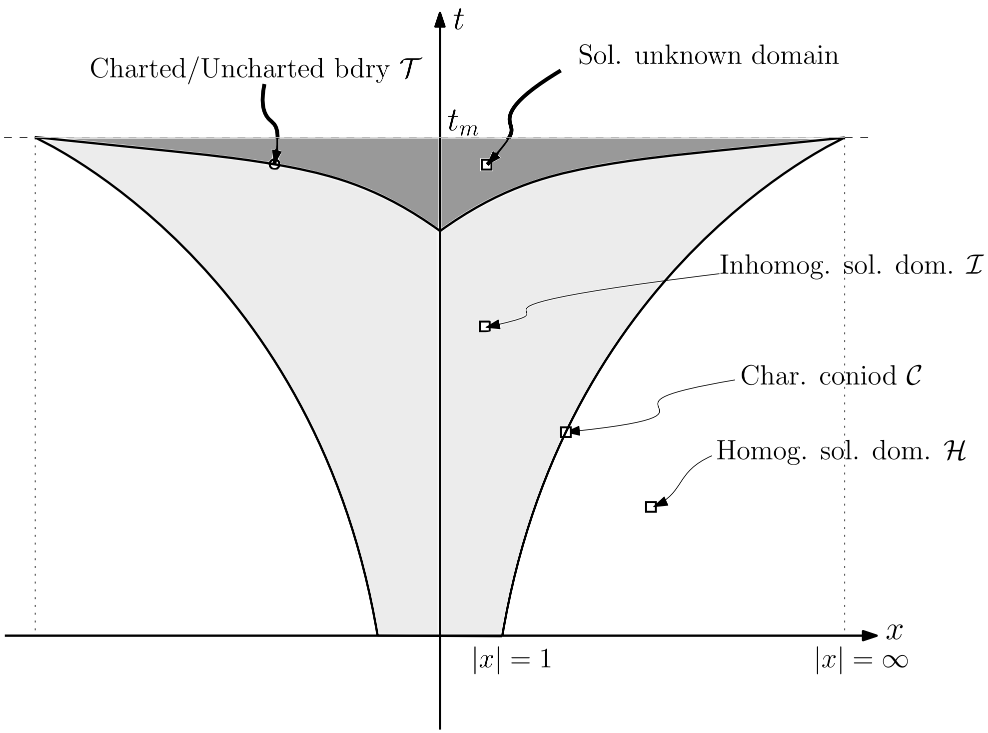


Figure: $q^i \propto x^i/|x|$

Main Theorem

Suppose $k \in \mathbb{Z}_{\frac{n}{2}+3}$, $\mathtt{A}$, $\mathtt{B}$, $\mathtt{C}$, $\mathtt{D}$ are constants depending on the initial data β and β_0, as defined in the article, and that Assumptions hold. Let $(\psi, \psi_0) \in C_0^1(\mathbb{R}^n)$ be given functions with $\mathrm{supp}(\psi, \psi_0) = B_1(0)$, $f(\mathfrak{t})$ be the solution to the key reference ODE (later!). Then there exist sufficiently small constants $\sigma_0 > 0$ and $\delta_0 > 0$, such that if the initial data satisfy

$$\|\psi\|_{H^k(B_1(0))} + \|\partial_i \psi\|_{H^k(B_1(0))} + \|\psi_0\|_{H^k(B_1(0))} \leq e^{-\frac{153}{\delta_0}} \sigma_0^2,$$

then there exists a hypersurface $t = \mathcal{T}(x, \delta_0)$ satisfying

$$\Gamma_{\delta_0} := \{(t, x) \in [t_0, t_m) \times \mathbb{R}^n \mid t = \mathcal{T}(x, \delta_0)\} \subset \mathcal{I}, \qquad \lim_{a \to +\infty} \mathcal{T}(a\delta_1^i, \delta_0) = t_m$$

$$\lim_{\delta_0 \to 0+} \mathcal{T}(x, \delta_0) = t_m,$$

such that there is a solution $\varrho \in C^2(\mathcal{K} \cup \mathcal{H})$ to the main equation where $\mathcal{K} := \{(t, x) \in [t_0, t_m) \times \mathbb{R}^n \mid t < \mathcal{T}(x, \delta_0)\}$ satisfying

Main Theorem (conti.)

- if we denote

$$\mathbf{1}_-(x^1) := 1 - C\sigma_0^2 e^{-\frac{50}{\delta_0}} e^{-\frac{x^1}{2}} \searrow 1 \quad \text{and} \quad \mathbf{1}_+(x^1) := 1 + C\sigma_0^2 e^{-\frac{50}{\delta_0}} e^{-\frac{x^1}{2}} \searrow 1, \quad \text{as } x^1 \to +\infty,$$

then there are estimates for $(t,x) \in \mathcal{K} \cap \mathcal{I}$,

$$\mathbf{1}_-(x^1) f_0\big(t_0 + \mathbf{1}_-(x^1)(t-t_0)\big) \leq \varrho_0(t,x) \leq \mathbf{1}_+(x^1) f_0\big(t_0 + \mathbf{1}_+(x^1)(t-t_0)\big),$$
$$-C\sigma_0^2 e^{-\frac{50}{\delta_0}} e^{-\frac{x^1}{2}} \big(1 + f\big(t_0 + \mathbf{1}_-(x^1)(t-t_0)\big)\big) \leq \varrho_i(t,x) \leq C\sigma_0^2 e^{-\frac{50}{\delta_0}} e^{-\frac{x^1}{2}} \big(1 + f\big(t_0 + \mathbf{1}_+(x^1)(t-t_0)\big)\big),$$
$$\mathbf{1}_-(x^1) f\big(t_0 + \mathbf{1}_-(x^1)(t-t_0)\big) \leq \varrho(t,x) \leq \mathbf{1}_+(x^1) f\big(t_0 + \mathbf{1}_+(x^1)(t-t_0)\big).$$

Moreover, both ϱ_0 and ϱ reach the self-increasing singularities at the point $p_m := (t_m, +\infty, 0, \cdots, 0)$:

$$\lim_{\mathcal{K} \ni (t,x) \to p_m} \varrho = \lim_{\mathcal{K} \ni (t,x) \to p_m} f = +\infty,$$
$$\lim_{\mathcal{K} \ni (t,x) \to p_m} \varrho_0 = \lim_{\mathcal{K} \ni (t,x) \to p_m} f_0 = +\infty \quad \text{and} \quad \lim_{\mathcal{K} \ni (t,x) \to p_m} \varrho_i = 0.$$

- $\varrho \equiv f$ for $(t,x) \in \mathcal{H}$.

Main Theorem (conti.)

- the growth rate of ϱ can be estimated using visualizable functions

$$\varrho(t,x) \geq \mathbf{1}_-(x^1) f\big(t_0 + \mathbf{1}_-(x^1)(t-t_0)\big) > \mathbf{1}_-(x^1)\big(e^{\mathtt{C}(t_0+\mathbf{1}_-(x^1)(t-t_0))^{\frac{2}{3}}} - 1\big)$$

  and

$$\varrho(t,x) \leq \mathbf{1}_+(x^1) f\big(t_0 + \mathbf{1}_+(x^1)(t-t_0)\big) < \frac{3}{2}\left(\frac{1}{1+\frac{\mathtt{A}}{t_0+\mathbf{1}_+(x^1)(t-t_0)} + \mathtt{B}(t_0+\mathbf{1}_+(x^1)(t-t_0))^{\frac{2}{3}}} - 1\right)$$

  for all $(t,x) \in \mathcal{K} \cap \mathcal{I}$.

- if the initial data satisfy $\breve{\beta} := \frac{t_0\beta_0}{1+\beta} - 1 > 0$, then ϱ has an improved lower bound, indicating finite-time blowups.

$$\varrho(t,x) \geq \mathbf{1}_-(x^1) f\big(t_0 + \mathbf{1}_-(x^1)(t-t_0)\big) > \mathbf{1}_-(x^1)\left(\frac{1+\beta}{\Big(\frac{\beta_0 t_0^{\frac{4}{3}}}{1+\beta}\big(t_0+\mathbf{1}_-(x^1)(t-t_0)\big)^{-\frac{1}{3}} - \breve{\beta}\Big)^3} - 1\right)$$

  for all $(t,x^k) \in \mathcal{K} \cap \mathcal{I}$.

§2. Backgrounds: classical Jeans instabilities

(only include classical ones without further developments in astrophy.)

Classical Jeans instability (Static, Euler–Poisson system)

$$\begin{cases} \partial_t \rho + \partial_i(\rho v^i) = 0 \\ \partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi = 0 \\ \delta^{ij}\partial_i\partial_j \phi = 4\pi G \rho \end{cases}$$

Euler – Poisson system

let $\rho = \underset{\substack{\uparrow \\ \text{constant.}}}{\underline{\rho_0}} + \hat{\rho}$, $v^i = \underset{\substack{\uparrow \\ 0}}{\underline{v_0}} + \tilde{v}$, $\phi = \underset{\substack{\uparrow \\ \nabla\phi_0 = 0}}{\underline{\phi_0}} + \tilde{\phi}$, $\tilde{p} = c_s^2 \tilde{\rho}$

$\nabla\phi_0 = 0$ Contradicts Poisson.

① $\boxed{\text{linearization}}$ $\Longrightarrow$

$$\begin{cases} \partial_t \tilde{\rho} + \rho_0 \partial_i \tilde{v}^i = 0 \\ \partial_t \tilde{v}^i + \frac{c_s^2}{\rho_0} \partial^i \tilde{\rho} + \partial^i \tilde{\phi} = 0 \\ \delta^{ij}\partial_i\partial_j \tilde{\phi} = 4\pi G \tilde{\rho} \end{cases} \xrightarrow{②\ \partial_i} \boxed{\partial_t^2 \tilde{\rho} - \underset{\substack{\uparrow \\ \text{Const.}}}{\underline{c_s^2}}\, \delta^{ij}\partial_i\partial_j \tilde{\rho} - \underset{\substack{\uparrow \\ \text{Const.}}}{\underline{4\pi G \rho_0}}\, \tilde{\rho} = 0}$$

$\Box\tilde{\rho} - k\tilde{\rho} = 0.$ $(k>0)$

$$\boxed{\partial_t^2\tilde{\rho}-c_s^2\delta^{ij}\partial_i\partial_j\tilde{\rho}-4\pi G\rho_0\tilde{\rho}=0}$$

$$\Downarrow \text{ Fourier transform.}$$

$$\tilde{\rho}_k''+(k^2c_s^2-4\pi G\rho_0)\tilde{\rho}_k=0 \Rightarrow \begin{cases}\tilde{\rho}_k \propto \exp(\pm i\omega(k)t)\\ \omega(k)=\sqrt{k^2c_s^2-4\pi G\rho_0}\end{cases}$$

if $k^2c_s^2-4\pi G\rho_0<0.$ $\Leftrightarrow$ $k<\dfrac{\sqrt{4\pi G\rho_0}}{c_s}=:k_J$

$$\Rightarrow \boxed{\tilde{\rho}_k\propto\exp(\pm|\omega|t)}$$

exponentially growth.

Classical Jeans instability (expansion)

Expanding Newtonian Universe

$$\begin{cases} \rho = \rho_0(t) = \dfrac{1}{6\pi G t^2}, \quad v^i = v_0^i = \dfrac{2}{3t} x^i \quad \left(H(t) = \dfrac{2}{3t} \text{ Hubble's law} \right) \\ \phi = \phi_0 = \dfrac{2}{3}\pi G \rho_0 |x|^2 \end{cases}$$

let $\rho = \rho_0 + \tilde{\rho}$, $v^i = v_0^i + \tilde{v}^i$, $\phi = \phi_0 + \tilde{\phi}$

$$\boxed{H = \frac{\dot{a}}{a}}$$

① Lagrangian coord. (comoving with Hubble flow) : $x^i = a(t)\, q^i$

② Linearization Euler–Poisson

③ ∂_i(momentum conservation) ← Poisson eq. ← continuity eq.

density contrast

let $e = \dfrac{\tilde{\rho}}{\rho_0}$ $\Longrightarrow$

$$\boxed{\partial_t^2 e + 2\underbrace{H}_{=\frac{4}{3t}}\dot{e} - \frac{c_s^2}{a^2}\delta^{ij}\partial_i\partial_j e - 4\pi G \rho_0 e = 0}$$

$$\boxed{\partial_t^2 e + \frac{4}{3t}\dot{e} - \frac{c_s^2}{a^2}\delta^{ij}\partial_i\partial_j e - \frac{2}{3t^2}e = 0}$$

$$\Big\Downarrow \text{ Fourier transform}$$

$$e_k'' + \frac{4}{3t}e_k' + \left(\frac{c_s^2k^2}{a^2} - \frac{2}{3t^2}\right)e_k = 0.$$

$$\Big\Downarrow \ c_s \text{ small (pressure small)}$$

$$e_k'' + \frac{4}{3t}e_k' - \frac{2}{3t^2}e_k = 0$$

$$\Big\Downarrow \text{ Euler ODE}$$

$$e_k = c_1 t^{-1} + c_2 t^{\frac{2}{3}} \Rightarrow |e| \sim t^{\frac{2}{3}}.$$

§3. Our previous works on Jeans instabilities

Our previous works on nonlinear Jeans instability

- (Extremely simplified nonlinear model) Rigorous proof of slightly nonlinear Jeans instability in the expanding Newtonian universe. Physical Review D (PRD), 2022, 105(4): 043519.
- (Mathematical model) Blowups for a class of second order nonlinear hyperbolic equations: A reduced model of nonlinear Jeans instability. arXiv:2208.06788.
- (New exact Jeans instable solution to Euler-Poisson) Fully nonlinear gravitational instabilities for expanding Newtonian universes with inhomogeneous pressure and entropy: Beyond the Tolman's solution. Physical Review D (PRD), 2023, 107(12): 123534.
- (Nonlinear Jeans instability for Euler-Poisson with specific source) Fully nonlinear gravitational instabilities for expanding spherical symmetric Newtonian universes with inhomogeneous density and pressure. arXiv:2305.13211.

Article 2: Composite nonlinearities but with synchronizable sources

$$\boxed{\Box\varrho(x^\mu)+\frac{a}{t}\partial_t\varrho(x^\mu)-\frac{b}{t^2}\varrho(x^\mu)(1+\varrho(x^\mu))-\frac{c-k}{1+\varrho(x^\mu)}(\partial_t\varrho(x^\mu))^2=kF(t),}$$

$$\varrho|_{t=t_0}=\mathring{\varrho}(x^i)>0 \quad \text{and} \quad \partial_t\varrho|_{t=t_0}=\mathring{\varrho}_0(x^i)>0,$$

where $\Box:=\partial_t^2-\Delta_g=\partial_t^2-g^{ij}(t)\partial_i\partial_j$,

$$a>1, \quad b>0, \quad 1<c<3/2$$

$$g^{ij}(t):=\frac{m^2(\partial_t f(t))^2}{(1+f(t))^2}\delta^{ij} \quad \text{and} \quad F(t):=\frac{(\partial_t f(t))^2}{1+f(t)},$$

where $m\in\mathbb{R}$ is a given constant and $f(t)$ solves an ODE,

$$\boxed{\partial_t^2 f(t)+\frac{a}{t}\partial_t f(t)-\frac{b}{t^2}f(t)(1+f(t))-\frac{c}{1+f(t)}(\partial_t f(t))^2=0,}$$

$$f(t_0)=\beta>0 \quad \text{and} \quad \partial_t f(t_0)=\beta_0>0.$$

① Synchronizing blowup times;
② Partially kill the effective Riccati term s.t. no shocks!

The solutions of ODEs

$$\partial_t^2 f(t) + \frac{\mathrm{a}}{t}\partial_t f(t) - \frac{\mathrm{b}}{t^2} f(t)(1+f(t)) - \frac{\mathrm{c}}{1+f(t)}(\partial_t f(t))^2 = 0,$$
$$f(t_0) = \beta > 0 \quad \text{and} \quad \partial_t f(t_0) = \beta_0 > 0.$$

Theorem

1. $t_\star \in [0, \infty)$ exists and $t_\star > t_0$;
2. (Blowups) there is a constant $t_m \in [t_\star, \infty]$, such that there is a unique solution $f \in C^2([t_0, t_m))$ to the ODE, and
$$\lim_{t \to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \to t_m} f_0(t) = +\infty.$$
3. (Estimates of growth rates of f) f satisfies upper and lower bound estimates,
$$1 + f(t) > \exp\left(\mathtt{C} t^{\frac{\bar{\mathtt{a}}+\triangle}{2}} + \mathtt{D} t^{-1}\right) \qquad \text{for} \quad t \in (t_0, t_m);$$
$$1 + f(t) < \left(\mathtt{A} t^{\frac{\bar{\mathtt{a}}-\triangle}{2}} + \mathtt{B} t^{\frac{\bar{\mathtt{a}}+\triangle}{2}} + 1\right)^{-1} \qquad \text{for} \quad t \in (t_0, t_\star).$$

$$\partial_t^2 f(t) + \frac{\mathrm{a}}{t}\partial_t f(t) - \frac{\mathrm{b}}{t^2} f(t)(1+f(t)) - \frac{\mathrm{c}}{1+f(t)}(\partial_t f(t))^2 = 0,$$
$$f(t_0) = \beta > 0 \quad \text{and} \quad \partial_t f(t_0) = \beta_0 > 0.$$

Theorem

Furthermore, if the initial data satisfies $\beta_0 > \bar{\mathrm{a}}(1+\beta)/(\bar{\mathrm{c}} t_0)$, then

4. $t_\star$ and $t^\star$ exist and finite, and $t_0 < t_\star < t^\star < \infty$;
5. there is a finite time $t_m \in [t_\star, t^\star)$, such that there is a solution $f \in C^2([t_0, t_m))$ to the ODE, and

$$\lim_{t \to t_m} f(t) = +\infty \quad \text{and} \quad \lim_{t \to t_m} f_0(t) = +\infty.$$

6. (Improved lower bounds, finite time blowups) the solution f has improved lower bound estimates, for $t \in (t_0, t_m)$,

$$(1+\beta)\left(1 - \mathrm{E} t_0^{\bar{\mathrm{a}}} + \mathrm{E} t^{\bar{\mathrm{a}}}\right)^{1/\bar{\mathrm{c}}} < 1 + f(t).$$

The solutions to the PDEs

$$\boxed{\Box\varrho(x^\mu)+\frac{\mathrm{a}}{t}\partial_t\varrho(x^\mu)-\frac{\mathrm{b}}{t^2}\varrho(x^\mu)(1+\varrho(x^\mu))-\frac{\mathrm{c}-\mathrm{k}}{1+\varrho(x^\mu)}(\partial_t\varrho(x^\mu))^2=\mathrm{k}F(t),}$$

$$\varrho|_{t=t_0}=\mathring{\varrho}(x^i)>0 \quad \text{and} \quad \partial_t\varrho|_{t=t_0}=\mathring{\varrho}_0(x^i)>0,$$

- Conclusions: ϱ has self-increasing singularities at $t=t_m$ and has the growth rates $\sim f$.

The ideas

- Methods: Fuchsian formulations.

$$
\begin{aligned}
B^\mu \partial_\mu u =& \frac{1}{t}\mathbf{BP}u + G \quad && \text{in } [-1,0) \times \mathbb{T}^n, \\
u =& u_0 && \text{on } \{-1\} \times \mathbb{T}^n.
\end{aligned}
$$

  Thm by Oliynyk (2016) then by Beyer, Olvera-Santamaría (2020) implies the solution exists globally in $[-1,0) \times \mathbb{T}^n$.

- The compactified time

$$
\begin{aligned}
\tau := -g(t) =& -\exp\Big(-A\int_{t_0}^{t} \frac{f(s)(f(s)+1)}{s^2 f_0(s)} ds\Big) \\
=& -\Big(1 + \mathrm{b}B\int_{t_0}^{t} s^{\mathrm{a}-2} f(s)(1+f(s))^{1-\mathrm{c}} ds\Big)^{-\frac{A}{\mathrm{b}}} \in [-1,0).
\end{aligned}
$$

Summary of tools

- Fuchsian GIVP;
- The reference ODE of f;
- Hidden quantities and identities of f (distinguishing Singular and Regular-τ terms);
- Time compactifications.

Article 3&4: Nonlinear gravitational instabilities

The dimensionless and normalized Euler–Poisson system

$$\partial_t \rho + \partial_i(\rho v^i) = 0,$$
$$\partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi = 0,$$
$$\partial_t s + v^i \partial_i s = 0,$$
$$\Delta \phi = \delta^{ij} \partial_i \partial_j \phi = 4\pi \rho.$$

The *equation of state* becomes

$$p = K e^s \rho^{\frac{4}{3}} + \mathfrak{p}, \quad \text{for } K \geq 0.$$

There is an exact solution on $(t, x^k) \in [t_0, \infty) \times \mathbb{R}^3$,

$$\mathring{\rho}(t) = \frac{\iota^3}{6\pi t^2}, \quad \mathring{p}(t) = K t^{-\frac{4}{3}} \delta_{kl} x^k x^l \mathring{\rho}^{\frac{4}{3}} + \mathfrak{p}, \quad \mathring{v}^i(t, x^k) = \frac{2}{3t} x^i,$$
$$\mathring{\phi}(t, x^k) = \frac{2}{3} \pi \mathring{\rho} \delta_{ij} x^i x^j = \frac{\iota^3}{9t^2} \delta_{ij} x^i x^j, \quad \mathring{s}(t, x^k) = \ln(t^{-\frac{4}{3}} \delta_{kl} x^k x^l)^{\mathrm{sgn}(1-\iota^3)},$$

We construct solutions

$$\rho(t) = (1+f(t))\mathring{\rho}(t) = \frac{\iota^3(1+f(t))}{6\pi t^2},$$
$$v^i(t,x^i) = \frac{2}{3t}x^i - \frac{f'(t)}{3(1+f(t))}x^i,$$
$$\phi(t,x^i) = \frac{2}{3}\pi\mathring{\rho}(1+f(t))|\boldsymbol{x}|^2 = \frac{\iota^3(1+f(t))|\boldsymbol{x}|^2}{9t^2},$$
$$s(t,x^k) = \ln\left(t^{-\frac{4}{3}}(1+f)^{\frac{2}{3}}\delta_{kl}x^k x^l\right)^{\mathrm{sgn}(1-\iota^3)}.$$

and the *density contrast* $\varrho(t) = f(t)$ where $|\boldsymbol{x}|^2 := \delta_{ij}x^i x^j$ and $f(t)$ is a solution of the following nonlinear ODE,

$$f''(t) + \frac{4}{3t}f'(t) - \frac{2}{3t^2}f(t)(1+f(t)) - \frac{4(f'(t))^2}{3(1+f(t))} = 0,$$
$$f|_{t=t_0} = \beta \quad \text{and} \quad f'|_{t=t_0} = 3(1+\beta)\gamma.$$

Moreover, the pressure becomes $p(t) = \frac{K\iota^4}{(6\pi)^{\frac{4}{3}}t^4}(1+f)^2\delta_{kl}x^k x^l$.

- Result: Self-increasing singularities.

Article 3&4: Nonlinear gravitational instabilities

The dimensionless and normalized Euler–Poisson system

$$\begin{aligned}
\partial_t \rho + \partial_i(\rho v^i) =& 0, \\
\partial_t v^i + v^j \partial_j v^i + \frac{\partial^i p}{\rho} + \partial^i \phi =& \mathcal{D}^i(t, x^j, \rho, v^k, s, \phi), \\
\partial_t s + v^i \partial_i s =& \mathcal{S}(t, x^j, \rho, v^k, s, \phi), \\
\Delta \phi = \delta^{ij} \partial_i \partial_j \phi =& 4\pi \rho.
\end{aligned}$$

EoS is

$$p = K e^s \rho^{\frac{4}{3}} \quad \text{for} \quad K > 0.$$

- $\mathcal{S}$ and $\mathcal{D}$ provide the synchronizable source like F.
- Transform to a type of Article 2;
- Self-increasing singularities.

Eventually, we arrive at

$$\Box_{\mathrm{g}}\hat{\varrho} + \Big(\frac{4}{3t} + \frac{\kappa f_0}{1+f}\Big)\partial_t\hat{\varrho} - \frac{2}{3t^2}\hat{\varrho}(1+\hat{\varrho}) - \frac{4(\partial_t\hat{\varrho})^2}{3(1+\hat{\varrho})} = F_1, \quad \text{Article 2's eq.}$$

$$\partial_t\nu + \frac{f_0}{3(1+f)}\nu\partial_\zeta\nu = G_1, \quad \text{Transport eq.},$$

where the wave operator is

$$\Box_{\mathrm{g}} := \partial_t^2 - \mathrm{g}^{\zeta\zeta}\partial_\zeta^2 + 2\mathrm{g}^{0\zeta}\partial_\zeta\partial_t,$$

$$\mathrm{g}^{\zeta\zeta} := \frac{(2+\omega)(1-\iota^3)}{9t^2}\frac{(1+\hat{\varrho})^{\omega+1}}{(1+f)^\omega} - \frac{f_0^2}{9(1+f)^2}\nu^2, \quad \mathrm{g}^{0\zeta} := \frac{f_0}{3(1+f)}\nu.$$

§4. Emergence of nonlinear Jean-type instabilities for QNLW

After time transform $t \to \ln t$, the equation becomes:

$$\partial_t^2 \varrho - \mathrm{g}^{ij}\partial_i\partial_j\varrho = \frac{2}{3t^2}\varrho(1+\varrho) - \frac{4}{3t}\partial_t\varrho + \frac{4}{3}\frac{(\partial_t\varrho)^2}{1+\varrho} + \mathrm{g}q^i\partial_i\varrho$$
$$- \frac{1}{t^2}\mathrm{K}^{ij}(t,\varrho,\partial_\mu\varrho)\partial_i\varrho\partial_j\varrho, \quad \text{in } [t_0, t^\star)\times\mathbb{R}^n,$$
$$\varrho|_{t=t_0} = \beta + \psi(x^k) \quad \text{and} \quad \partial_t\varrho|_{t=t_0} = \beta_0 + \psi_0(x^k), \quad \text{in } \{t_0\}\times\mathbb{R}^n,$$

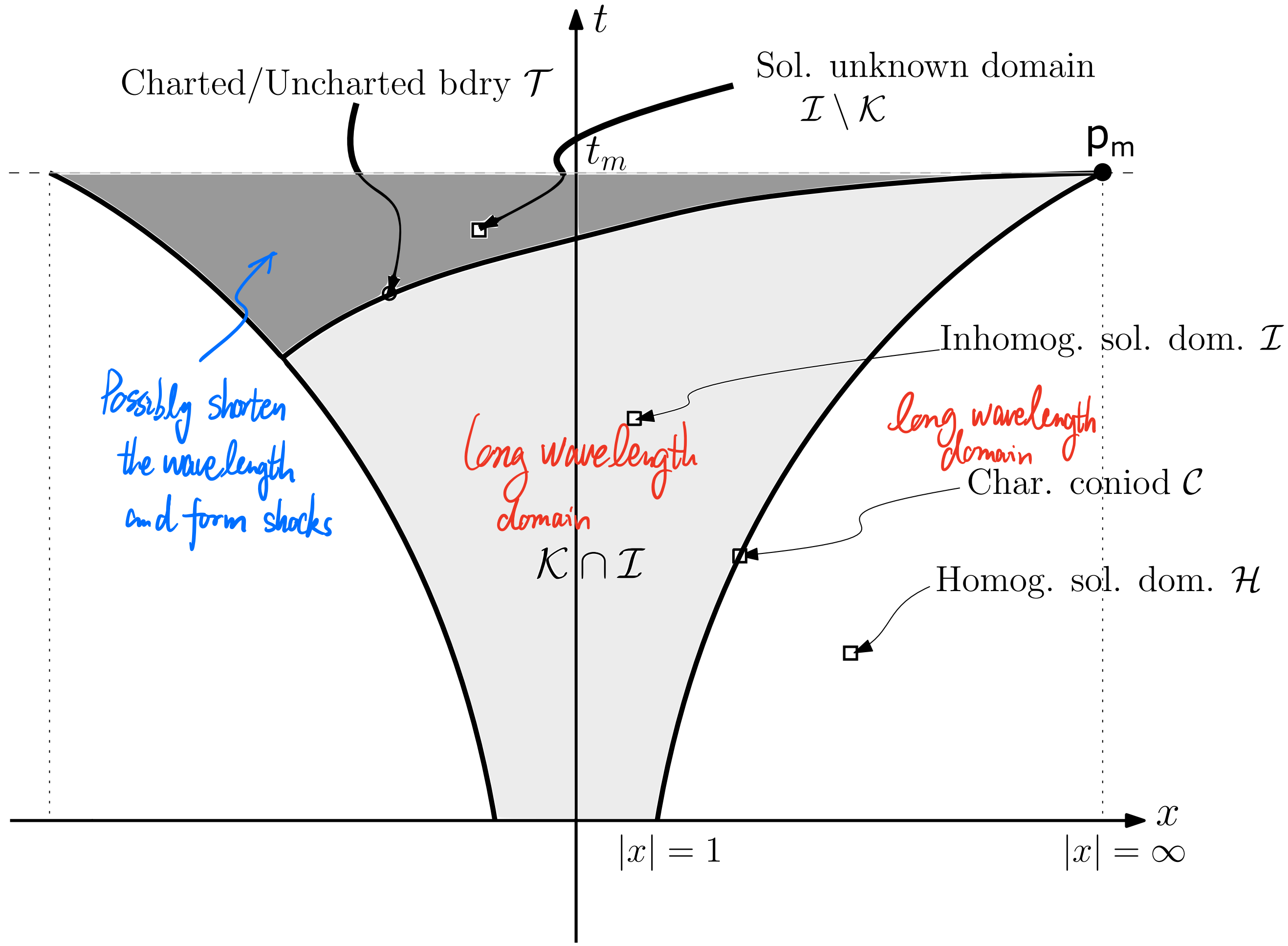
where

$$\mathrm{g}^{ij} = \mathrm{g}^{ij}(t,\varrho,\partial_t\varrho) := \mathrm{g}(t,\varrho,\partial_t\varrho)\delta^{ij} = \left(\mathrm{m}^2\frac{(\partial_t\varrho)^2}{(1+\varrho)^2} + 4(\mathrm{k}-\mathrm{m}^2)\frac{1+\varrho}{t^2}\right)\delta^{ij}.$$

Remark

$$\text{Self-increasing/Riccati terms (dominant)} \sim \frac{4}{3}\frac{(\partial_{\mathfrak{t}} f)^2}{1+f} \sim \frac{2}{3\mathfrak{t}^2}f(1+f);$$
$$\text{but damping terms} \sim -\frac{4}{3\mathfrak{t}}\partial_{\mathfrak{t}} f \sim \frac{1}{\mathfrak{t}^2}f^{\frac{1}{2}}(1+f).$$

Main Theorem shown by one picture



A few words about the Proof

- Want to transform the wave eq. to a Fuchsian formulation by previous techniques, but fails;
- No synchronizing terms, compactified time fails;
- Construct a companion system which can be transformed to a Fuchsian formulation. It shares the same solution with the wave eq. in the lightgrey domain.

Thank you for your attention!

§5. Ideas of the proofs (Extra materials below)

Ideas and Fuchsian direction

- Basic direction: the Fuchsian method (need time compactifications and Fuchsian variables)!
- Require time compactifications $[t_0, t_m) \to [-1, 0)$. How?
  Idea: Intro. a compactified time like "Try 2"?
  Difficulty: Fail! Since there is no synchronized term (synchronizing source term synchronize the blowup time to 0), and it is high possible that the solution blows up at different time (if it blows up!). The compactified time works only if the blow up time can be synchronized and the perturbations do not change the blowup times (if blwoup at infinity, it may still work)
- Define the compacitified time $\tau = g(t, x)$ for t by solving the equation

$$
\begin{aligned}
\partial_t g(t, x^i) = & \frac{AB\varrho(t, x^i)\left(-g(t, x^i)\right)^{\frac{2}{3A}+1}}{t^{\frac{2}{3}}(\varrho(t, x^i)+1)^{\frac{1}{3}}}, \\
g(t_0, x^i) = & -1.
\end{aligned}
$$

Compactified time $[t_0, t_m) \to [-1, 0)$

- Overcome diff.: Intro. two compactified time:
  (1) for the reference solution (sol. to ref. ODE), use "try 2" compactified time;

$$\underbrace{\tau = \mathfrak{g}(\mathfrak{t})}_{\text{Increasing}} = -\left(1 + \frac{2}{3}B\int_{t_0}^{\mathfrak{t}} s^{-\frac{2}{3}} f(s)(1+f(s))^{-\frac{1}{3}} ds\right)^{-\frac{3A}{2}} \in [-1, 0).$$

  It synchronizes the blowup time of the reference solution. However, the perturbations may not blowup at this time, blowup time may deviate it.

- In order to be comparable (this may not hold!), we intro the compactified time analogue to this

$$\tau = g(t, x^i) = -\left(1 + \frac{2}{3}B\int_{t_0}^{t} s^{-\frac{2}{3}} \varrho(s, x^i)(1+\varrho(s, x^i))^{-\frac{1}{3}} ds\right)^{-\frac{3A}{2}} \in [-1, 0).$$

- Wrong compactified time leads wrong structures and fails. It is crucial how to choose it. Need guess and experiments!

Remark: Comparisons of ϱ and f

Wrong: $\varrho(t, x) - f(t)$ (when there is a synchronizing term as Try 2);

Correct: $\underline{\varrho}(\tau, \zeta^k) - \underline{f}(\tau)$ (align the variables by compactified time τ).

ODE equivalence of the compactification

The compacitified time can be reexpressed in terms of two ODEs:

$$\partial_t g(t,x^i) = \frac{AB\varrho(t,x^i)\left(-g(t,x^i)\right)^{\frac{2}{3A}+1}}{t^{\frac{2}{3}}(\varrho(t,x^i)+1)^{\frac{1}{3}}},$$
$$g(t_0,x^i) = -1.$$

and

$$\partial_{\mathfrak{t}}\mathfrak{g}(\mathfrak{t}) = -A\mathfrak{g}(\mathfrak{t})\frac{f(\mathfrak{t})(f(\mathfrak{t})+1)}{\mathfrak{t}^2 f_0(\mathfrak{t})} = \frac{ABf(\mathfrak{t})(-\mathfrak{g}(\mathfrak{t}))^{1+\frac{2}{3A}}}{\mathfrak{t}^{\frac{2}{3}}(1+f(\mathfrak{t}))^{\frac{1}{3}}},$$
$$\mathfrak{g}(t_0) = -1.$$

- These ODEs provide the Jacobian and determine how the coordinate transforms develop. We must solve the variant of the main equation concurrently with the coordinate equation.
- They provide some hidden identities.

The first coordinate transform

We express the main equation to a singular hyperbolic system (1st order) in terms of (τ, ζ) given by

$$\tau = g(t, x^i) \quad \text{and} \quad \zeta^i = x^i$$

Its inverse transformation denote

$$t = \mathrm{b}(\tau, \zeta^i) \quad \text{and} \quad x^i = \zeta^i$$

and satisfies a ODE (Why? Since it is Fuchsianable)

$$\begin{aligned}
\partial_\tau \mathrm{b}(\tau, \zeta^i) =& \frac{\mathrm{b}^{\frac{2}{3}}(\tau, \zeta^i)(1 + \underline{\varrho}(\tau, \zeta^i))^{\frac{1}{3}}}{AB\underline{\varrho}(\tau, \zeta^i)\,(-\tau)^{\frac{2}{3A}+1}}, \\
\mathrm{b}(-1, \zeta^i) =& t_0
\end{aligned}$$

- We do not give the coordinate transform directly but give it by an evolution equation (similar to the wave coordinates, perturbed Lagrangian coordinates, etc.)
- b and $\mathbf{b}_\zeta := \partial_\zeta \mathbf{b}$ become unknown variables since they describe the coordinate transform and this transform has been solved from an equation.

Singular symmetric hyperbolic system

- Intro. perturbation variables: e.g. $u(\tau, \zeta^k) = \frac{\underline{\varrho}(\tau,\zeta^k) - \underline{f}(\tau)}{\underline{f}(\tau)}$
- Using a lot of hidden relations derived from the reference ODE and the quantities χ and ξ in "try 2" we can have a singular symmetric hyperbolic equation (similar to "try 2").
- Comparing with "Try 2", this method has already lead to the Fuchsian system and it is done! However, now it can not be achieved.

Lemma

$$\mathbf{A}^0\partial_\tau U + \frac{1}{A\tau}\mathbf{A}^i\partial_{\zeta^i}U = \frac{1}{A\tau}\mathbf{A}U + \mathbf{F},$$

where $U := \big(u_0, u_j, u, \mathcal{B}_l, z\big)^T$, $\mathbf{F} = \big(\mathfrak{F}_{u_0}, \mathfrak{F}_{u_j}, \mathfrak{F}_u, \mathfrak{F}_{\mathcal{B}_i}, \mathfrak{F}_z\big)^T$,

$$\mathbf{A}^0 = \begin{pmatrix} 1 & \mathscr{R}^j & 0 & 0 & 0 \\ \mathscr{R}_k & (S+\mathscr{L})\delta^j_k & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & \delta^l_s & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{A}^i = \begin{pmatrix} 0 & H^{ij} & 0 & 0 & 0 \\ H^i_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^i \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} -\frac{14}{3} + \mathscr{Z}_{11} & -4\mathrm{k}q^j + \mathscr{Z}^j_{12} & 8 + \mathscr{Z}_{13} & 0 & -8 + \mathscr{Z}_{15} \\ 0 & (4\mathrm{k} + \mathscr{Z}_{22})\delta^j_k & 0 & (24\mathrm{k} + \mathscr{Z}_{24})\delta^l_k & 0 \\ -8 + \mathscr{Z}_{31} & 0 & \frac{40}{3} + \mathscr{Z}_{33} & 0 & -16 + \mathscr{Z}_{35} \\ 0 & (\frac{2}{3} + \mathscr{Z}_{42})\delta^j_s & 0 & (\frac{2}{3} + \mathscr{Z}_{44})\delta^l_s & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Difficulty: **A** can not be positive definite whatever you do!

Ideas from algebraic observations

- $\tilde{\zeta}^1 = \frac{\alpha}{A}\ln(-\tau) + \zeta^1$ (where $\alpha > 0$) and $\tilde{\tau} = \tau$, we obtain the following transformations: $\partial_\tau = \partial_{\tilde{\tau}} + \frac{\alpha}{A\tilde{\tau}}\partial_{\tilde{\zeta}^1}$ and $\partial_{\zeta^1} = \partial_{\tilde{\zeta}^1}$.

$$\mathbf{A}^0\partial_{\tilde{\tau}}U + \frac{1}{A\tilde{\tau}}\underbrace{(\alpha\mathbf{A}^0 + \mathbf{A}^1)}_{=\begin{pmatrix}\alpha & 1+\alpha U\\ 1+\alpha U & \frac{1}{4}\alpha\end{pmatrix}}\partial_{\tilde{\zeta}^1}U = \frac{1}{A\tilde{\tau}}\mathbf{A}U + \mathbf{F}(U).$$

- A *variable transformation* $\mathfrak{U} = e^{\theta\tilde{\zeta}^1}U$ (where $\theta > 0$), the derivative transforms as follows: $\frac{1}{A\tilde{\tau}}\partial_{\tilde{\zeta}^1}U = e^{-\theta\tilde{\zeta}^1}\frac{1}{A\tilde{\tau}}\partial_{\tilde{\zeta}^1}\mathfrak{U} - \theta e^{-\theta\tilde{\zeta}^1}\frac{1}{A\tilde{\tau}}\mathfrak{U}$.

$$\underbrace{\begin{pmatrix}1 & e^{-\theta\tilde{\zeta}^1}\mathfrak{U}\\ e^{-\theta\tilde{\zeta}^1}\mathfrak{U} & \frac{1}{4}\end{pmatrix}}_{=:\mathbf{B}^0}\partial_{\tilde{\tau}}\mathfrak{U} + \frac{1}{A\tilde{\tau}}\underbrace{\begin{pmatrix}\alpha & 1+\alpha e^{-\theta\tilde{\zeta}^1}\mathfrak{U}\\ 1+\alpha e^{-\theta\tilde{\zeta}^1}\mathfrak{U} & \frac{1}{4}\alpha\end{pmatrix}}_{=:\mathbf{B}^1}\partial_{\tilde{\zeta}^1}\mathfrak{U}$$

$$=\frac{1}{A\tilde{\tau}}\underbrace{\begin{pmatrix}\alpha\theta - \frac{14}{3} & \theta + \alpha\theta e^{-\theta\tilde{\zeta}^1}\mathfrak{U} - q\\ \theta + \alpha\theta e^{-\theta\tilde{\zeta}^1}\mathfrak{U} & 1 + \frac{1}{4}\alpha\theta\end{pmatrix}}_{=:\mathbf{B}}\mathfrak{U} + e^{\theta\tilde{\zeta}^1}\mathbf{F}(e^{-\theta\tilde{\zeta}^1}\mathfrak{U}).$$

Remark

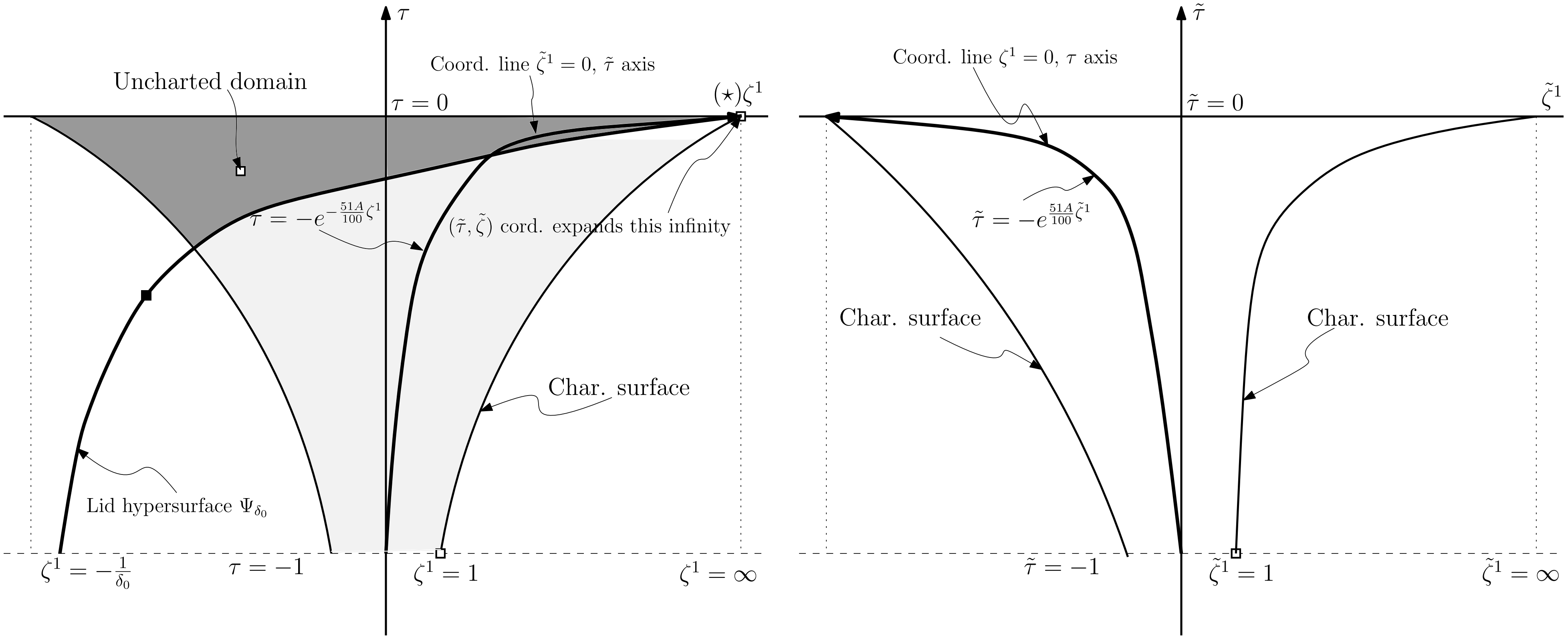
An intuitive idea to overcoming this difficulty is to solve a global existence problem for a revised hyperbolic system and ensure that this revised system is consistent with the main eq. (or its variant) within a sufficiently large lens-shaped domain. Outside the lens-shaped domain, we have considerable flexibility to modify the hyperbolic equation to align with the Fuchsian formulations, allowing us to use the Fuchsian GIVP for the revised system to achieve a global solution.

Two revisions (geometric meanings)

(1) The second coordinate transform: the tilted coordinate

$$\tilde{\tau} = \tilde{\tau}(\tau, \zeta^k) = \tau \quad \text{and} \quad \tilde{\zeta}^i = \tilde{\zeta}^i(\tau, \zeta^k) = \frac{\mathrm{ac}^i}{A} \ln(-\tau) + \zeta^i,$$

- Motivations: Expand the "null infinity" (not precisely) and upright a timelike direction (close to null) to be the time axis. since our analysis can only work in this "closed to null" domain.
- From the equation point of view, (1) generate more terms in $\frac{1}{A\tau}\mathbf{A}^i$ and will help compensate $\frac{1}{A\tau}\mathbf{A}$ to achieve the positive definiteness.
- From the geometric point of view, they tilt the characteristic conoid and expand the "near-null" domain.


τ
Coord. line $\tilde{\zeta}^1 = 0$, $\tilde{\tau}$ axis
Uncharted domain
$\tau = 0$
$(\star)\zeta^1$
$\tau = -e^{-\frac{51A}{100}\zeta^1}$
$(\tilde{\tau}, \tilde{\zeta})$ cord. expands this infinity
Char. surface
Lid hypersurface Ψ_{δ_0}
$\zeta^1 = -\frac{1}{\delta_0}$
$\tau = -1$
$\zeta^1 = 1$
$\zeta^1 = \infty$
$\tilde{\tau}$
Coord. line $\zeta^1 = 0$, τ axis
$\tilde{\tau} = 0$
$\tilde{\zeta}^1$
$\tilde{\tau} = -e^{\frac{51A}{100}\tilde{\zeta}^1}$
Char. surface
Char. surface
$\tilde{\tau} = -1$
$\tilde{\zeta}^1 = 1$
$\tilde{\zeta}^1 = \infty$

(2) rescale all the variables by spatial factors, e.g., $\mu(\tilde{\zeta}^1) := \sigma_0 e^{-\frac{153}{\delta_0}} e^{-51\tilde{\zeta}^1}$ and the variable, e.g., becomes

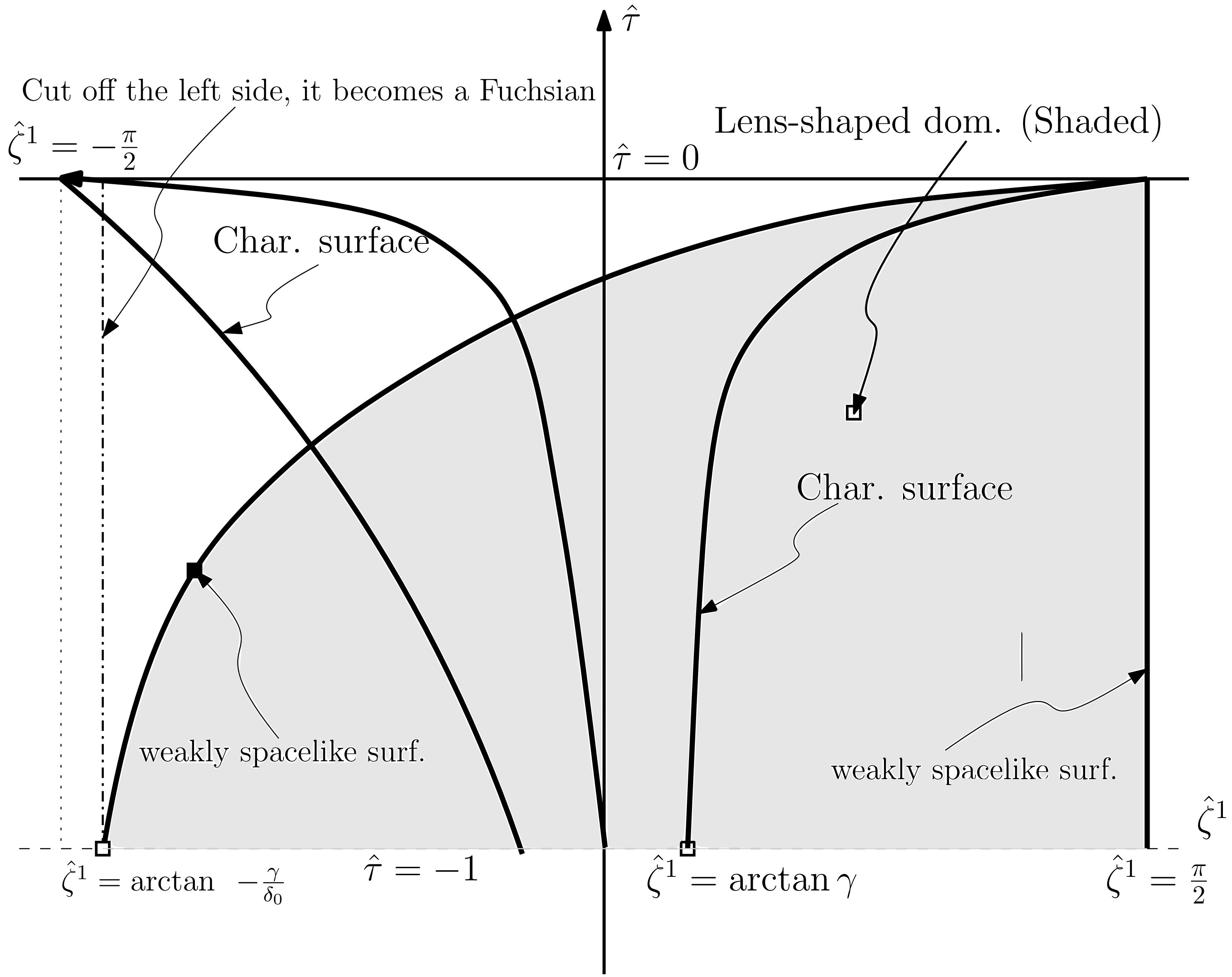
$$u_0(\tilde{\tau}, \tilde{\zeta}) = \frac{1}{\sigma_0 e^{-\frac{153}{\delta_0}} e^{-51\tilde{\zeta}^1}} \widetilde{u_0}(\tilde{\tau}, \tilde{\zeta})$$

- Motivation: Spatial factors like μ will separate a new singular remainder term $\frac{1}{A\tau}\mathbf{A}_{\text{remainder}} U$ from $\frac{1}{A\tau}\mathbf{A}^i \partial_i U$, and $\frac{1}{A\tau}\mathbf{A}_{\text{remainder}} U$ compensate $\frac{1}{A\tau}\mathbf{A} U$ to obtain a positive definite singular lower order term (consists with the Fuchsian).
- Along the boundary of the char. cone, $\tilde{\tau} \sim e^{-\tilde{\zeta}^1}$ gives decay factors.
- Defect: $\mu \sim e^{-51\tilde{\zeta}^1}$ introduce infinities to the equation as $\tilde{\zeta}^1 \to -\infty$. Break the structures.
- Idea to overcome: Revise the equation by cutoff function ϕ such that the infinities vanish. However, the equation fails to equivalent to the original equation due to the revision.

$$\phi \in C^\infty\big(\mathbb{R}; [0, 1]\big), \quad \phi|_{[-\delta_0^{-1}, +\infty)} = 1 \quad \text{and} \quad \operatorname{supp}\phi \subset [-2\delta_0^{-1}, +\infty) \subset \mathbb{R}.$$

How to recover the solution of the original one?

- To recover original solution, only use the lens-shaped domain (determination domain, see Fig. to explain)

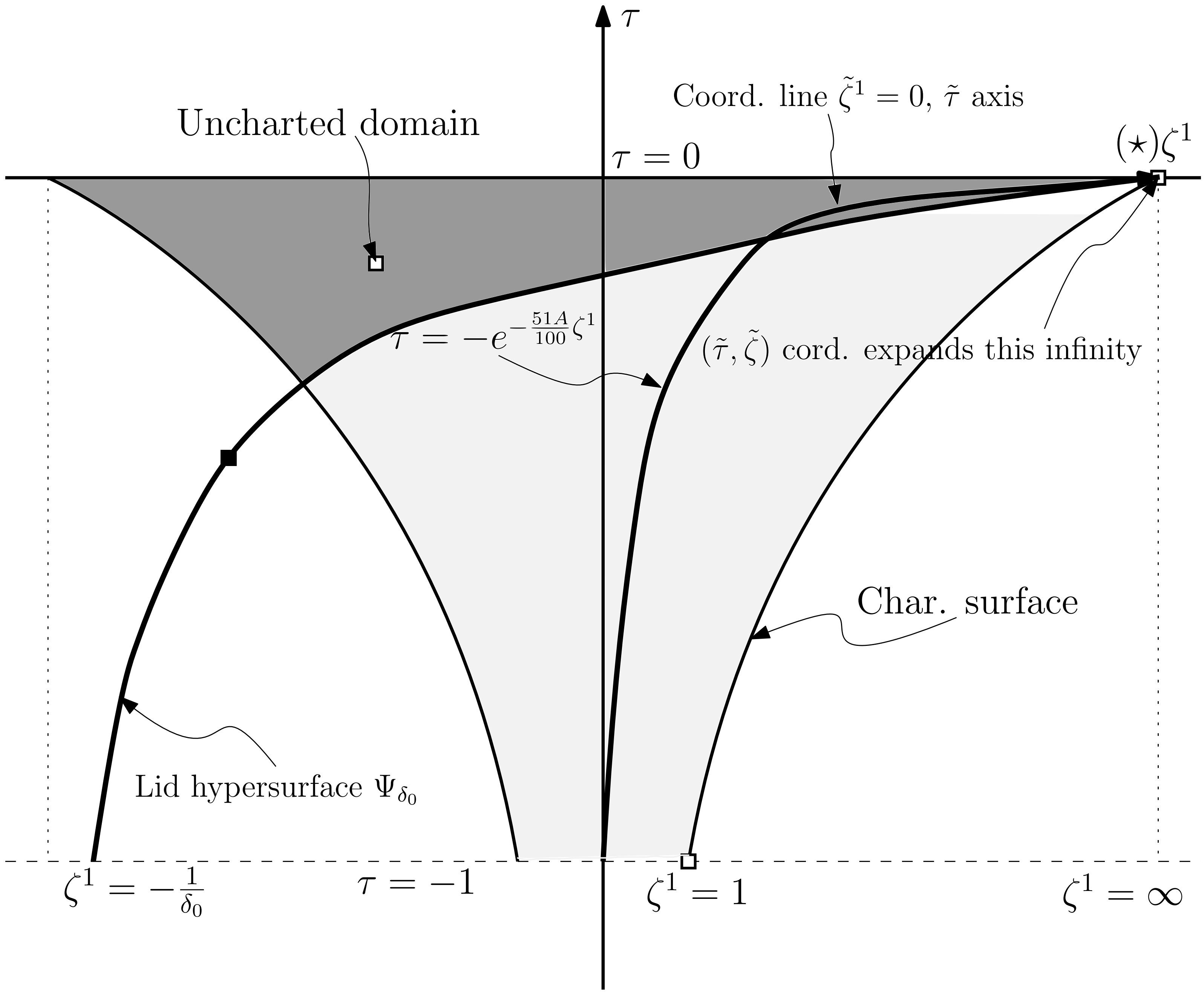


The revised system becomes a Fuchsian system by compactifying space

The third coordinate transform (compactifying the space)

$$\hat{\tau} = \tilde{\tau} \in [-1, 0) \quad \text{and} \quad \hat{\zeta}^i = \arctan(\gamma \tilde{\zeta}^i) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

- $\mathbb{R}^n$ becomes $\mathbb{T}^n$, a closed manifold which is required by the Fuchsian analysis.
- After this coordinate transform, we have Fuchsian formulation and can derive the global existence and stability result for this revised system.
- Using determination domain obtain the main theorem.


τ
Coord. line $\tilde{\zeta}^1 = 0$, $\tilde{\tau}$ axis
Uncharted domain
$(\star)\zeta^1$
$\tau = 0$
$\tau = -e^{-\frac{51A}{100}\zeta^1}$
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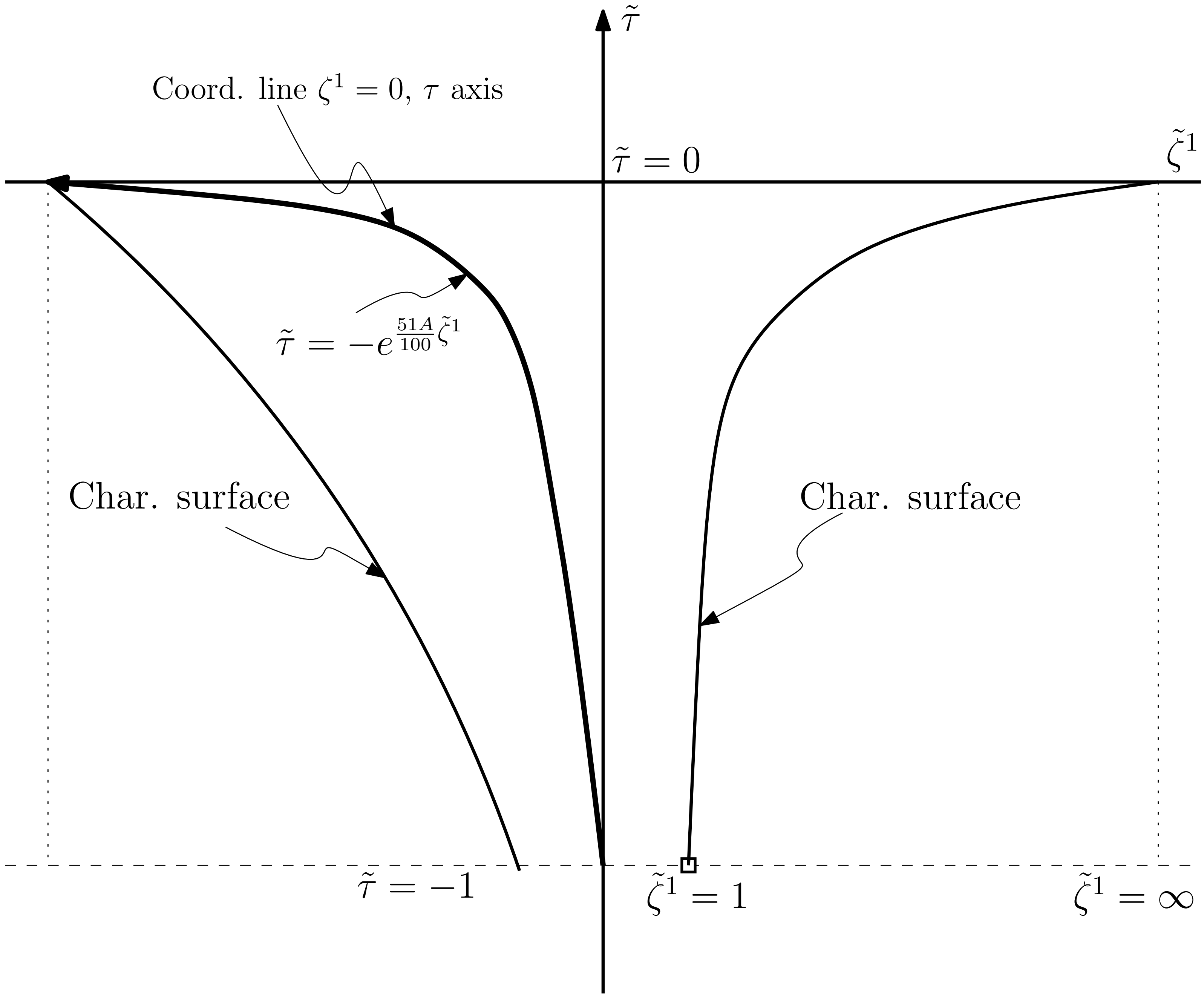

$\tilde{\tau}$
Coord. line $\zeta^1 = 0$, τ axis
$\tilde{\tau} = 0$
$\tilde{\zeta}^1$
$\tilde{\tau} = -e^{\frac{51A}{100}\tilde{\zeta}^1}$
Char. surface
Char. surface
$\tilde{\tau} = -1$
$\tilde{\zeta}^1 = 1$
$\tilde{\zeta}^1 = \infty$

Thank you
for your attention!