Causal Structure of Singularities formed in Gravitational Collapse

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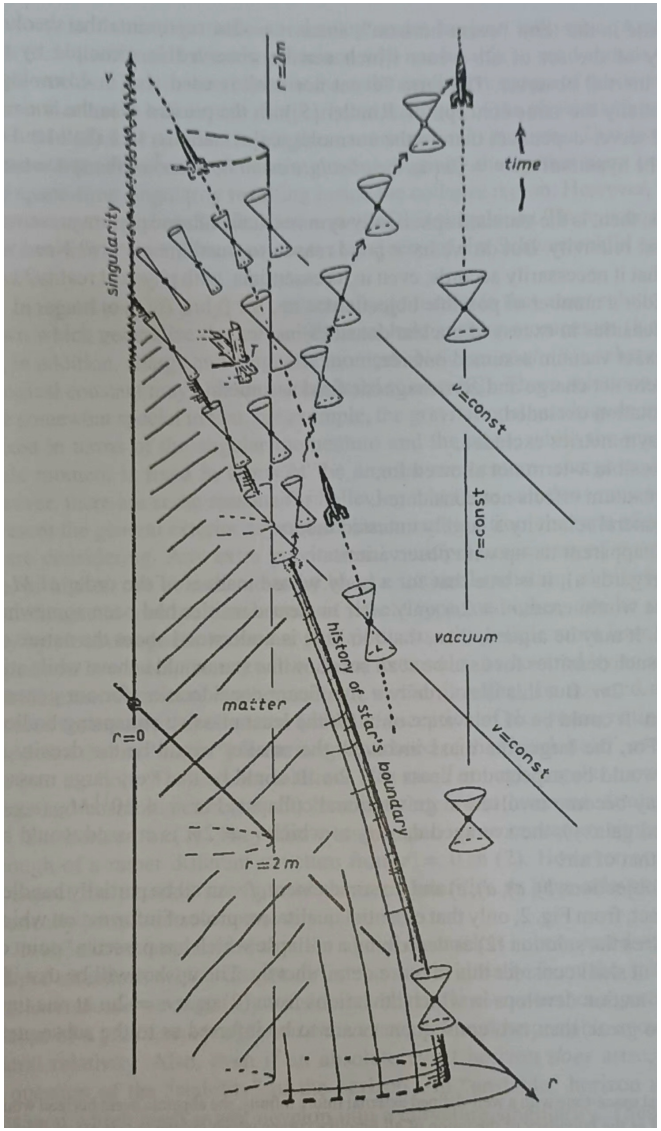


FIGURE: Penrose (1969)

Outline

1. Global causality condition
2. Singularity theorems
3. Cosmic Censorship Conjecture
4. Causal Structure of Apparent horizon
5. Causal structure of AH in a gravitational collapse
6. Causal structure of singularities formed in a spherical gravitational collapse
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GLOBAL CUSALITY CONDITION: AVOIDANCE OF CLOSE TIMELIKE LOOP

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- The global causal conditions are imposed on a spacetime manifold $(\mathcal{M}, g)$ to avoid the occurrence of closed timelike curves (CTC) in that manifold.
- The existence of CTCs can cause chronology violation and which further causes serious causal paradoxes which needs to be avoided to maintain a causally well behaved spacetime manifold.
- If $\mathcal{M}$ is a chronology violating set then $I^+(q) \cap I^-(q) \neq \varnothing \; \forall q \in \mathcal{M}$.

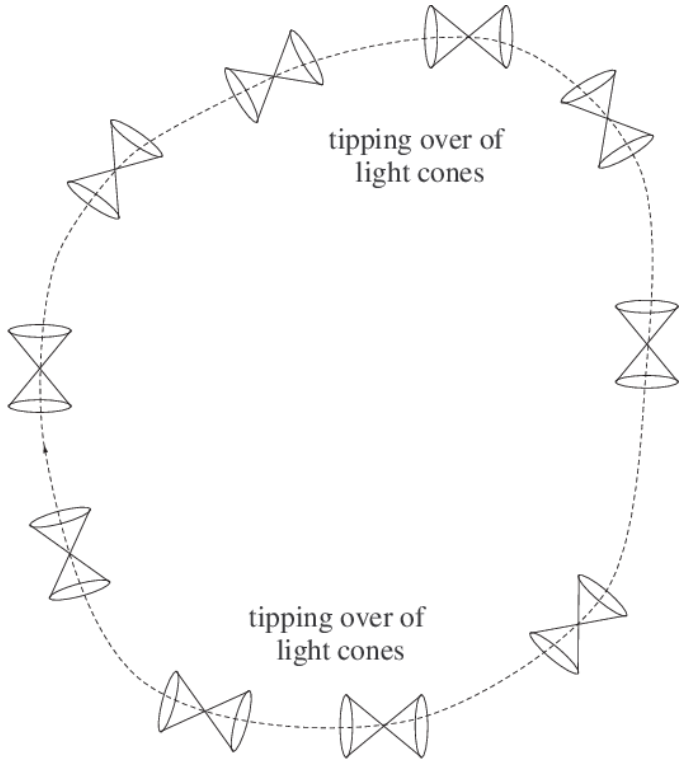


FIGURE: CTC

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- **Strong causality condition:** A spacetime $(\mathcal{M}, g)$ is said to be strongly causal if for all point $p \in \mathcal{M}$ and for every neighborhood $\mathcal{O}$ of p, there exists a neighborhood $\mathcal{V}$ of p such that $\mathcal{V} \subset \mathcal{O}$ and no non-spacelike curves intersect $\mathcal{V}$ more than once.

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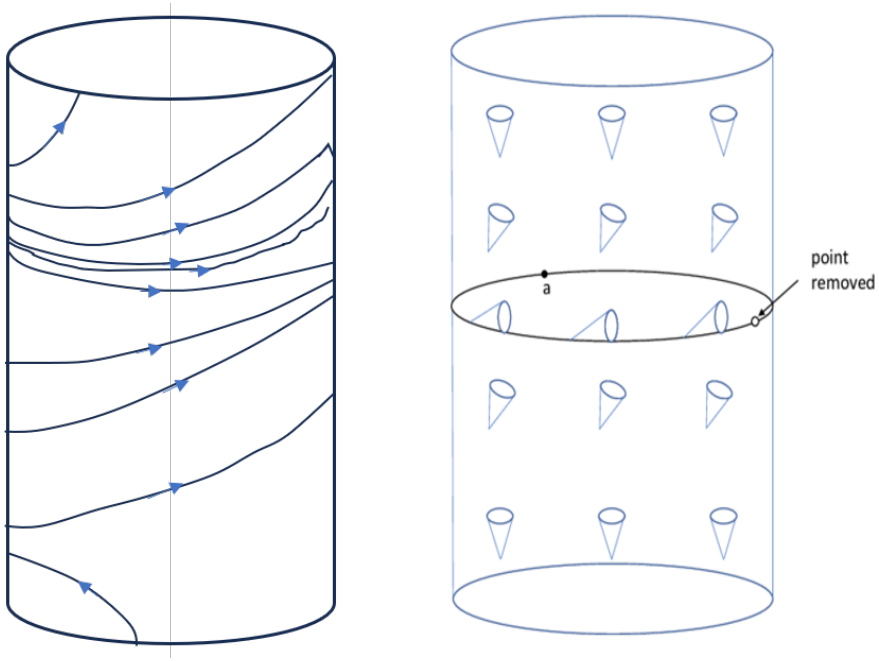


FIGURE: Strong causality violation (e.g., Taub-Nut spacetime)

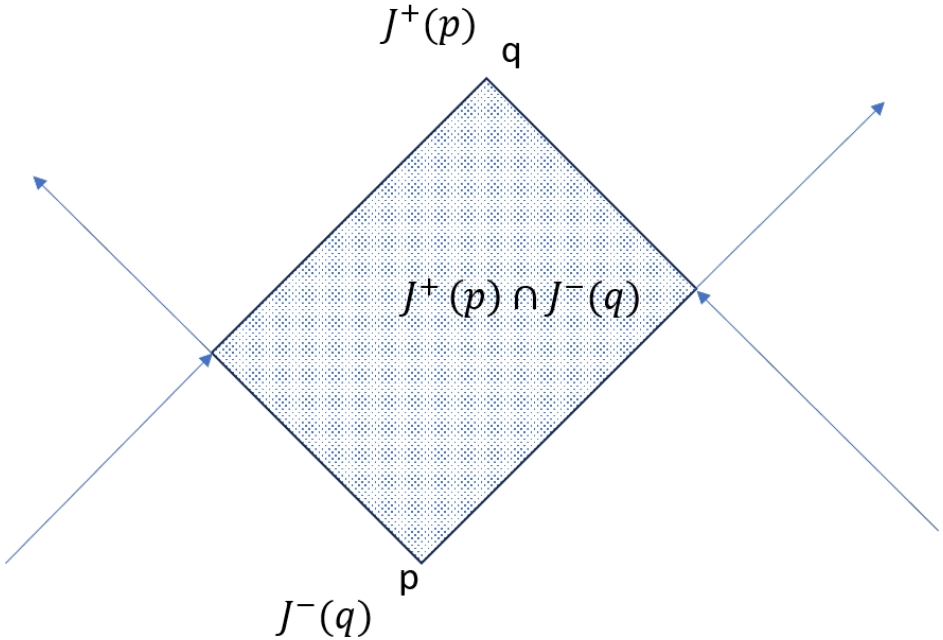
Global Causality condition: Global Hyperbolicity

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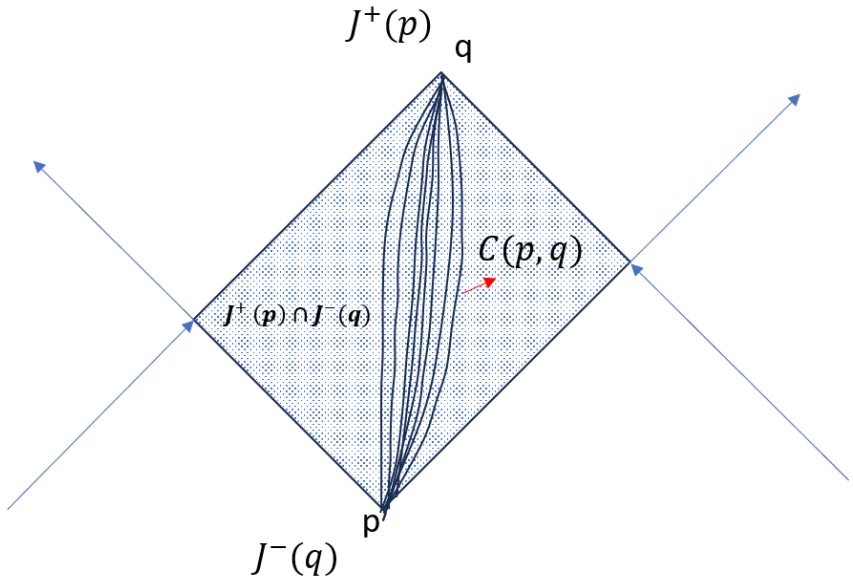
Global hyperbolicity is the strongest condition on the global causal structure of a spacetime manifold or of a Lorentzian manifold.

- A set $\mathcal{L}$ is called globally hyperbolic if:
  1. strong causality condition holds on $\mathcal{L}$.
  2. for any two points $p, q \in \mathcal{L}$, the set $\mathcal{V} = J^+(p) \cap J^-(q)$ is compact and $\mathcal{V} \subset \mathcal{L}$. (Leray (1952))

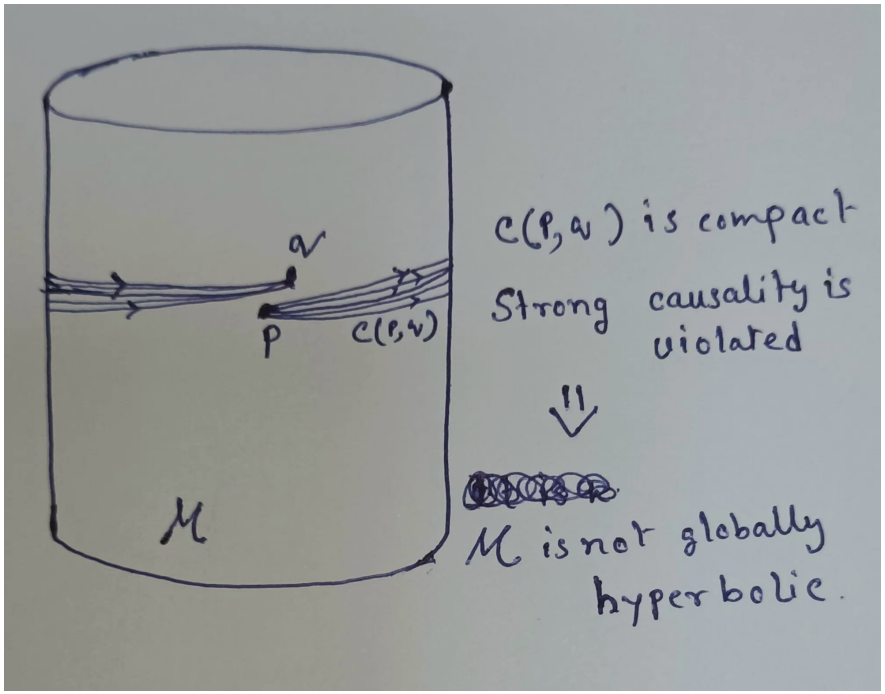


From the above definition, one can also prove that:

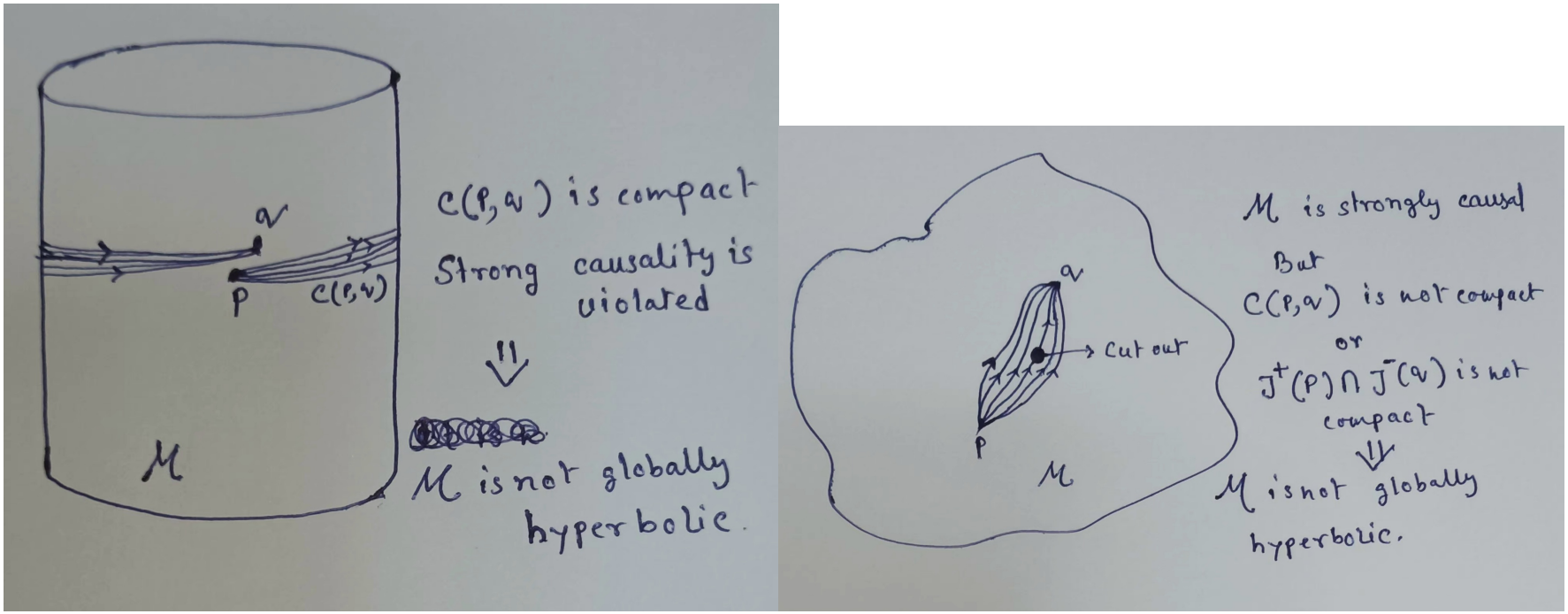
- If in an open set $\mathcal{L}$ is globally hyperbolic if the global causal condition holds throughout the set and the space of causal curves $C(p, q)$ is compact for all $p, q \in \mathcal{L}$. (Seifert (1967), Geroch (1970))



Global Causality condition: Global Hyperbolicity



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- If $\mathcal{L}$ is a closed, achronal, C^{1-} Lipschitz submanifold in $\mathcal{M}$, then the interior $int(D(\mathcal{L})) \equiv D(\mathcal{L}) - \dot{D}(\mathcal{L})$ if non-empty, is globally hyperbolic.

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- If the achronal set $\mathcal{L}$ is a spacelike submanifold, then it is referred to as a Cauchy surface. If such a surface exists within int($D(\mathcal{L})$), then int($D(\mathcal{L})$) must be globally hyperbolic. When a global Cauchy surface exists, we have $D(\mathcal{L}) = \mathcal{M}$. (Geroch (1970))

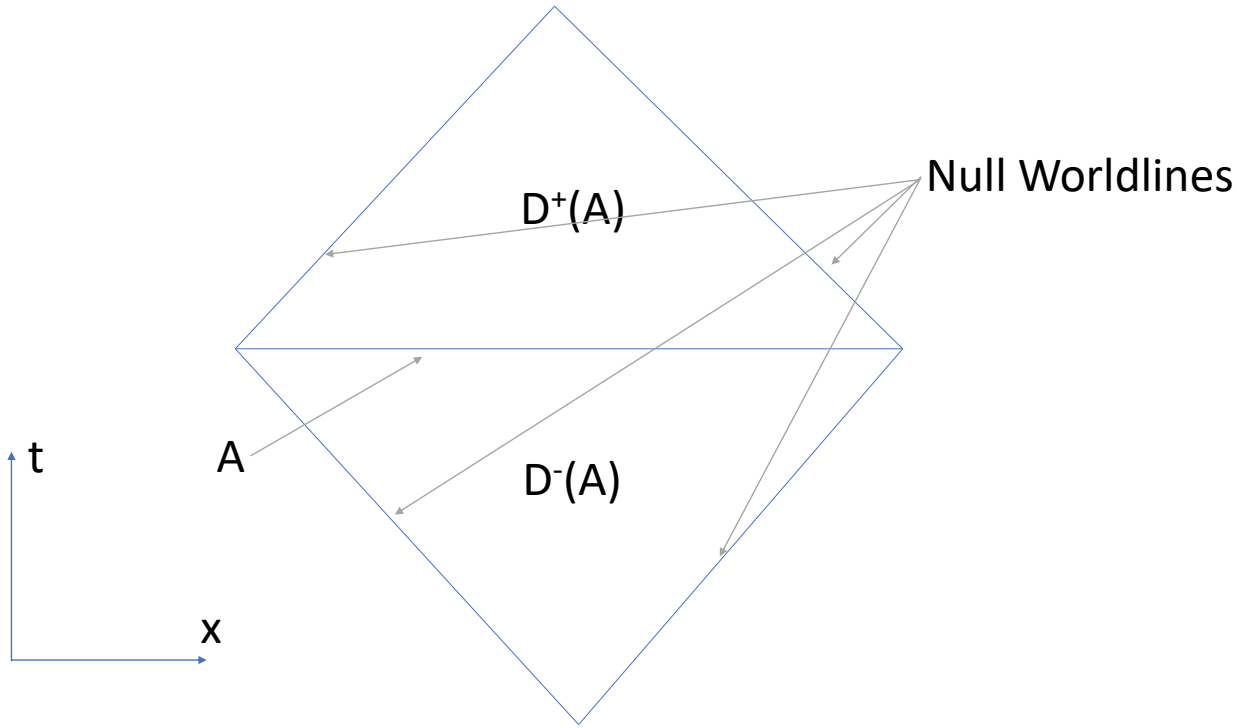


FIGURE: *A* is a Cauchy surface

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- Penrose, Hawking singularity theorems prove that under certain conditions, a spacetime manifold must be geodesically incomplete.

- These theorems physically imply that, in the gravitational collapse of generic matter, the formation of a singularity is inevitable under certain conditions, regardless of the symmetry of the collapsing system.

Singularity theorem 1 (Penrose (1965)):
Space-time $(\mathcal{M}, g)$ cannot be null geodesically complete if:

1. $R_{ab}l^a l^b \geq 0$ for all null vectors l^a;
2. There is a closed trapped surface $\mathcal{T}$ in $\mathcal{M}$;
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Using the null version of the Raychaudhuri equation, i.e.,

$$D\theta = -\frac{\theta^2}{2} - \sigma^2 + \omega^2 - R_{ab}l^a l^b,$$

along with conditions one and two mentioned above, one can show that there exists a conjugate point to the trapped surface $\mathcal{T}$ along every future-directed null geodesic orthogonal to $\mathcal{T}$, within an affine distance of $\frac{2}{\theta_0}$, where $-\theta_0$ is the null expansion at the intersection of the null geodesic with $\mathcal{T}$.

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Singularity theorem 2 (Hawking and Penrose (1970)):
Space-time $(\mathcal{M}, g)$ is time and null geodesically incomplete if:

1. $R_{ab}k^a k^b \geq 0$ for all non-spacelike vectors k^a;
2. $(\mathcal{M}, g)$ is algebraically generic, i.e., $k_{[a}R_{b]cd[e}k_{f]}k^c k^d \neq 0$;
3. the strong causality condition holds in $\mathcal{M}$;
4. a closed trapped surface $\mathcal{T}$ exits.

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- The theorem only specifies the conditions under which a singularity is inevitable; however, it provides no information about the causal structure of the singularity.

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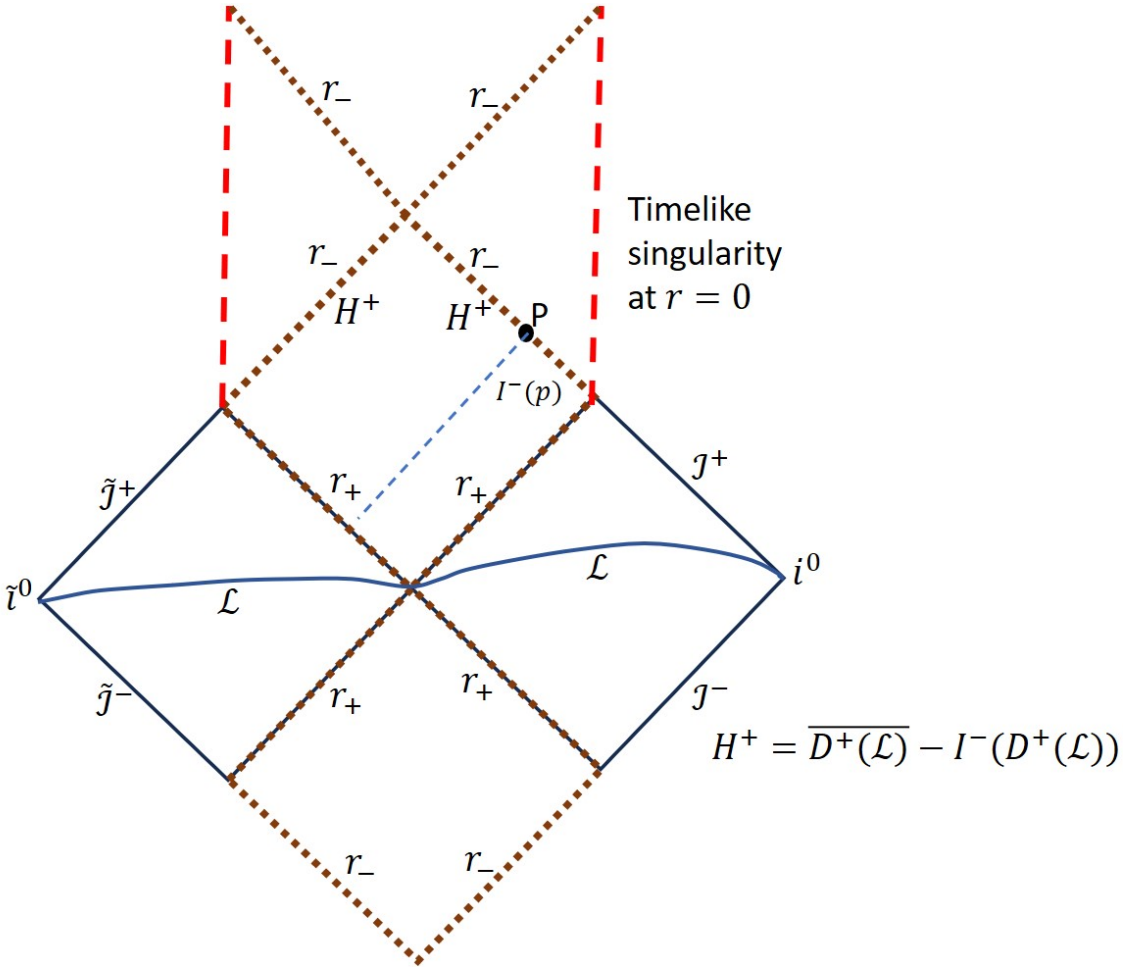
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Strong Cosmic Censorship Conjecture

If $(\mathcal{L}, h_{ab}, K_{ab})$ is an initial data set for Einstein's equation with $(\mathcal{L}, h_{ab})$ a complete Riemanian manifold and with the Einstein-matter equation of the hyperbolic form with T_{ab} satisfying the dominant energy condition, then if the maximal Cauchy development of this initial data is extendable, for each $p \in H^+(\mathcal{L})$ in any extension, either strong causality is violated or $\overline{I^-(p) \cap \mathcal{L}}$ is non-compact. (Penrose, 1979)

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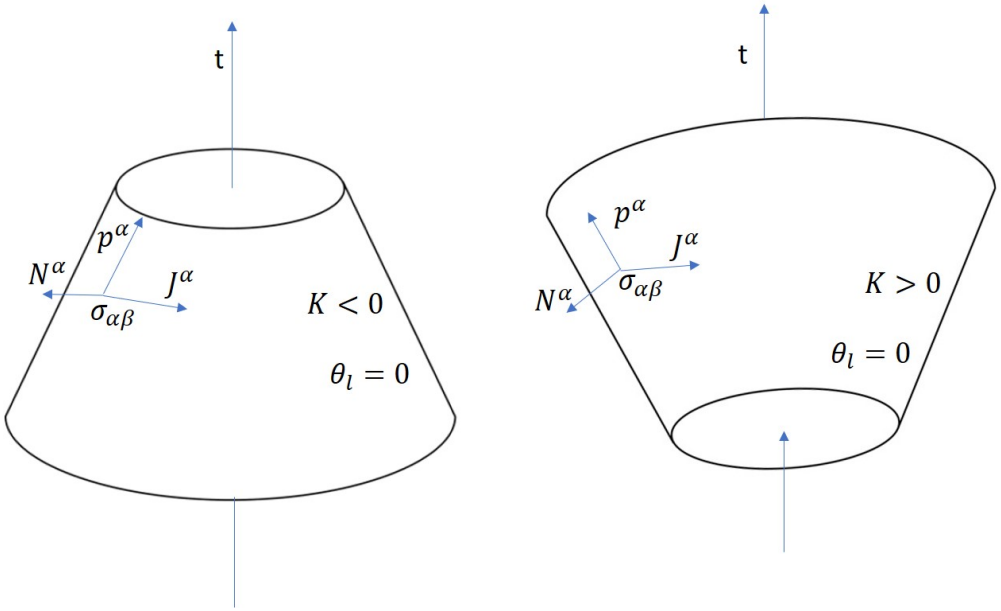


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APPARENT HORIZON

The marginally trapped surface can be defined by: $\theta_l = 0$ where the θ_l is the expansion scalar of outgoing null geodesics congruence. The apparent horizon is a surface that can be constructed by the foliation of the marginally trapped surface. Therefore, it is a three dimension hypersurface.

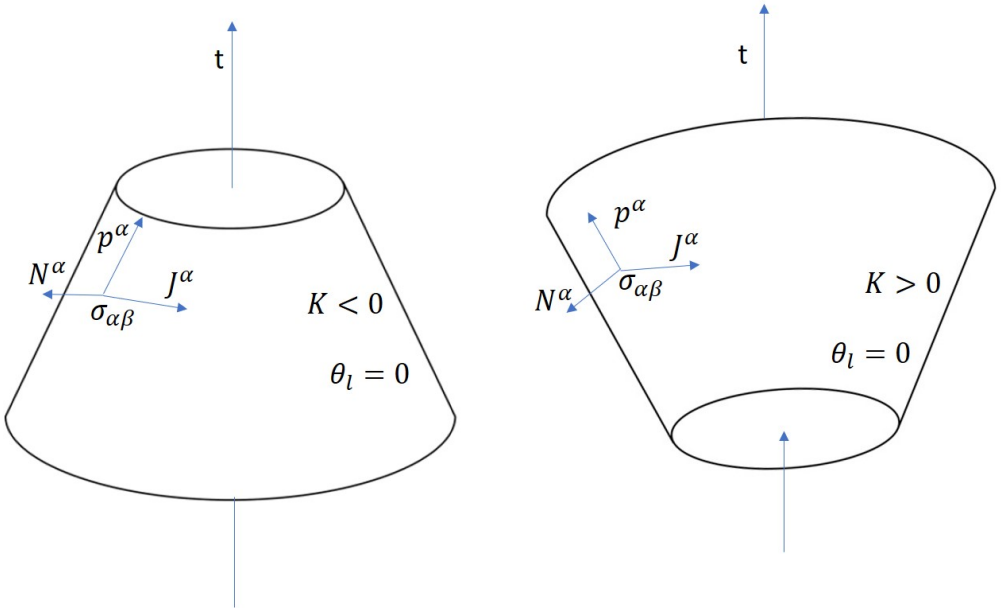


- The normal (N^μ) on the apparent horizon can be defined as: $N_\mu = \partial_\mu \theta_l|_{\theta_l=0}$. Now, we can write down the induced metric ($h^{\alpha\beta}$) on the apparent horizon as:

$$h^{\alpha\beta} = \frac{1}{\mathcal{N}} p^\alpha p^\beta + \lambda^\alpha \lambda^\beta + J^\alpha J^\beta, \tag{1}$$

where λ^α, J^α are the spacelike vectors tangent to the apparent horizon (AH) and p^α is also a tangent to the AH but its nature depends upon the causal nature of the AH.

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- If N^α is space-like i.e., p^α is time-like then the AH would be a time-like hyper-surface and for the opposite nature of n^α and p^α, the AH would be space-like hyper-surface.

- We can describe the metric of the four-dimensional manifold $\mathcal{M}$ in the following two ways:

$$g^{\alpha\beta} = -l^\alpha n^\beta - n^\alpha l^\beta + \sigma^{\alpha\beta} \ , \tag{2}$$

$$g^{\alpha\beta} = \frac{N^\alpha N^\beta}{N^\gamma N_\gamma} + h^{\alpha\beta} = \frac{N^\alpha N^\beta}{N^\gamma N_\gamma} + \frac{1}{\mathcal{N}} p^\alpha p^\beta + \sigma^{\alpha\beta} \ , \tag{3}$$

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- Therefore, $l^\alpha \sigma_{\alpha\beta} = n^\alpha \sigma_{\alpha\beta} = 0$ and $N^\alpha \sigma_{\alpha\beta} = p^\alpha \sigma_{\alpha\beta} = 0$, and we can write N^α, p^α as a linear combination of the null bases l^α, n^α:

$$\begin{aligned} N^\alpha &= -(n^\gamma N_\gamma) l^\alpha - (l^\gamma N_\gamma) n^\alpha \\ &= -\delta\theta_l|_H l^\alpha - D\theta_l|_H n^\alpha \ , \qquad (4) \\ p^\alpha &= \mathcal{C}\left(\delta\theta_l|_H l^\alpha - D\theta_l|_H n^\alpha\right) \ , \qquad (5) \end{aligned}$$

where $\mathcal{C}$ is the proportionality constant.

- As we discussed, $N^\alpha N_\alpha > 0$ or $p^\alpha p_\alpha < 0$ implies the apparent horizon (AH) is timelike and for the opposite case, it is spacelike.
- We can write:

$$N^\alpha N_\alpha = -2\, \delta\theta_l|_H\, D\theta_l|_H\,, \tag{6}$$

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- The above expressions implies,

$$\text{Timelike AH: } N_\mu N^\mu > 0 \implies \begin{cases} D\theta_l|_H > 0, & \text{and } \delta\theta_l|_H < 0 \\ D\theta_l|_H < 0, & \text{and } \delta\theta_l|_H > 0, \end{cases} \tag{8}$$

and

$$\text{Spacelike AH: } N_\mu N^\mu < 0 \implies \begin{cases} D\theta_l|_H > 0, & \text{and } \delta\theta_l|_H > 0 \\ D\theta_l|_H < 0, & \text{and } \delta\theta_l|_H < 0. \end{cases} \tag{9}$$

- One can calculate how the area ($\mathcal{A}$) of the spacelike surface (σ, $\sigma^{\alpha\beta}$) changes in the direction of p^α which is a tangent to the three dimensional hypersurface AH. One can write:

$$\begin{aligned}\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial \lambda_p} = \sigma^{\alpha\beta}\nabla_\alpha p_\beta &= \left(\lambda^\alpha \lambda^\beta + J^\alpha J^\beta\right)\left[-\partial_\alpha\left(\frac{D\theta_l}{\delta\theta_l}\right)_H n_\beta + \nabla_\alpha l_\beta - \frac{D\theta_l|_H}{\delta\theta_l|_H}\nabla_\alpha n_\beta\right] \\ &= (\theta_l|_H - \frac{D\theta_l|_H}{\delta\theta_l|_H}\theta_n|_H) \\ &= -\frac{D\theta_l|_H}{\delta\theta_l|_H}\theta_n|_H\end{aligned}$$

In those cases, where $\theta_n|_H < 0$, the expansion (i.e., $\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial \lambda_p} > 0$) and contraction (i.e., $\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial \lambda_p} < 0$) of the two dimensional spacelike surface σ depend upon the signature of $\frac{D\theta_l|_H}{\delta\theta_l|_H}$.

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- The above results imply an expanding AH is always spacelike and a contracting AH is always timelike.

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COLLAPSE

- $$ds^2 = -dt^2 + \frac{R'^2}{1 + f(r)} dr^2 + R^2 d\omega^2$$

Velocity function: f

- $f < 0$ bound collapse
- $f = 0$ marginally bound collapse
- $f > 0$ unbound collapse

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- For homogeneous collapse:
  - Necessary condition: $F = F_0 r^3 \quad \longrightarrow$ initial density is homogeneous.
  - Sufficient condition: Suitable velocity function f.

COLLAPSE

- **One model of homogeneous collapse:**
  - $F = F_0 r^3$,
  - $f = 0$. Marginally bound
- Equation of motion: $\dot{R}^2 = \frac{F}{R} \quad \Rightarrow \quad t(r) = t_s(r) - \frac{2}{3}\frac{R^{3/2}}{\sqrt{F}}$, where $t_s(r)$ is the singularity formation time for co-moving radius r.
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- For above model of homogeneous collapse: $t_s(r) = \frac{2}{3}\frac{1}{\sqrt{F_0}} \quad \Rightarrow$ All shells collapse to central singularity simultaneously.

COLLAPSE

- **One model of homogeneous collapse:**
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  - $f = 0$. Marginally bound
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- Apparent horizon formation time $t_{AH}(r)$ is:

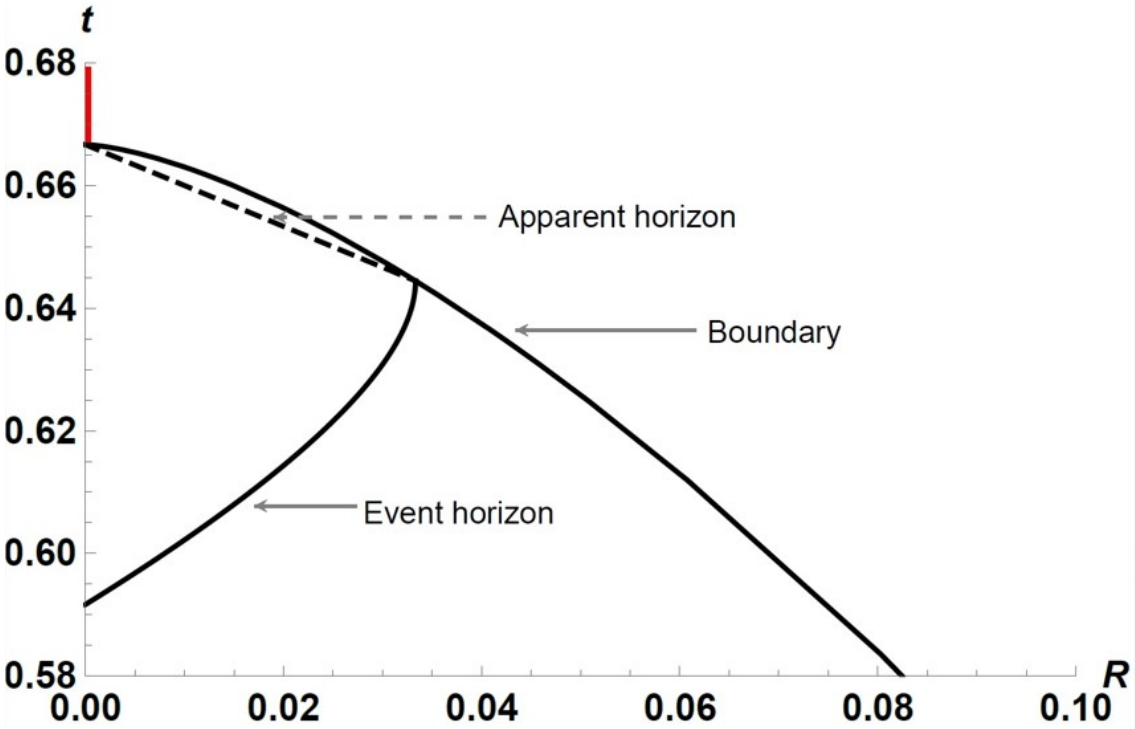
$$t_{AH}(r) = t_s(r) - \frac{2}{3}\frac{R^{3/2}}{\sqrt{F}},$$

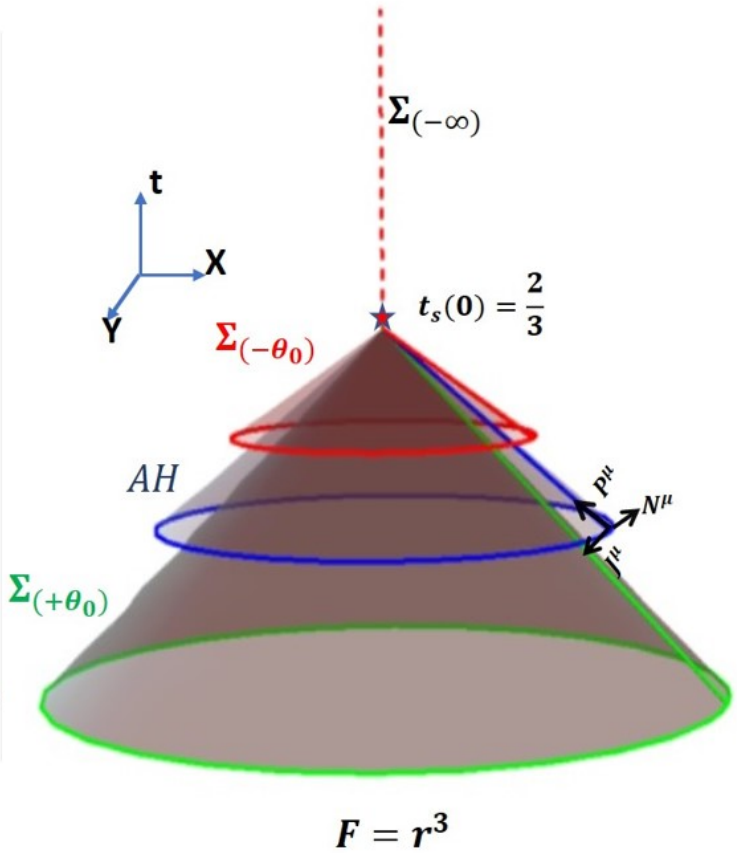
with $F = R$ since $\theta_l = 0$ on AH

$$= t_s(r) - \frac{2}{3}F$$

$$= \frac{2}{3}\frac{1}{\sqrt{F_0}} - \frac{2}{3}F_0 r^3 \Rightarrow t_s(r = 0) > t_{AH}(r)\ \forall r \in (0, r_c].$$

AH DYNAMICS IN OPPENHEIMER-SNYDER-DATT (OSD) COLLAPSE





The results above suggest that the apparent horizon (AH) in OSD collapse is always timelike, as it consistently contracts toward the center.

COLLAPSE

- **One model of inhomogeneous LTB collapse ([1] [2]):**
  - $F = F_0 r^3 - F_3 r^6$,
  - $f = 0$. Marginally bound
- Considering $R(t = 0, r) = r \quad \Rightarrow \quad t_s(r) = \frac{2}{3}\frac{1}{\sqrt{F_0}}\left(1 - \frac{F_3}{F_0} r^3\right)^{-\frac{1}{2}}$
- The expression for $t_s(r)$ indicates that different shells collapse to the central singularity at different times, unlike in the homogeneous case. The central shell collapses first at $t_s(r = 0) = \frac{2}{3}\frac{1}{\sqrt{F_0}}$.

[1] G Lemaitre, Ann. Soc. Sci. Bruxelles I A **53**, 51 (1933).

[2] R. C. Tolman, Proc. Natl. Acad. Sci. USA **20**, 410 (1934), H. Bondi, Mon. Not. Astron. Soc. **107**, 343 (1947).

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- The AH formation time: $t_{\rm AH}(r) = \frac{2}{3}\frac{1}{\sqrt{F_0}}\left(1 - \frac{F_3}{F_0}r^3\right)^{-\frac{1}{2}} - \frac{2}{3}F_0 r^3 + \frac{2}{3}F_3 r^6$.
- If we take $r = 0$ limit of $t_{AH}(r)$ then we get: $\lim_{r\to 0} t_{\rm AH}(r) = t_s(0) + \frac{r^3}{3}\left(\frac{F_3}{F_0^{3/2}} - 2F_0\right) + O(r^6)$

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COLLAPSE

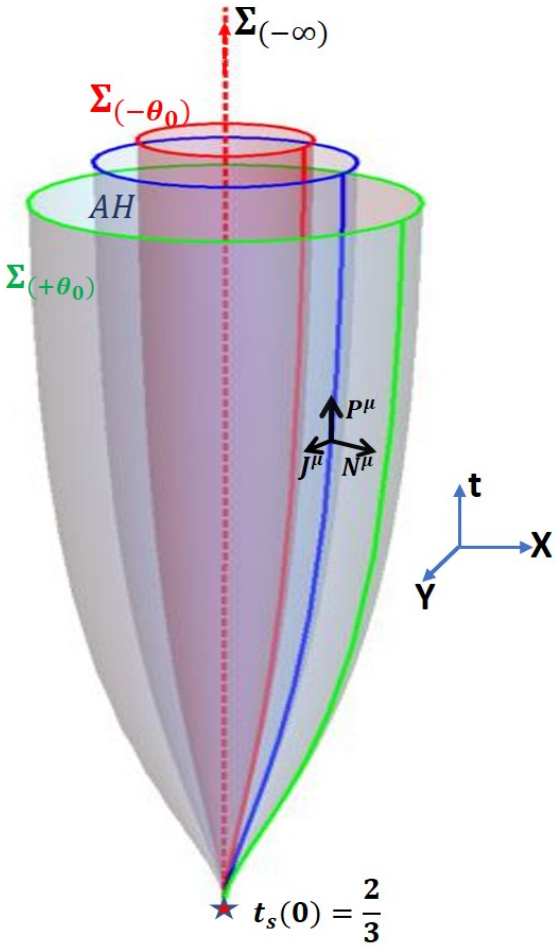
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- Therefore, in LTB collapse, the apparent horizon (AH) expands outward from the central singularity, indicating its spacelike nature when $F_3 > 2F_0^{5/2}$. Conversely, when $F_3 < 2F_0^{5/2}$, it contracts toward the center, demonstrating its timelike character.

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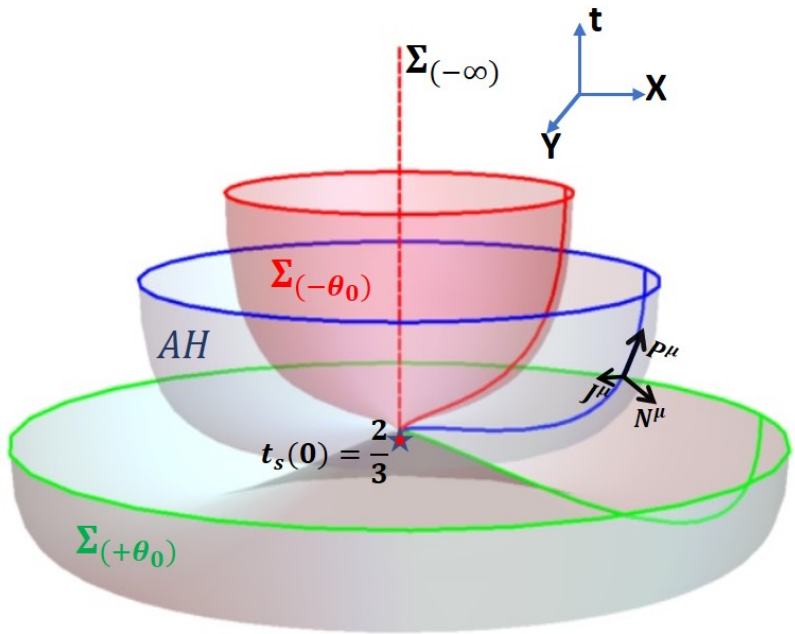
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COLLAPSE



$F = r^3 - 3.2r^5$



$F = r^3 - 0.7r^5$

COLLAPSE

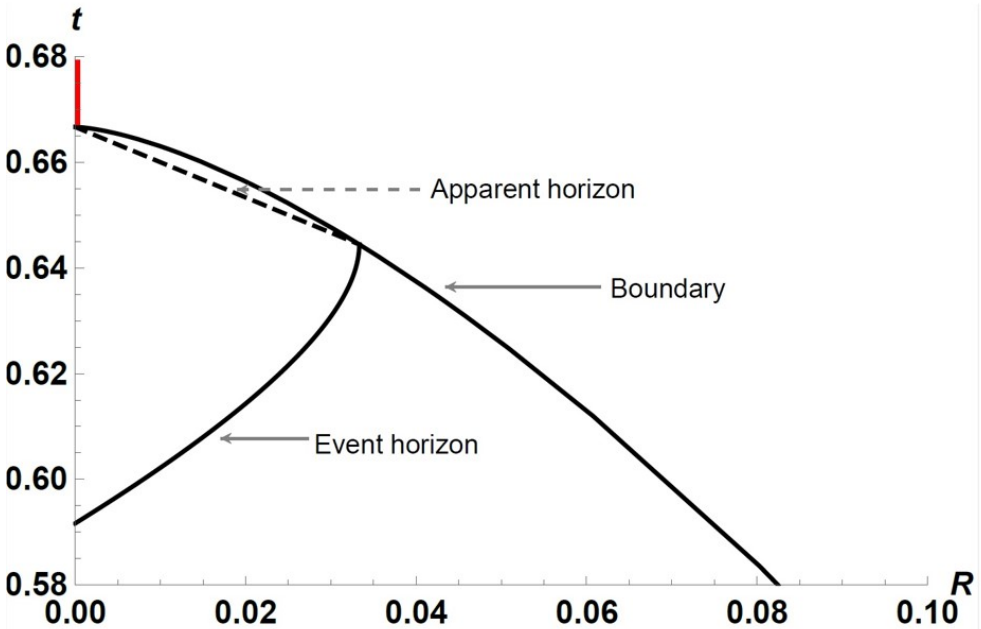
Therefore, the introduction of inhomogeneity in the collapsing spherical dust can alter the dynamics, thereby affecting the causal nature of the apparent horizon (AH).

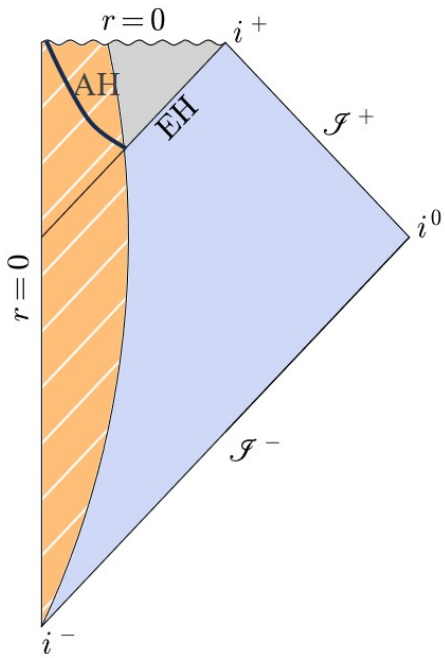
Outline

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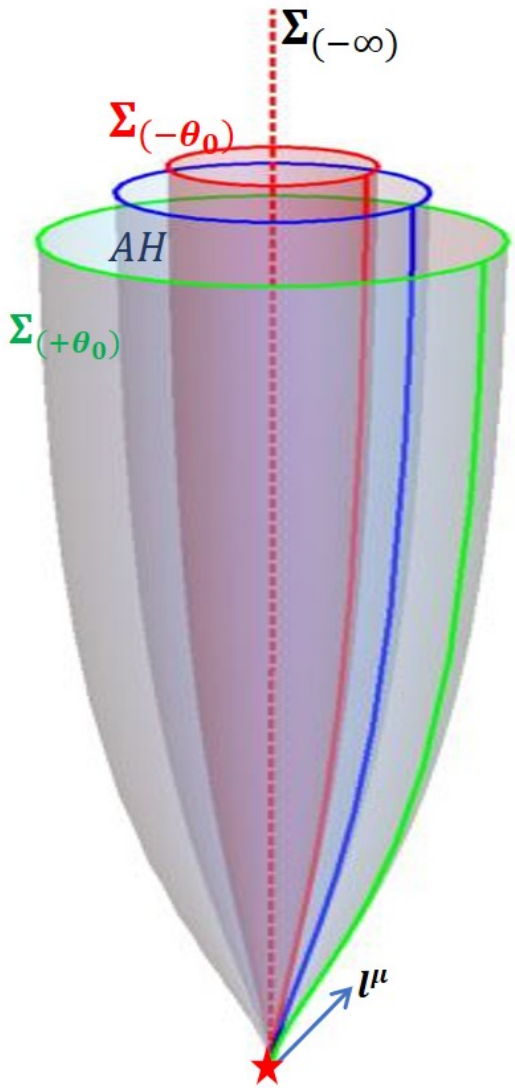
- In the case of homogeneous collapse, trapped surfaces and the apparent horizon (AH) form near the center prior to the emergence of the central singularity. The AH, being a timelike surface for homogeneous scenario, moves inward and reaches the center at the moment the singularity forms, indicating that the singularity is spacelike.

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- Therefore, a black hole is the end state of the OSD collapse, where a black hole B is defined as: $B = \mathscr{M} - J^-(\mathscr{I}^+)$.

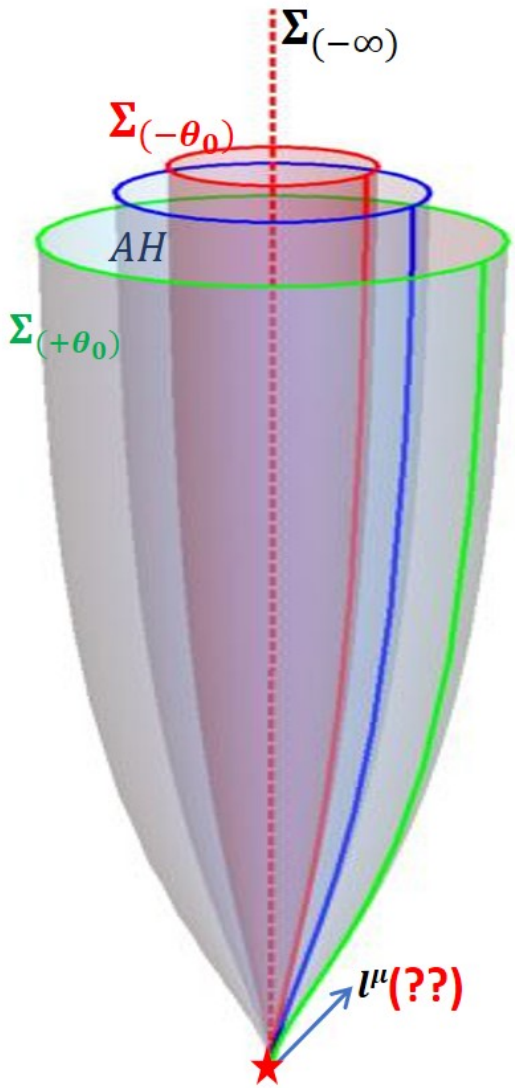




INHOMOGENEOUS LTB COLLAPSE



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As a result, a singularity forms in this spacetime. However, the existence of a Cauchy surface depends on the causal structure of the singularity.

INHOMOGENEOUS LTB COLLAPSE

For a singularity to be at least locally naked, there should exist past incomplete causal curves. We have the following result that ensures the existence of such curves.

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THEOREM

Consider an unhindered gravitational collapse of a spherically symmetric perfect fluid. The singularity formed as an end state of such collapse is at least locally naked if and only if $\exists$ $X_0 \in \mathbb{R}^+$ *as a root of* $V(X)$*, where*

$$V(X) = X - \frac{1}{\alpha}\left(X + \sqrt{\frac{F_0(0)}{X}}\left(\chi_1(0) + 2r\chi_2(0) + 3r^2\chi_3(0)\right) r^{\frac{5-3\alpha}{2}}\right)\left(1 - \sqrt{\frac{F_0(0)}{X}} r^{\frac{3-\alpha}{2}}\right). \tag{10}$$

Here, $X = \lim_{r,v\to 0} \frac{R}{r^\alpha}$ *and*

$$\alpha \in \left\{\frac{2n}{3} + 1; \quad n \in \mathbb{N}\right\}, \qquad \chi_i(v) = \frac{1}{i!}\frac{d^i t(r,v)}{dr^i}\bigg|_{r=0}, \qquad F(r,v) = \sum_{i=0}^{\infty} F_i(v) r^{i+3}. \tag{11}$$

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Note: The above theorem also applies to inhomogeneous, non-dust, perfect fluid collapse, as the Misner-Sharp mass term is considered in the theorem as a function of r and t.

Proof: In the case of nonmarginally bound dust, we have $\chi_i(v)$ as follows:

$$\chi_1(v) = -\frac{1}{2}\int_v^1 \frac{\frac{F_1}{v} + f_1}{\left(\frac{F_0}{v} + f_0\right)^{\frac{3}{2}}} dv, \tag{12}$$

$$\chi_2(v) = \int_v^1 \left(\frac{3}{8}\frac{\left(\frac{F_1}{v} + f_1\right)^2}{\left(\frac{F_0}{v} + f_0\right)^{\frac{5}{2}}} - \frac{1}{2}\frac{\frac{F_2}{v} + f_2}{\left(\frac{F_0}{v} + f_0\right)^{\frac{3}{2}}} \right) dv, \tag{13}$$

and

$$\chi_3 = \int_v^1 \left(\frac{f_1}{\left(\frac{F_0}{v} + f_0\right)^{\frac{3}{2}}} \left(-\frac{5}{16}\left(\frac{f_1}{\frac{F_0}{v} + f_0}\right)^2 + \frac{3}{4}\left(\frac{\frac{F_2}{v} + f_2}{\frac{F_0}{v} + f_0}\right) \right) - \frac{1}{2}\frac{\left(\frac{F_3}{v} + f_3\right)}{\left(\frac{F_0}{v} + f_0\right)^{\frac{3}{2}}} \right) dv. \tag{14}$$

Here, we Taylor expand f around $r = 0$ as

$$f(r) = \sum_{i=0}^{\infty} f_i r^{i+2}. \tag{15}$$

The above expressions for χ are derived by taking the limiting value of the null condition, $\frac{dt}{dr} = \frac{R'}{\sqrt{1+f}}$, and the equation of motion, $\dot{R}^2 = \frac{F}{R} + f(r)$, near $r = 0$.

- Let us consider the following functional form of the Misner-Sharp mass function and the velocity function:

$$F(r) = F_0 r^3 - F_3 r^6; \quad F_0 > 0, \quad F_3 > 0 \quad f(r) = f_0 r^2. \tag{16}$$

Eq.(10) for such $F(r)$ and $f(r)$ can be rewritten as $(r, v) \to (0, 0)$:

$$V(X) = 2X^2 + \sqrt{F_0} X^{3/2} - 3\sqrt{F_0}\chi_3(0)\sqrt{X} + 3F_0\chi_3(0), \tag{17}$$

where

$$\chi_3(0) = \frac{1}{2}\int_0^1 \frac{F_3/v}{(F_0/v)^{3/2}}\, dv. \tag{18}$$

We can see numerically for $F_0 = 1$, Eq.(17) has a positive real root if and only if $F_3 > 25.967$ [3]: [4]:

[3] K. Mosani, D. Dey and P. S. Joshi, Phys. Rev. D **102**, no.4, 044037 (2020).

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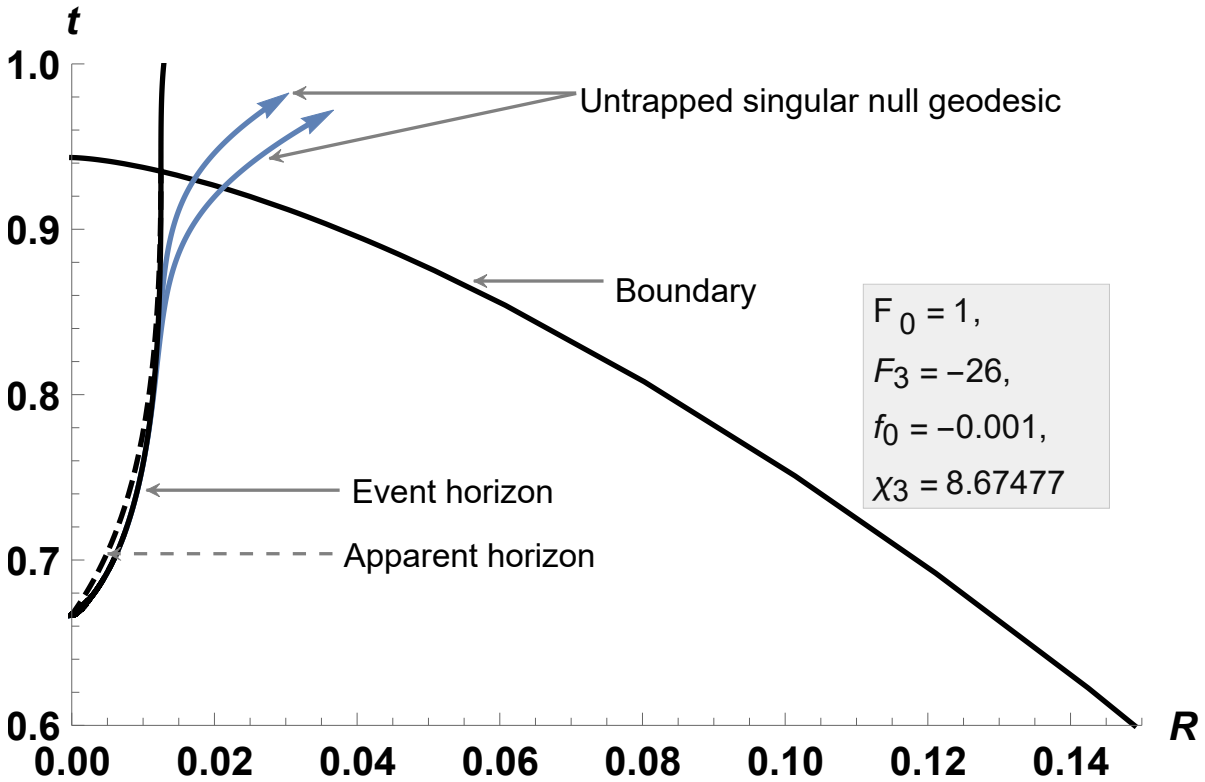
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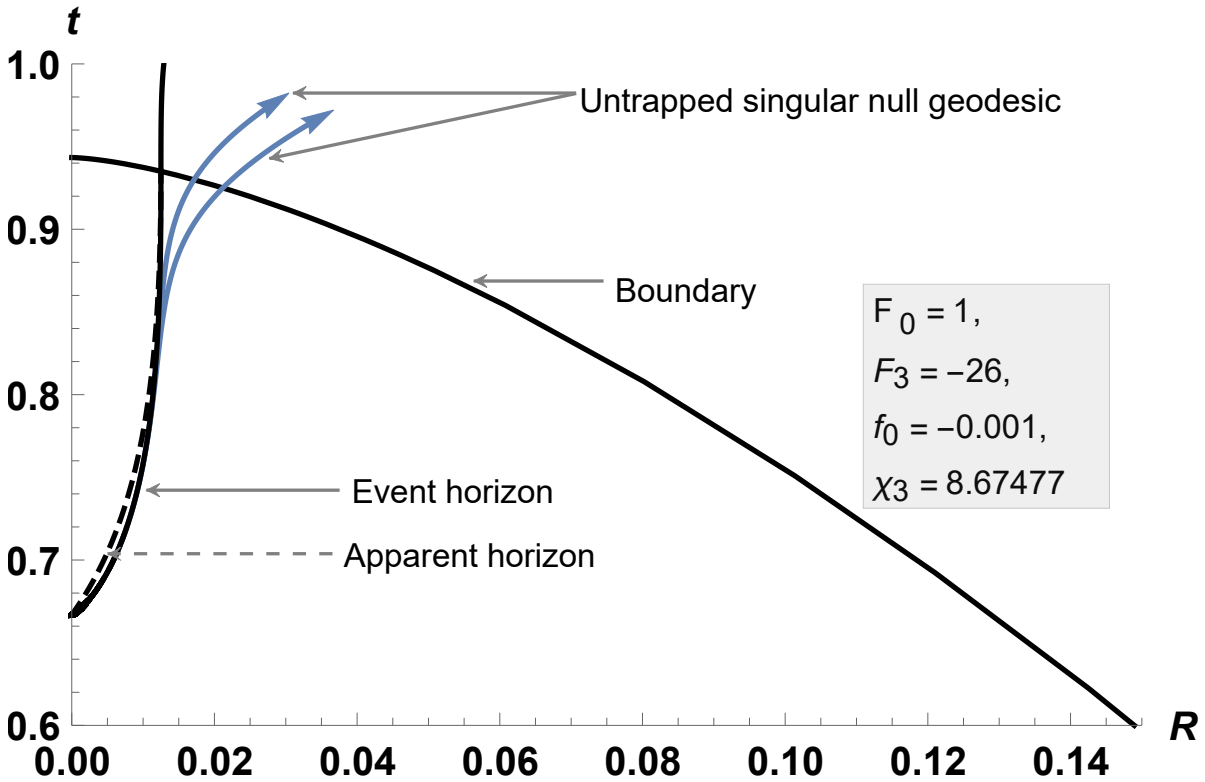
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- It can be shown that with the $F_3 > 25.967$ values event horizon (EH) and apparent horizon (AH) starts expanding from the center at the same time, i.e., $t_{AH}(0) = t_{EH}(0)$. Therefore, the null curves can reach the asymptotic null infinity.

[3] K. Mosani, D. Dey and P. S. Joshi, Phys. Rev. D **102**, no.4, 044037 (2020).

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t
1.0
0.9
0.8
0.7
0.6
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
R
Untrapped singular null geodesic
Boundary
Event horizon
Apparent horizon
F 0 = 1,
F3 = −26,
f0 = −0.001,
χ3 = 8.67477



- Thus, the final state of this type of inhomogeneous gravitational collapse can be expressed as: $\tilde{B} = \mathcal{M} - J^{-}(\mathscr{I}^{+}) + \dot{J}^{+}(p)$, where p denotes the event corresponding to the formation of the singularity at the center.

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- The collapsing aspherical matter cloud is governed by the particular Szekeres metric [5]:

$$ds^2 = -dt^2 + M^2(t, r, \theta)dr^2 + R^2(t, r, \theta)(d\theta^2 + \sin^2(\theta)d\phi^2),$$

where

$$M = Q\frac{\left(\frac{S}{Q}\right)'}{\sqrt{1 + f(r)}},$$

where

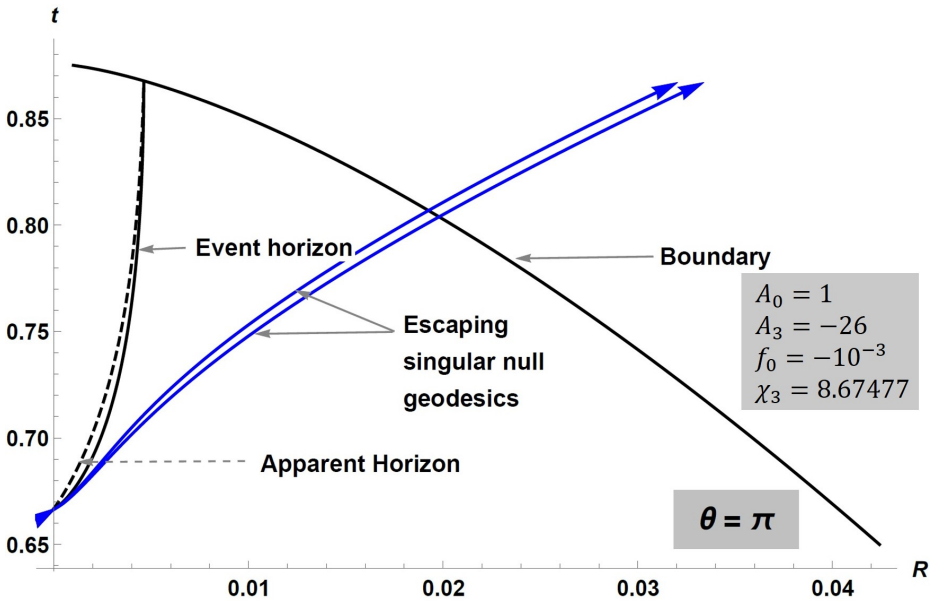
$$Q = a(r)\tan^2\left(\frac{\theta}{2}\right) + c(r); \text{ and } a(r)c(r) = \frac{1}{4}.$$

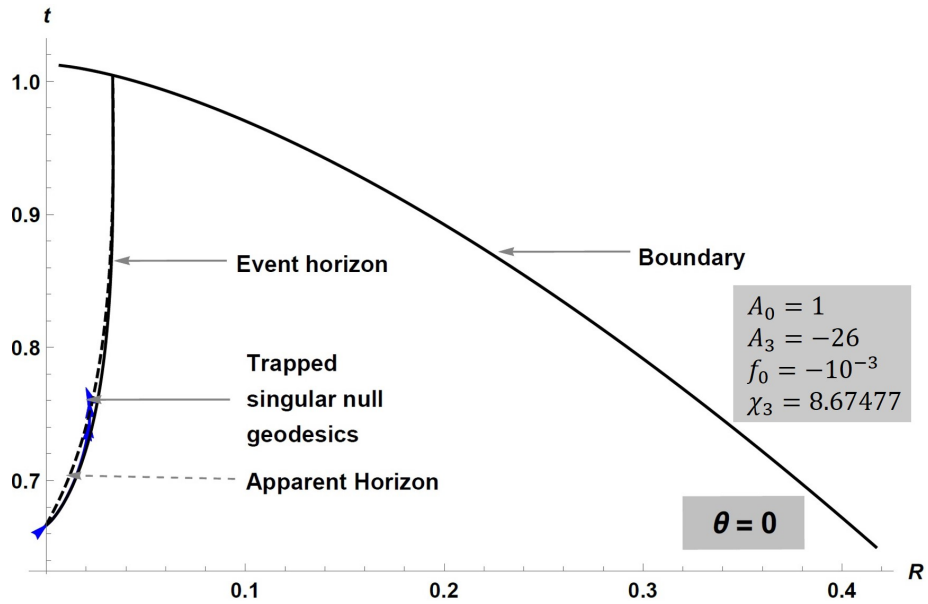
The physical radius of the collapsing cloud is

$$R(t, r, \theta) = \frac{S(t, r)\sec^2(\frac{\theta}{2})}{2(a(r)\tan^2(\frac{\theta}{2}) + c(r))}$$

[5] P. Szekeres, Phys. Rev. D **12**, 2941 (1975)

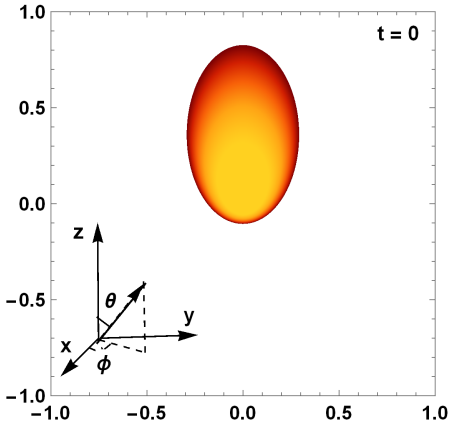
INHOMOGENEOUS, NON-SPHERICAL SZEKERES COLLAPSE

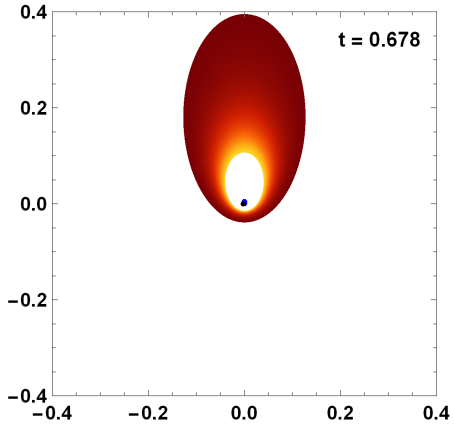


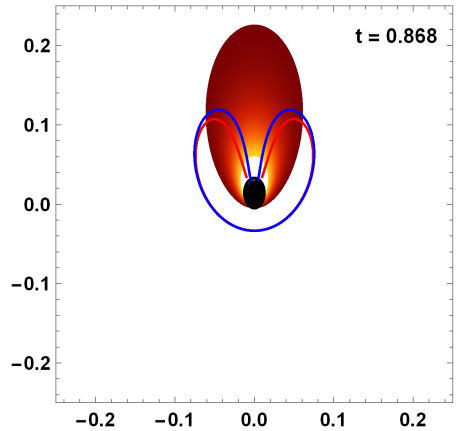


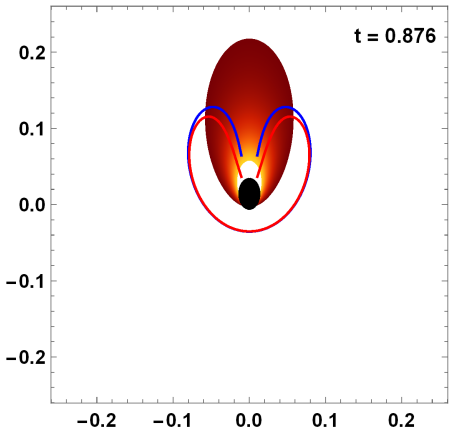
D.Dey, P.S.Joshi, K.Mosani and V.Vertogradov, Eur. Phys. J. C **82**, no.5, 431 (2022).

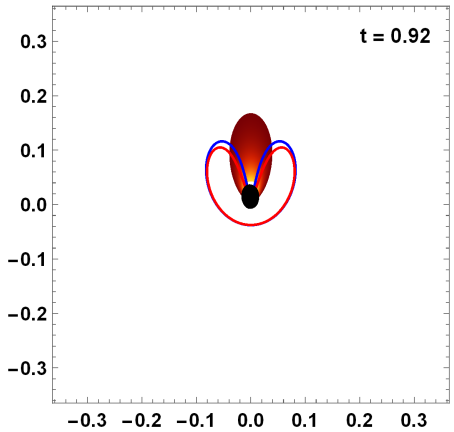
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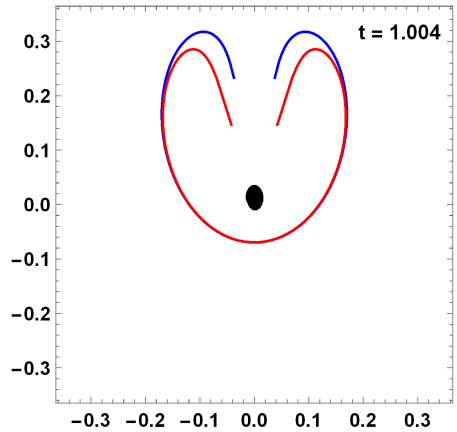












D.Dey, P.S.Joshi, K.Mosani and V.Vertogradov, Eur. Phys. J. C **82**, no.5, 431 (2022).

- **Definition** The volume element defined by wedge product of independent Jacobi fields along γ, should approach to zero as one approaches the singularity. [6]

[6] F. J. Tipler, Physics Letters A, **64**, 1 (1977).

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- A locally naked singularity, formed due to gravitational collapse of a spherically symmetric pressured cloud, is strong in the sense of Tipler if

$$\alpha \in \left\{ \frac{2n+1}{3}; \quad n \geq 4; \quad n \in \mathbb{N} \right\}.$$

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8. Conclusion

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- Using the Szekeres metric, it is demonstrated that similar possibilities for the existence of non-spacelike singularities arise in non-spherical gravitational collapse.

- Additionally, it is shown that there exists a continuous space of initial data for which these non-spacelike singularities form, implying that small changes in the initial data do not alter the final state's properties.

SOME UNSOLVED PROBLEMS:

- It can be shown that the condition $\frac{D\theta_I|_H}{\delta\theta_I|_H} > 0$ (indicating a spacelike apparent horizon, AH) is necessary for the singularity to be non-spacelike. However, the sufficient condition remains unknown. As we discussed, in marginally bound LTB collapse, when $\frac{F_3}{F_0^{\frac{5}{2}}} > 2$, the apparent horizon becomes spacelike and begins expanding from the central singularity. Nevertheless, the singularity becomes locally visible only when $\frac{F_3}{F_0^{\frac{5}{2}}} \geq 26$. It is still unclear why the singularity remains hidden in the intermediate range, $2 < \frac{F_3}{F_0^{\frac{5}{2}}} < 26$.

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- Although our results are coordinate-independent, there is no coordinate-independent method to define the condition for locally visible singularities. The approach we used requires considering a dynamic metric with a specific coordinate system, followed by calculating the tangent of the null curve. Similar to the singularity theorems, there is no generic way (applicable for any symmetry and for any generic matter) to formulate these conditions.

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- Another important issue is the stability of these types of singularities. While some studies in the literature address this, the authors often use overly simplified scenarios to address the stability problem.

Thank You