A tour around the Gan–Gross–Prasad conjecture

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$$\begin{aligned} L(s, A) &:= \prod_{p \mid N^+} \frac{1}{1+p^{-s}} \prod_{p \mid N^-} \frac{1}{1-p^{-s}} \prod_{p \nmid N} \frac{1}{1-a_p(A)p^{-s}+p^{1-2s}} \\ &= \prod_{p} \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}. \end{aligned}$$

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$$= \prod_{p} \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}.$$

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$$L(s,\mathrm{Sym}^k A) \coloneqq \prod_p \frac{1}{(1-\alpha_p^k p^{-s})(1-\alpha_p^{k-1}\beta_p p^{-s})\cdots(1-\alpha_p\beta_p^{k-1}p^{-s})(1-\beta_p^k p^{-s})}.$$

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Suppose that the number of prime factors of N^- is *odd*. Let B_{N^-} be the *definite* quaternion algebra ramified exactly at primes dividing N^-. Let $\mathcal{S}_{N^-,N^+}$ be the (finite) set of cyclic isogenies of oriented maximal orders of B_{N^-} of order N^+, up to conjugation. Then there is a unique nonzero $\mathbb{Q}$-valued function f_A on $\mathcal{S}_{N^-,N^+}$, unique up to a scalar, such that $\mathrm{T}_p f_A = a_p(A) f_A$ for every $p \nmid N$.

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If

$$\mathcal{P}(f_A) := \sum_{s \in \mathcal{S}_{N^-,N^+}} \frac{f_A(s)^3}{\#\mathrm{Stab}(s)} \neq 0,$$

then both $L(1, A)$ *and* $L(2, \mathrm{Sym}^3 A)$ *are nonvanishing.*

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Example

Consider A: $y^2 + y = x^3 - x^2 - 10x - 20$, which has conductor 11 with split multiplicative reduction at 11 (so that $N^+ = 1$ and $N^- = 11$). The set $\mathcal{S}_{11,1}$ has two elements s, t with $\#\mathrm{Stab}(s) = 2$ and $\#\mathrm{Stab}(t) = 3$. Then we may take

$$f_A(s) = 2, \quad f_A(t) = -3.$$

It follows that

$$\mathcal{P}(f_A) = \frac{2^3}{2} + \frac{(-3)^3}{3} = -5 \neq 0,$$

which by the theorem implies that both $L(1, A)$ and $L(2, \mathrm{Sym}^3 A)$ are nonvanishing.

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- F: number field,
- c: automorphism of F of order either 1 or 2,
- $F^+ \coloneqq F^{c=1}$,
- V: an F-vector space of dimension n equipped with a nondegenerate c-symmetric pairing $(\ ,\ )_V \colon V \times V \to F$ that is F-linear in the first variable (known as c-hermitian space),
- $V^\sharp \coloneqq V \oplus Fe$ with $(e,e)_{V^\sharp} = 1$,
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For every cusp form $f \in \pi$, we define its **Bessel period**:

$$\mathcal{P}(f) := \int_{H(F^+)\backslash H(\mathbb{A}_{F^+})} f(h)\mathrm{d}h$$

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Example

Take $F = \mathbb{Q}$, $c = \mathrm{id}$, and $V = B^{\mathrm{tr}=0}$ for a quaternion algebra B over $\mathbb{Q}$ (so that $n = 3$). Then the pair $H \subseteq G$ is the triple diagonal subgroup $\Delta\mathrm{P}B^\times \subseteq \mathrm{P}B^\times \times \mathrm{P}B^\times \times \mathrm{P}B^\times$.

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If $\mathrm{disc}B \mid N$ and we take $\pi = \pi_A \boxtimes \pi_A \boxtimes \pi_A$, then $L(s,\pi) = L(s+1/2, A)^2 \cdot L(s+3/2, \mathrm{Sym}^3 A)$.

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The following two statements are equivalent:

(1) $L(1/2, \pi) \neq 0$;

(2) there exist another c-hermitian space V' *of dimension* n *(with the corresponding* $H' \subseteq G'$*) and an irreducible cuspidal automorphic representation* π' *of* $G'(\mathbb{A}_{F^+})$ *that is nearly equivalent to* π*, such that* $\mathcal{P}$ *is nontrivial on* π'*.*

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Suppose that π *is everywhere tempered. For a decomposable vector* $f = \otimes_v f_v$ *of* π*, we have*

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The generalized Ramanujan conjecture predicts that a generic irreducible cuspidal representation is automatically everywhere tempered.

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When $c = \mathrm{id}$, Ginzburg–Jiang–Rallis (JAMS 2004) and Jiang–(Lei) Zhang (Annals 2020) proved one direction of the GGP conjecture, namely, (2)⇒(1).

Generalized Bessel periods

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- r: a nonnegative integer,
- $V^\sharp := V \oplus Fe \oplus P^{\oplus r}$ with $(e,e)_{V^\sharp} = 1$ and P the hyperbolic plane,
- $\mathcal{F}$: a chosen complete filtration of a Lagrangian of $P^{\oplus r}$,
- $\mathrm{U}_\mathcal{F}$: the stabilizer of $\mathcal{F}$ in $\mathrm{U}(V^\sharp)$, which contains $\mathrm{U}(V)$ and admits a projection to $\mathrm{U}(V)$,
- $\psi\colon \mathrm{U}_\mathcal{F}(F)\backslash\mathrm{U}_\mathcal{F}(\mathbb{A}_F) \to \mathbb{C}^\times$: a chosen generic character trivial on $\mathrm{U}(V)(\mathbb{A}_F)$ and invariant by the conjugation of $\mathrm{U}(V)(\mathbb{A}_F)$,
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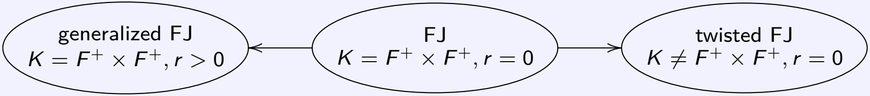

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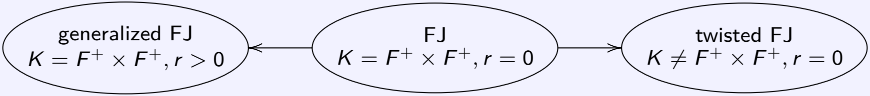

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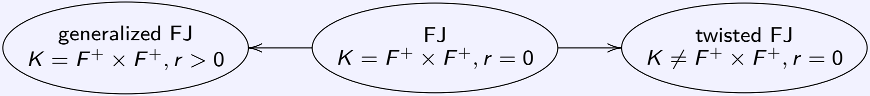

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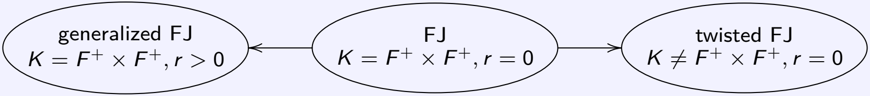

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The first step of the proof is the implication (1)⇒(2) in the GGP conjecture.

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$$\Lambda_{\mathcal{F}} := \mathbb{Z}_p[[\mathrm{Gal}(\mathcal{F}/F)]], \quad \Lambda_{\mathcal{F},\mathbb{Q}_p} := \Lambda_{\mathcal{F}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

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Application: Iwasawa's main conjecture

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An **anticyclotomic extension** of F is an abelian extension $\mathcal{F}/F$ (contained in $\overline{F}$) such that $\mathcal{F}/F^+$ is Galois for which c acts on $\mathrm{Gal}(\mathcal{F}/F)$ by -1. An anticyclotomic extension $\mathcal{F}/F$ is **free** if $\mathrm{Gal}(\mathcal{F}/F)$ is torsion free. For an anticyclotomic extension $\mathcal{F}/F$, we denote by $\Sigma_{\mathcal{F}}$ the set of primes of F ramified in $\mathcal{F}$.
An anticyclotomic extension $\mathcal{F}/F$ is *p*-**ordinary** (with respect to A, A') if for $w \in \Sigma_{\mathcal{F}}$, both A_w and A'_w have good ordinary reduction (for example, the trivial extension F/F). Put

$$\Lambda_{\mathcal{F}} := \mathbb{Z}_p[[\mathrm{Gal}(\mathcal{F}/F)]], \quad \Lambda_{\mathcal{F},\mathbb{Q}_p} := \Lambda_{\mathcal{F}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

known as Iwasawa algebras.

Theorem (L., arXiv 2023)

Let $\mathcal{F}/F$ be a free p-ordinary anticyclotomic extension $\mathcal{F}/F$. There exists an element $\mathscr{L}_{\mathcal{F}}(\mathrm{Sym}^{n-1} A_F \times \mathrm{Sym}^n A'_F) \in \Lambda_{\mathcal{F},\mathbb{Q}_p}$, unique up to a scalar in $\mathbb{Q}_p^\times$, satisfying the following interpolation property: There exists a constant $C \in \mathbb{C}^\times$ such that for every finite order character $\chi\colon \mathrm{Gal}(\mathcal{F}/F) \to \overline{\mathbb{Q}}_p^\times$ that is ramified at all places in $\Sigma_{\mathcal{F}}$ and every embedding $\iota\colon \overline{\mathbb{Q}}_p \to \mathbb{C}$,

$$\begin{aligned}&\iota\mathscr{L}_{\mathcal{F}}(\mathrm{Sym}^{n-1} A_F \times \mathrm{Sym}^n A'_F)(\chi) \\ &= C \cdot \iota \prod_{w \in \Sigma_{\mathcal{F}}} \left(\frac{q_w^{\frac{n(n+1)(2n+1)}{12}}}{(\alpha_w)^{\psi(n-1)}(\alpha'_w)^{\psi(n)}} \right)^{\mathfrak{c}_w(\chi)} \cdot L(n, \mathrm{Sym}^{n-1} A_F \times \mathrm{Sym}^n A'_F \otimes \iota\chi).\end{aligned}$$

Application: Iwasawa's main conjecture (cont.)

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In the theorem,

- $\psi(N) \coloneqq N^2 + (N-2)^2 + \cdots + 1^2/0^2$,
- q_w denotes the residue cardinality of F_w,
- α_w and α'_w denote the unique root in $\mathbb{Z}_p^\times$ of the polynomials $T^2 - a_w(A)T + q_w$ and $T^2 - a_w(A')T + q_w$, respectively,
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We now define Selmer groups over the Iwasawa algebra. For every finite extension $\tilde{F}/F^+$, there are similarly defined Bloch–Kato Selmer groups

$$\mathrm{H}^1_f(\tilde{F}, \mathrm{T}(A, A', n)_p) \subseteq \mathrm{H}^1(\tilde{F}, \mathrm{T}(A, A', n)_p),$$
$$\mathrm{H}^1_f(\tilde{F}, \mathrm{W}(A, A', n)_p) \subseteq \mathrm{H}^1(\tilde{F}, \mathrm{W}(A, A', n)_p),$$

where $\mathrm{W}(A, A', n)_p \coloneqq \mathrm{T}(A, A', n)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$.

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Put

$$\mathscr{S}(\mathcal{F}, \mathrm{V}(A, A', n)_p) \coloneqq \left(\varprojlim_{\mathcal{F}/\tilde{F}/F} \mathrm{H}^1_f(\tilde{F}, \mathrm{T}(A, A', n)_p) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

$$\mathscr{X}(\mathcal{F}, \mathrm{V}(A, A', n)_p) \coloneqq \left(\varprojlim_{\mathcal{F}/\tilde{F}/F} \mathrm{Hom}\left(\mathrm{H}^1_f(\tilde{F}, \mathrm{W}(A, A', n)_p), \mathbb{Q}_p/\mathbb{Z}_p \right) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

both being finitely generated modules over $\Lambda_{\mathcal{F},\mathbb{Q}_p}$.

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$$M \simeq \bigoplus_i \Lambda_{\mathcal{F},\mathbb{Q}_p}/(f_i)$$

for finitely many elements $f_i \in \Lambda_{\mathcal{F},\mathbb{Q}_p}$, unique up to units and permutations. Define the **characteristic ideal** $\operatorname{char}_{\mathcal{F}}(M)$ of M to be the ideal of $\Lambda_{\mathcal{F},\mathbb{Q}_p}$ generated by $\prod_i f_i$.

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The characteristic ideal $\operatorname{char}_{\mathcal{F}}(\mathscr{X}(\mathcal{F}, \mathrm{V}(A, A', n)_p))$ *is generated by* $\mathscr{L}_{\mathcal{F}}(\operatorname{Sym}^{n-1} A_F \times \operatorname{Sym}^n A'_F)$.

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Suppose that $F^+ \neq \mathbb{Q}$ *when* $n \geq 3$. *Suppose that* A *and* A' *are both modular,* $\operatorname{End}(A_{\overline{F}}) = \operatorname{End}(A'_{\overline{F}}) = \mathbb{Z}$, *and* $\operatorname{Hom}(A_{\overline{F}}, A'_{\overline{F}}) = 0$. *The following holds for all but (effectively) finitely many prime numbers* p*:*

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By taking $\mathcal{F} = F$, it recovers the theorem of [LTXZZ].

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- π_v is distinguished (that is, $\mathrm{Hom}_{H(F_v^+)}(\pi_v, \mathbb{C}) \neq 0$) for every place v of F^+,
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Theorem (Nelson, Invent. 2023)

For $\pi \in \mathscr{F}$, the subconvex bound

$$L(1/2, \pi) \ll C(\pi)^{1/4-\delta}$$

holds for each fixed

$$\delta < \frac{1}{8n^5 + 28n^4 + 42n^3 + 36n^2 + 14n}.$$

Here, $C(\pi)$ denotes the global analytic conductor of π.

Application: nonvanishing of central *L*-values

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Suppose that $F^+ = \mathbb{Q}$ and F is imaginary quadratic. For every integer $k \geq 1$, denote by

- $\mathcal{A}_{2,k}$ the set of everywhere unramified cohomological hermitian automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ of weight $(-k+1/2, k-1/2)$,
- $\mathcal{A}_{3,k}$ the set of everywhere unramified cohomological hermitian automorphic representations of $\mathrm{GL}_3(\mathbb{A}_F)$ of weight $(-2k-1, k, k+1)$.

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Theorem (Michel–Ramakrishnan–Yang, arXiv 2023)

Suppose that $k > 16$. For every member $\Pi_2 \in \mathcal{A}_{2,k}$, there exists a member $\Pi_3 \in \mathcal{A}_{3,k}$ such that

$$L(\tfrac{1}{2}, \Pi_2 \times \Pi_3) \neq 0.$$

Thank you!