Hopf algebras of dimension p^2 in positive characteristic

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Overview

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- The representation categories of finite-dimensional Hopf algebras provide fundamental examples of finite tensor categories. In particular, representation categories of semisimple Hopf algebras give fusion categories.

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- $k = \overline{k}$: base field
- p: prime number
- H: finite-dimensional Hopf algebra over k

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  - Reduce H to semisimple case by Nichols-Zoeller Theorem and Radford's S^4-formula .
  - Apply a generalization of a theorem of G. I. Kats on the character ring of semisimple Hopf algebras.

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- At that time, the only known non-semisimple Hopf algebras of dimension p^2 over k are the Taft algebras.

Taft algebras

Definition (Taft 1971)

Let integer $n \geq 2$ and ω be a primitive nth root of unity. The Taft algebra $T_\omega(n)$ is generated by x and g, subject to the relations

$$g^n = 1,\ x^n = 0,\ gx = \omega xg.$$

The Hopf algebra structure is given by

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \epsilon(g) = 1,$$
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- $T_\omega(n)$ is pointed of dimension n^2 with $\mathrm{ord}(S^2) = n$.
- $T_{-1}(2)$ is the smallest non-commutative and non-cocommutative Hopf algebra (Sweedler's 4-dimesnional Hopf algebra).

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  - By Larson and Radford's result: H is semisimple $\Leftrightarrow S^2 = \mathrm{id}$ to deduce $\mathrm{ord}(S^2) = p$.
  - An earlier result of Andruskiewitsch and Schneider: H is non-semisimple and $\mathrm{ord}(S^2) = p \Rightarrow H \cong T_\omega(p)$.

Complete classification of Hopf algebras of dimension p and p^2 in characteristic 0

Theorem (Zhu-Masuoka-Ng)

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From characteristic zero to positive characteristic

- p: prime number
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(3) $k[x]/(x^p - x)$ *with* $\Delta(x) = x \otimes 1 + 1 \otimes x$

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- *Estimate dimension of indecomposable projective* H*-modules to get either* H *is local (algebra) or connected (coalgebra).*

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Remark

- *Similar argument as in characteristic* 0 *reduces to* $S^4 = \mathrm{id}$.
- *Estimate dimension of indecomposable projective H-modules to get either H is local (algebra) or connected (coalgebra).*
- *Use classification of connected Hopf algebras in dimension p.*

From characteristic zero to positive characteristic

Conjecture

Let H be a Hopf algebra of prime dimension q over an algebraic closed field k of characteristic p

$$\boxed{p \neq q \implies H \cong k[C_q]}$$

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- *Ng and W. verified for $q < 4p$ by analyzing the Grothendieck rings with small ranks.*
- *The bound was later improved by Etingof by introducing Frobenius-Perron dimensions for integral $\mathbb{Z}_+$-rings.*

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(5) *Radford algebra $R(p)$.*

Remark

- *The classification follows an argument by Andruskiewitsch and Natale for Hopf algebras of low dimension.*

Radford algebra $R(p)$

Definition (Radford 1977)

The Radford algebra $R(p)$ ($\mathrm{char}(k) = p$) is generated by x and g, subject to the relations

$$g^p - 1, \quad x^p - x, \quad gx - xg - g(g-1).$$

The Hopf algebra structure is given by

$$\Delta(g) = g \otimes g, \quad S(g) = g^{-1}, \quad \epsilon(g) = 1,$$
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Remark

- *$R(p)$ was given by Radford as an example of a Hopf algebra with $\mathrm{ord}(S) = 2p$.*
- *Taft pointed out that $R(p)$ is an extension of $k[C_p]$ by $k[C_p]^*$, and its antipode is not semisimple.*

Hopf algebras of dimension p^2 in positive characteristic

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Remark

- *By classification result, $R(p)$ is the only non-commutative and non-cocommutative Hopf algebra among the pointed ones.*
- *An affirmative answer to the above question will lead to a complete classification of p^2-dimensional Hopf algebras in characteristic p.*

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The following statements are equivalent:

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The following statements are equivalent:

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(iv) *H is a nontrivial extension, i.e., not isomorphic to the tensor Hopf algebra $k[C_p] \otimes k[C_p]^*$, that fits into the exact sequence*

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In particular, the Radford algebra is self-dual.

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- *H is an extension of p-dimensional Hopf algebras.*

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Theorem (Ng-W. 2023)

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- *H contains a non-trivial group-like element.*

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- *H contains a non-trivial group-like element.*
- $p \leq 5$.

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- k: algebraically closed field of characteristic p
- $\mathfrak{g} = \mathrm{span}_k\{x, y\}$: nonabelian 2-dimensional p-restricted Lie algebra such that $[x, y] = y$, $x^p = x$ and $y^p = 0$
- $u(\mathfrak{g}) = k\langle x, y\rangle/([x, y] = y, x^p = x, y^p = 0)$: restricted universal enveloping algebra of $\mathfrak{g}$ (with x, y primitive)

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Theorem (Ng-W. 2023)

The Radford algebra $R(p)$ and the restricted universal enveloping algebra $u(\mathfrak{g})$ are gauge equivalent. In particular, $R(p)$ is isomorphic to the twist $u(\mathfrak{g})^J$ by the Drinfeld twist element:

$$J = \sum_{0 \le i \le p-1} \frac{x(x-1)\cdots(x-i+1) \otimes y^i}{i!}$$

Remark

- *The Drinfeld element J in $u(\mathfrak{g})$ was given earlier by Gelaki.*

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Theorem (Ng-W. 2023)

Then H admits a normal Hopf subalgebra of dimension p if and only if H is pointed. In particular, every pointed p^2-dimensional Hopf algebra is an extension of p-dimensional ones.

Main Results

- $\mathfrak{a}_1 : x^p = x$; $\mathfrak{a}_2 : x^p = 0$: 1-dim'l restricted Lie algebra (rla)
- $\mathfrak{g}_1 : x^p = x, y^p = y$, $\mathfrak{g}_2 : x^p = x, y^p = 0$, $\mathfrak{g}_3 : x^p = y, y^p = 0$, $\mathfrak{g}_4 : x^p = 0, y^p = 0$: 2-dim'l abelian rla
- $\mathfrak{g}_5 : [x, y] = y, x^p = x, y^p = 0$: 2-dim'l nonabelian rla
- $\mathcal{G}_p(2) = k[x]/(x^{p^2})$, $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{i=1}^{p-1} \frac{x^{pi}}{i!} \otimes \frac{x^{p(p-i)}}{(p-i)!}$.

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- $\mathfrak{g}_5 : [x, y] = y, x^p = x, y^p = 0$: 2-dim'l nonabelian rla
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Table: Nontrivial exact sequences $1 \to K \to H \to L \to 1$

K	H	L
$k[C_p]$	$k[C_{p^2}]$	$k[C_p]$
$k[C_p]$	$R(p)$	$u(\mathfrak{a}_1)$
$k[C_p]$	$u(\mathfrak{g}_5)^*$	$u(\mathfrak{a}_2)$
$u(\mathfrak{a}_1)$	None	$k[C_p]$
$u(\mathfrak{a}_1)$	$k[C_{p^2}]^*$	$u(\mathfrak{a}_1)$
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$u(\mathfrak{a}_2)$	$u(\mathfrak{g}_5)$	$u(\mathfrak{a}_1)$
$u(\mathfrak{a}_2)$	$u(\mathfrak{g}_3)$, $u(\mathfrak{g}_3)^*$, $\mathcal{G}_p(2)$	$u(\mathfrak{a}_2)$

Extensions of Hopf algebras

A short exact sequence of finite-dimensional Hopf algebras A, H and K, denoted by

$$\boxed{1 \to K \xrightarrow{\iota} A \xrightarrow{\pi} H \to 1}$$

consists of

- ι is a Hopf algebra injection
- π is a Hopf algebra surjective
- $\iota(K)$ is a normal Hopf subalgebra of A such that $\ker \pi = \iota(K^+)A$

In this case, we say that A is an extension of K by H.

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Let

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(1) *Suppose K is local. Then H is basic if and only if L is basic.*

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(1) *Suppose K is local. Then H is basic if and only if L is basic.*

(2) *Suppose L is connected. Then K is pointed if and only if H is pointed.*

Sweedler Cohomology

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- $\mathrm{Reg}^n(H,A)$: all convolution invertible linear maps $H^{\otimes n} \to A$
- The coface operator $d^i : \mathrm{Reg}^n(H,A) \to \mathrm{Reg}^{n+1}(H,A)$

$$d^i(f)(h_0 \otimes \cdots \otimes h_n) = \begin{cases} h_0 f(h_1 \otimes \cdots \otimes h_n) & \text{for } i = 0 \\ f(h_0 \otimes \cdots \otimes h_{i-1}h_i \otimes \cdots \otimes h_n) & \text{for } 1 \leq i \leq n \\ f(h_0 \otimes \cdots \otimes h_{n-1})\epsilon(h_n) & \text{for } i = n+1 \end{cases}$$

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The Sweedler cohomology of H with coefficients in A is the cohomology of the cochain complex $(\mathrm{Reg}^\bullet(H,A), \partial)$

$$\boxed{H^n_{sw}(H,A) \;=\; \mathrm{Ker}\,\partial^n / \mathrm{Im}\,\partial^{n-1}}$$

with differential $\partial(f) = d^0(f) * d^1(f^{-1}) * \cdots * d^{n+1}(f^{\pm 1})$.

Sweedler Cohomology

Theorem (Ng-W. 2022)

- *k: an algebraically closed field of characteristic $p > 0$*
- *$H = (k[G])^*$ for some elementary abelian p-group G*
- *A: H-module commutative algebra such that A^H is a complete local ring with residue field k*

Sweedler Cohomology

Theorem (Ng-W. 2022)

- *k: an algebraically closed field of characteristic $p > 0$*
- *$H = (k[G])^*$ for some elementary abelian p-group G*
- *A: H-module commutative algebra such that A^H is a complete local ring with residue field k*

Then we have

$$\boxed{H^2_{sw}(H, A) = 0}$$

Remark

- *The assumption that k is algebraically closed is essential. If k is of characteristic $p = 2$ (not necessarily algebraically closed), Guillot proved that $H^2_{sw}((k[G]^*, k) = (k/\{x + x^2 \mid x \in k\})^{\oplus r}$.*

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(1) k *is an algebraically closed field of characteristic* $p > 0$.

(2) G *is an abelian* p*-group.*

(3) G' *is an elementary abelian* p*-group.*

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Then

$$\boxed{H \;\cong\; k[G]\#(k[G'])^* \text{ as algebras}}$$

where $k[G]$ is a $(k[G'])^*$-module algebra.

Group-like elements in positive characteristic

Lemma (Ng-W. 2022)

- *H: a finite-dimensional Hopf algebra over k of arbitrary characteristic.*
- *V: a simple H-module.*
- *$G \leq G(H^*)$ such that $k_\beta \otimes V \cong V$ for all $\beta \in G$.*

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we have $|G| \mid \dim(V)$.

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Lemma (Ng-W. 2022)

- *H: a finite-dimensional Hopf algebra over k of characteristic $p > 0$*
- *$x \in H$: a primitive element*
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- $$\boxed{xC \subset C \Longrightarrow \dim(k[x])^2 \mid \dim(C_i) \textit{ for all } i}$$
- $$\boxed{C \textit{ is simple, } p \nmid \dim(C) \Longrightarrow k[x]C \textit{ free over } k[x] \textit{ of rank } \dim(C)}$$

Group-like elements in positive characteristic

Theorem (Ng-W. 2022)

- *k is an algebraically closed field of characteristic odd prime p*
- *H is non-cosemisimple of dimension p^n*
- $|G(H)| = p^{n-1}$

Then H is pointed.

Affirmative answer to Radford algebra for $p \leq 5$

- k: an algebraically closed field of characteristic $p > 2$
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Remark

- *Part (a) extends the analysis of the Grothendieck ring from dimension p to p^2 in characteristic p.*
- *Part (b) extends the use of the quantum coordinate ring $\mathcal{O}_q(SL_2)$ by Bichon and Natale to positive characteristic.*

Affirmative answer to Radford algebra for $p \leq 5$

Theorem (Ng-W. 2023)

Let k be an algebraically closed field of positive characteristic $p \leq 5$, and H a p^2-dimensional Hopf algebra over k. Then H or H^* is pointed.

Thank You!