Lemma 1. $\boxed{f(x) \le \sum_{y_1,\dots,y_4} k_{y_i} f(y_i)}$ (*)

y_1

$y_4 \cdot x$

$\underline{\Pi_k}(\sigma_v^1, \sigma_v^2 \mid \bar\sigma_v^1, \bar\sigma_v^2)$ — optimal coupling.

Idea of the proof of (*).

put instead $Q(\cdot,\cdot \mid \dots)$ $\underline{\underline{\Pi_k}}$

Suppose that at some x

$$\boxed{f(x) > (1+\varepsilon)\sum_{y_i \sim x} k_{y_i} f(y_i)}$$

$\Longrightarrow$ then $\underline{\underline{\Pi_k}}$ is not optimal:

there is a better coupling:

$q(\sigma_x^1 \mid \bar\sigma_{y_i}^1)$, $q(\sigma_x^2 \mid \bar\sigma_{y_i}^2)$ —

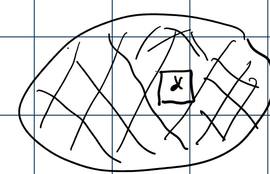
— let us couple :

$Q(\sigma^1, \sigma^2 \mid \bar\sigma^1, \bar\sigma^2)$

$$\boxed{Q\left(\sigma_x^1, \sigma_x^2 \,\middle|\, \bar\sigma^1_{y_\cdot}, \bar\sigma^2_{y_\cdot}\right)}$$

$$\boxed{f(x) < \sum k_{y_i} \left|\bar\sigma^1_{y_i} - \bar\sigma^2_{y_i}\right|} \quad \text{— for } Q.$$

Surgery.



Π_k

$$\underbrace{\Pi_k\left(\sigma_V^1, \sigma_V^2 \,\middle|\, \bar\sigma_V^1, \bar\sigma_V^2\right)}$$

"cut away a bad piece" $\longrightarrow$

$$\boxed{\Pi_k^*\left(\sigma^1_{V\setminus x}, \sigma^2_{V\setminus x} \,\middle|\, \bar\sigma_V^1, \bar\sigma_V^2\right) =}$$

$$= \sum_{\sigma_x^1, \sigma_x^2} \Pi_k\left(\sigma_V^1, \sigma_V^2 \,\middle|\, \bar\sigma_V^1, \bar\sigma_V^2\right)$$

$$\Downarrow$$

$$\tilde\Pi_k\left(\sigma_x^1, \sigma^1_{V\setminus x}; \sigma_x^2, \sigma^2_{V\setminus x} \,\middle|\, \bar\sigma_V^1, \bar\sigma_V^2\right) =$$

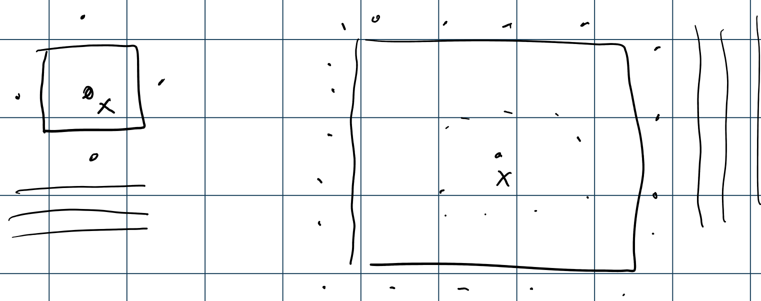
$$= Q_x\left(\sigma_x^1, \sigma_x^2 \,\middle|\, \sigma^1_{V\setminus x}, \sigma^2_{V\setminus x}\right) \; \Pi_k^*\left(\sigma^1_{V\setminus x}, \sigma^2_{V\setminus x} \,\middle|\, \bar\sigma_V^1, \bar\sigma_V^2\right)$$

This condition: — $C_{\{\cdot\}}$

(**) $\gamma = \sum k_y < 1$ — is sufficient for uniqueness,

but is not necessary.

but is not necessary.



Generalisation.

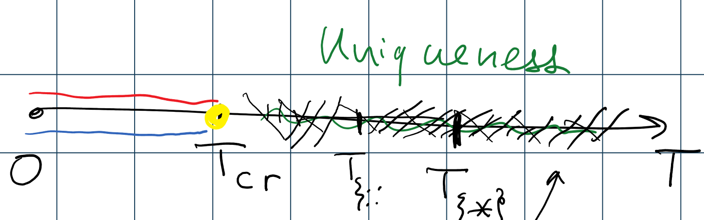
$\Lambda \longrightarrow$ ~~there~~ exists a condition C_Λ.

s. that if for some Λ $\underline{\underline{C_\Lambda}}$ - is satisfied,

then $\longrightarrow$ uniqueness.

For $\Lambda = \{pt\}$ - this is (**): $\gamma < 1$.

For 2D Ising:



$\underline{C_{\{*\}}}$ - holds for $T > T_{\{*\}}$

$\underline{\underline{C_{\{::\}}}} \longrightarrow T_{\{::\}}$

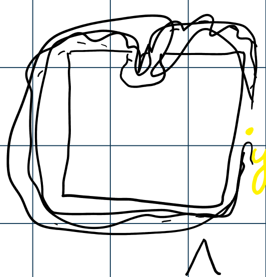
For every $\Lambda \longrightarrow T_\Lambda$:

For every $\Lambda \longrightarrow T_\Lambda$:

uniqueness for all $\underline{T > T_\Lambda}$ follows from C_Λ.

$T_\Lambda \searrow T_{cr}$ as $\Lambda \longrightarrow \mathbb{Z}^2$.

$\underline{C_\Lambda}$:



$q_\Lambda(\sigma^1_\Lambda \mid \bar{\sigma}_{\partial\Lambda \setminus y}, \sigma_y)$;

$q_\Lambda(\sigma^2_\Lambda \mid \bar{\sigma}_{\partial\Lambda \setminus y}, -\sigma_y)$

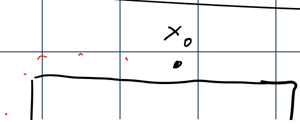
$\Pi_k(\sigma^1_\Lambda, \sigma^2_\Lambda \mid \bar{\sigma}_{\partial\Lambda \setminus y}, \sigma_y, -\sigma_y)$ — optimal coupling.

$$\text{Price}\,\Pi_k = \sum_{\sigma^1_\Lambda, \sigma^2_\Lambda} \left(\left\{ \sum_{x \in \Lambda} |\sigma^1_x - \sigma^2_x| \right\} \Pi_k(\sigma^1_\Lambda, \sigma^2_\Lambda \mid \dots) \right) =$$

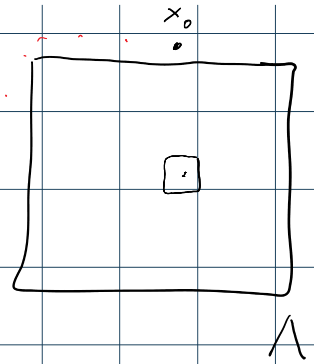
$$= k(y, \sigma_{\partial\Lambda \setminus y}) \longrightarrow k(y) = \max_{\sigma_{\partial\Lambda \setminus y}} k(y, \sigma_{\partial\Lambda \setminus y})$$

$$\boxed{C_\Lambda : \sum_{y \in \partial\Lambda} k(y) < |\Lambda|}$$

$|\Lambda| = 1$ for $\Lambda = (x)$ (point)



$f(x)$, $x \in \Lambda \cup \partial\Lambda$;



$f(x)$, $\quad x \in \Lambda \cup \partial\Lambda$;

$f(x_0) = 1$

$f(y) = 0$, $\quad y \in \partial\Lambda$

And inside Λ

$$f(x) \le \sum_{y:\, y\sim x} k_y f(x),$$

with $\sum_{y:\, y\sim x} k_y \le \gamma < 1$

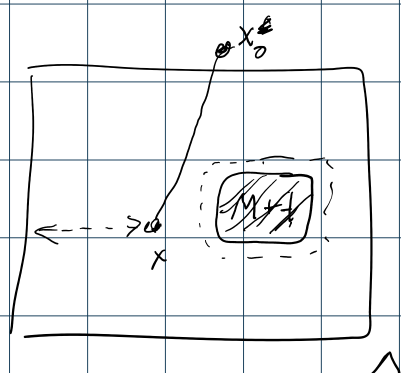
$$\Longrightarrow \quad f(x) \le \exp\{-c \; dist(x, x_0)\}$$

$c>0$

Generalization $\quad \{x\} \longrightarrow M \subset \mathbb{Z}^2$ (corr. to C_M)

And inside Λ:

$\Longrightarrow$

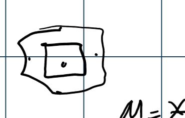


$M+t \subset \Lambda$

$\forall t: \quad M+t \subset \Lambda$

$$\boxed{\sum_{x\in M+t} f(x)} \le \sum_{x\in\partial(M+t)} k_x f(x)$$

$$\sum_{x\in\partial(M+t)} k_x < |M|$$



Th... $\longrightarrow \quad f(x) \le \cancel{\exp\{-c \; dist(x, x_0)\}}$

Then $\longrightarrow$ $\cancel{f(x) \leq \exp\{-c \ \mathrm{dist}(x, x_0)\}}$

— NO!

$$f(x) \leq \exp\{-c \ \mathrm{dist}(x, \partial\Lambda)\}$$