Lecture 20: Computation of bordism groups I

Goal: Compute $\Omega_d^O(K(\mathbb{Z}/2,2))$ and $\Omega_d^{SO}(K(\mathbb{Z}/2,2))$ for $d \le 5$.

① $\Omega_d^O(K(\mathbb{Z}/2,2))$ for $d \le 5$

Known facts:

1. Pontryagin–Thom isomorphism: $\Omega_d^O(K(\mathbb{Z}/2,2)) = \pi_d(MO \wedge K(\mathbb{Z}/2,2)_+)$
2. MO is the wedge sum of suspensions of the Eilenberg-MacLane spectrum $H\mathbb{F}_2$
3. Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(MO,\mathbb{Z}/2) \otimes H^*(K(\mathbb{Z}/2,2),\mathbb{Z}/2), \mathbb{Z}/2) \Rightarrow \pi_{t-s}(MO \wedge K(\mathbb{Z}/2,2)_+)$$

($\pi_d(MO \wedge K(\mathbb{Z}/2,2)_+)$ has only 2-torsion.)

4. $H^*(K(\mathbb{Z}/2,2),\mathbb{Z}/2) = \mathbb{Z}/2[x_2, x_3, x_5, x_9, \cdots]$
   where $x_3 = Sq^1 x_2$, $x_5 = Sq^2 x_3$, $x_9 = Sq^4 x_5$ $\cdots$
5. $H^*(MO,\mathbb{Z}/2) = \mathcal{A} \otimes N$ where N is the dual vector space of
   $N_* = \mathbb{Z}/2[u_i \mid i > 1 \text{ and } i \ne 2^r - 1] = \pi_* MO = \Omega_*^O$
6. Manifold generators of Ω_*^O

   $u_2 = \mathbb{RP}^2$, $u_4 = \mathbb{RP}^4$, $u_5 = S^1 \times_{\mathbb{Z}} \mathbb{CP}^2$ the Dold manifold or

   $SU(3)/SO(3)$ the Wu manifold, or the real Milnor manifold Y_5 (the same as the manifold generator of Ω_5^{SO} because the bordism invariants for Ω_5^O and Ω_5^{SO} are both $w_2 w_3$)

The bordism invariant for Ω_2^O is w_1^2 or w_2, actually $w_1^2 = w_2$ on 2-manifolds. (By Wu formula, $Sq^2(1) = w_2 + w_1^2 = 0$ on 2-manifolds)

$\mathbb{C}P^2$ is bordant to $\mathbb{R}P^2 \times \mathbb{R}P^2$ in Ω_4^O

The bordism invariants for Ω_4^O are w_1^4 (for $\mathbb{R}P^4$) and w_2^2 (for $\mathbb{R}P^2 \times \mathbb{R}P^2$)

($\because$ The nonzero Stiefel-Whitney numbers of $\mathbb{R}P^4$ are w_1^4 and w_4,

the nonzero Stiefel-Whitney numbers of $\mathbb{R}P^2 \times \mathbb{R}P^2$ are w_2^2 and w_4,

and the bordism invariants α, β for Ω_4^O should satisfy

$\alpha(\mathbb{R}P^4) = 1$, $\alpha(\mathbb{R}P^2 \times \mathbb{R}P^2) = 0$, $\beta(\mathbb{R}P^4) = 0$, $\beta(\mathbb{R}P^2 \times \mathbb{R}P^2) = 1$. $\therefore \alpha = w_1^4, \beta = w_2^2$)

Then 1, w_1^2 (or w_2), w_1^4, w_2^2, $w_2 w_3$, $\cdots$ form a basis of N

$H^*(MO, \mathbb{Z}/2) = \mathcal{A} \otimes N$ = direct sum of suspensions of $\mathcal{A}$ with one copy for each basis elements of N.

$$\therefore H^*(MO, \mathbb{Z}/2) = \mathcal{A} \oplus \Sigma^2 \mathcal{A} \oplus 2\Sigma^4 \mathcal{A} \oplus \Sigma^5 \mathcal{A} \oplus \cdots$$

$$H^*(MO, \mathbb{Z}/2) \otimes H^*(K(\mathbb{Z}/2, 2), \mathbb{Z}/2)$$
$$= (\mathcal{A} \oplus \Sigma^2 \mathcal{A} \oplus 2\Sigma^4 \mathcal{A} \oplus \Sigma^5 \mathcal{A} \oplus \cdots) \otimes \mathbb{Z}/2[x_2, x_3, x_5, x_9, \cdots]$$
$$= \mathcal{A} \oplus \underbrace{2\Sigma^2 \mathcal{A}}_{w_1^2 \text{ (or } w_2) \text{ and } x_2} \oplus \underbrace{\Sigma^3 \mathcal{A}}_{x_3} \oplus \underbrace{4\Sigma^4 \mathcal{A}}_{w_1^4,\, w_2^2,\, w_1^2 x_2,\, x_2^2 \text{ (or } w_2 x_2)} \oplus \underbrace{4\Sigma^5 \mathcal{A}}_{w_2 w_3,\, w_1^2 x_3 \text{ (or } w_2 x_3),\, x_2 x_3,\, x_5} \oplus \cdots$$

Note that by Wu formula, $x_2^2 = Sq^2 x_2 = (w_2 + w_1^2) x_2$ on 4-manifolds,

$x_5 = Sq^2 x_3 = (w_2 + w_1^2) x_3$ on 5-manifolds.

Note that there is a map $f: M \to K(\mathbb{Z}/2, 2)$ in the definition of bordism groups,

$\therefore$ the cohomology classes of $K(\mathbb{Z}/2, 2)$ are pulled back to M.

d	$\Omega_d^O(K(\mathbb{Z}/2,2))$	bordism invariants
0	$\mathbb{Z}/2$	
1	0	
2	$\mathbb{Z}/2^2$	w_1^2 (or w_2), x_2
3	$\mathbb{Z}/2$	x_3
4	$\mathbb{Z}/2^4$	w_1^4, w_2^2, $w_1^2 x_2$, x_2^2
5	$\mathbb{Z}/2^4$	$w_2 w_3$, $w_1^2 x_3$, $x_2 x_3$, x_5

Now we determine the manifold generators of $\Omega_5^O(K(\mathbb{Z}/2,2))$.

Rewrite $x_2 = B$, $x_3 = Sq^1 B$, $x_5 = Sq^2 Sq^1 B$

The manifold generators are pairs (M, B).

$\alpha :=$ the generator of $H^1(\mathbb{RP}^2, \mathbb{Z}/2)$

$\beta :=$ the generator of $H^1(\mathbb{RP}^3, \mathbb{Z}/2)$

$\gamma :=$ the generator of $H^1(S^1, \mathbb{Z}/2)$

$\zeta :=$ the generator of $H^1(\mathbb{RP}^4, \mathbb{Z}/2)$

	$(\mathbb{RP}^2 \times \mathbb{RP}^3, \alpha \cup \beta)$	$(S^1 \times \mathbb{RP}^4, \gamma \cup \zeta)$	$(S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, \gamma \cup a_1)$	$(W, 0)$
$w_2 w_3$	0	0	0	1
$w_1^2 Sq^1 B$	0	1	1	0
$Sq^2 Sq^1 B$	0	1	0	0
$B Sq^1 B$	1	0	0	0

$Sq^1(\alpha \cup \beta) = \alpha^2 \cup \beta + \alpha \cup \beta^2$ $\quad Sq^1(\gamma \cup \zeta) = \gamma \cup \zeta^2$ $\quad Sq^1(\gamma \cup \alpha_1) = \gamma \cup \alpha_1^2$

$Sq^2 Sq^1(\alpha \cup \beta) = 0$ $\quad Sq^2 Sq^1(\gamma \cup \zeta) = \gamma \cup \zeta^4$ $\quad Sq^2 Sq^1(\gamma \cup \alpha_1) = 0$

$w_1(\mathbb{R}P^2 \times \mathbb{R}P^3) = \alpha$ $\quad w_1(S^1 \times \mathbb{R}P^4) = \zeta$ $\quad w_1(S^1 \times \mathbb{R}P^2 \times \mathbb{R}P^2) = \alpha_1 + \alpha_2$

$w_2(\mathbb{R}P^2 \times \mathbb{R}P^3) = \alpha^2$ $\quad w_2(S^1 \times \mathbb{R}P^4) = 0$ $\quad w_2(S^1 \times \mathbb{R}P^2 \times \mathbb{R}P^2) = \alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2$

$w_3(\mathbb{R}P^2 \times \mathbb{R}P^3) = 0$ $\quad w_3(S^1 \times \mathbb{R}P^4) = 0$ $\quad w_3(S^1 \times \mathbb{R}P^2 \times \mathbb{R}P^3) = \alpha_1^2\alpha_2 + \alpha_1\alpha_2^2$

2 $\Omega_d^{SO}(K(\mathbb{Z}/2, 2))$ for $d \leq 5$

Known facts:

1. Pontryagin-Thom isomorphism: $\Omega_d^{SO}(K(\mathbb{Z}/2, 2)) = \pi_d(MSO \wedge K(\mathbb{Z}/2, 2)_+)$
2. $MSO_{(2)}$ is the wedge sum of suspensions of the Eilenberg-MacLane spectra $H\mathbb{F}_2$ and $H\mathbb{Z}_{(2)}$
3. Adams spectral sequence
$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(MSO, \mathbb{Z}/2) \otimes H^*(K(\mathbb{Z}/2, 2), \mathbb{Z}/2), \mathbb{Z}/2) \Rightarrow \pi_{t-s}(MSO \wedge K(\mathbb{Z}/2, 2)_+)$$
($\pi_d(MSO \wedge K(\mathbb{Z}/2, 2)_+)$ has only 2-torsion.)

4. $H^*(H\mathbb{Z}, \mathbb{Z}/2) = \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$, where $\mathcal{A}(0)$ is the subalgebra of $\mathcal{A}$ generated by Sq^1

5. $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathcal{A} \otimes_{\mathcal{A}(0)} M, \mathbb{Z}/2) = \mathrm{Ext}_{\mathcal{A}(0)}^{s,t}(M, \mathbb{Z}/2)$.

6. For $d \leq 5$, $\pi_d(MSO \wedge K(\mathbb{Z}/2, 2)_+) = \pi_d(H\mathbb{Z} \wedge K(\mathbb{Z}/2, 2)_+) \oplus \pi_{d-4}(H\mathbb{Z} \wedge K(\mathbb{Z}/2, 2)_+)$
$\oplus \pi_{d-5}(H\mathbb{F}_2 \wedge K(\mathbb{Z}/2, 2)_+)$

7. $\mathrm{Ext}_{\mathcal{A}(0)}^{s,t}(H^*(K(\mathbb{Z}/2, 2), \mathbb{Z}/2), \mathbb{Z}/2) \Rightarrow \pi_{t-s}(H\mathbb{Z} \wedge K(\mathbb{Z}/2, 2)_+)$

8. $H^*(K(\mathbb{Z}/2,2),\mathbb{Z}/2) = \mathbb{Z}/2[x_2, x_3, x_5, x_9, \cdots]$

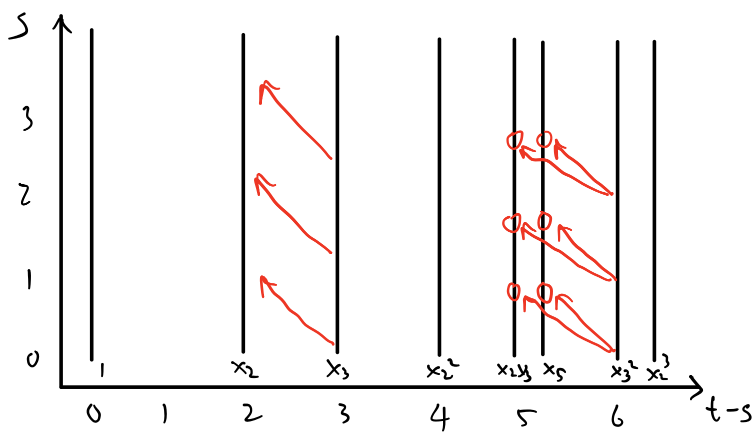
where $x_3 = Sq^1 x_2$, $x_5 = Sq^2 x_3$, $x_9 = Sq^4 x_5$ ...

$Sq^1 x_3 = 0$, $Sq^1(x_2^2) = 0$, $Sq^1(x_2 x_3) = Sq^1(x_5) = x_3^2$

Cartan formula Adem relation

The $A(0)$ module structure of $H^*(K(\mathbb{Z}/2,2),\mathbb{Z}/2)$ is the direct sum of $|Sq^1$

and $\cdot$ $\cdot \to |Sq^1 \to \cdot$ $Ext^{s,t}_{A(0)}(\mathbb{Z}/2, \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & t=s \\ 0 & t\neq s \end{cases}$



E_1 page for the Adams SS : $Ext^{s,t}_{A(0)}(H^*(K(\mathbb{Z}/2,2),\mathbb{Z}/2)) \Rightarrow \pi_{t-s}(H\mathbb{Z} \wedge K(\mathbb{Z}/2,2)_+)$

Prop) If $0 \to M' \to M \to M'' \to 0$, then

$\delta : Hom(M', M') \to Ext^1(M'', M')$

Sends $Id_{M'}$ to the class of the extension.

$M' = \Sigma\mathbb{F}_2$, $M'' = \mathbb{F}_2$, the differential $d_1 = \delta$: $Hom(\Sigma\mathbb{F}_2, \Sigma\mathbb{F}_2) \to Ext^1(\mathbb{F}_2, \Sigma\mathbb{F}_2)$

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maps the top class to $h_0 \cdot$(the bottom class) $\qquad Ext^{1,1}(\mathbb{F}_2, \mathbb{F}_2)$

(on $\Sigma\mathbb{F}_2$) $\qquad$ (on $\mathbb{F}_2$)

There is a differential d_2 corresponding to the Bockstein $\beta_{(2,4)}$ associated to the SES

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/8 \to \mathbb{Z}/4 \to 0$$

Let $P(x_2) \in H^4(K(\mathbb{Z}/2,2), \mathbb{Z}/4)$ be the Pontryagin square of x_2

$$P(x_2) = x_2 \cup x_2 + x_2 \underset{1}{\cup} \delta x_2$$

then $\beta_{(2,4)} P(x_2) = \frac{1}{4}\delta(P(x_2))$

$$= \frac{1}{4}\delta(x_2 \cup x_2 + x_2 \underset{1}{\cup} \delta x_2)$$

$$= \frac{1}{4}(\delta x_2 \cup x_2 + x_2 \cup \delta x_2 + \delta x_2 \underset{1}{\cup} \delta x_2)$$

$$= \frac{1}{4}(2x_2 \cup \delta x_2 + \delta x_2 \underset{1}{\cup} \delta x_2)$$

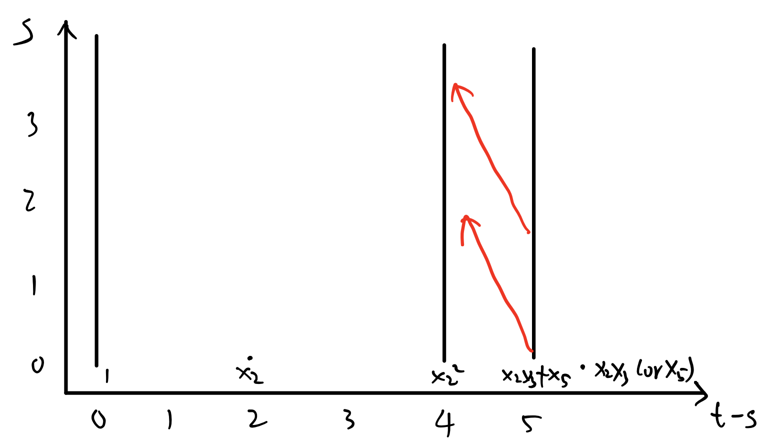
$$= x_2 \cup \tfrac{1}{2}\delta x_2 + \tfrac{1}{2}\delta x_2 \underset{1}{\cup} \tfrac{1}{2}\delta x_2$$

$(\because Sq^1 = \frac{1}{2}\delta \bmod 2)$ $\quad = x_2 \cup Sq^1 x_2 + Sq^1 x_2 \underset{1}{\cup} Sq^1 x_2$

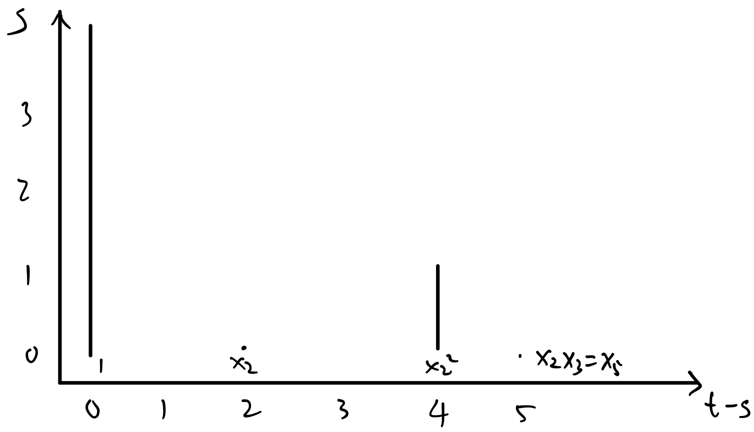
$(\because Sq^2(x_3) = x_3 \underset{1}{\cup} x_3)$ $\quad = x_2 \cup Sq^1 x_2 + Sq^2 Sq^1 x_2$

$$= x_2 x_3 + x_5 \quad \bmod 2$$

$$\therefore\ d_2(x_2 x_3 + x_5) = x_2^2 h_0^2$$



E_2 page for the Adams SS: $\mathrm{Ext}^{s,t}_{A(0)}(H^*(K(\mathbb{Z}/2,2),\mathbb{Z}/2)) \Rightarrow \pi_{t-s}(H\mathbb{Z} \wedge K(\mathbb{Z}/2,2)_+)$



E_∞ page for the Adams SS: $\mathrm{Ext}^{s,t}_{A(0)}(H^*(K(\mathbb{Z}/2,2),\mathbb{Z}/2)) \Rightarrow \pi_{t-s}(H\mathbb{Z}\wedge K(\mathbb{Z}/2,2)_+)$

d	$\Omega^{SO}_d(K(\mathbb{Z}/2,2))$	bordism invariants
0	$\mathbb{Z}$	
1	0	
2	$\mathbb{Z}/2$	x_2
3	0	
4	$\mathbb{Z}\times\mathbb{Z}/4$	Sign, $P(x_2)$
5	$\mathbb{Z}/2^2$	$x_2x_3=x_5$, w_2w_3

Lemma 2 If $\mathbb{Z}_{w_1}$ denotes the orientation local system, then

$$H^1(BO(1),\mathbb{Z}_{w_1})\cong\mathbb{Z}/2$$

Def 3 The pullback of the nonzero element of $H^1(BO(1),\mathbb{Z}_{w_1})$ under the determinant map $B\det: BO(n)\to BO(1)$ is called the twisted first Stiefel-Whitney class $\tilde{w}_1\in H^1(BO(n),\mathbb{Z}_{w_1})$.

Thm 4 Let M be a closed 5-manifold and $B\in H^2(M,\mathbb{Z}/2)$. Then

$$\langle BSq^1_{\mathbb{Z}}B+Sq^2_{\mathbb{Z}}Sq^1_{\mathbb{Z}}B,\ [M]\rangle=\tfrac{1}{2}\langle\tilde{w}_1\cup P(B),[M]\rangle. \qquad (1)$$

$\therefore\ x_2 x_3 + x_5 = \frac{1}{2}\tilde{w}_1 P(x_2) = 0$ on oriented 5-manifolds

In (1), we consider $\tilde{w}_1 \in H^1(M, (\mathbb{Z}/4)_{w_1})$. If $[M]$ denotes the fundamental class in twisted $\mathbb{Z}/4$ cohomology, RHS of (1) is

$$H^1(M,(\mathbb{Z}/4)_{w_1}) \times H^4(M,\mathbb{Z}/4) \xrightarrow{\cup} H^5(M,(\mathbb{Z}/4)_{w_1}) \xrightarrow{\frown[M]} \mathbb{Z}/4 .$$

Since $2\tilde{w}_1 = 0$, $\langle \tilde{w}_1 \cup P(B), [M]\rangle$ is even.

We'll prove Thm 4 in three steps:

1. Prove that both sides of (1) are bordism invariants of $\Omega_5^O(K(\mathbb{Z}/2,2))$ (LHS done!)
2. Determine the manifold generators of $\Omega_5^O(K(\mathbb{Z}/2,2))$ (Done!)
3. Verify (1) on the generators.

Step 1 for RHS:

Prop 5: The quantity $\langle \tilde{w}_1 \cup P(B), [M]\rangle$ is a bordism invariant.

proof: This quantity is additive under disjoint union, so it suffices to show that it vanishes when M bounds. Let (M,B) bound, i.e. M is a closed 5-manifold, $B \in H^2(M,\mathbb{Z}/2)$, and there is a compact manifold W and a $\hat{B} \in H^2(W,\mathbb{Z}/2)$ such that $M = \partial W$ and if $i: M \hookrightarrow W$ is inclusion $B = i^*\hat{B}$. Then, $TW|_M \cong TM \oplus \underline{\mathbb{R}}$, so $i^*\tilde{w}_1(W) = \tilde{w}_1(M)$. By naturality, $i^* P(\hat{B}) = P(B)$. In the LES for (W,M),

$$H^n(W,(\mathbb{Z}/4)_{w_1}) \xrightarrow{i^*} H^n(M,(\mathbb{Z}/4)_{w_1}) \xrightarrow{\delta} H^{n+1}(W,M,(\mathbb{Z}/4)_{w_1}),$$

so $\tilde{w}_1(M)P(B) \in \mathrm{Im}(i^*) = \ker\delta$.

Let $[W,M] \in H_{n+1}(W,M,(\mathbb{Z}/4)_{w_1})$ denote the fundamental class of the pair, and $[M] \in H_n(M, (\mathbb{Z}/4)_{w_1})$ denote the fundamental class. Under the connecting morphism $\partial: H_{n+1}(W, M, (\mathbb{Z}/4)_{w_1}) \to H_n(M, (\mathbb{Z}/4)_{w_1})$, $[W, M] \mapsto [M]$.

If $x \in H^n(M, (\mathbb{Z}/4)_{w_1})$, then

$$\langle x, \partial [W, M] \rangle = \langle \delta x, [W, M] \rangle.$$

Hence

$$\langle \widetilde{w_1}(M) P(B), [M] \rangle = \langle \widetilde{w_1}(M) P(B), \partial [W, M] \rangle = \langle \delta(\widetilde{w_1}(M) P(B)), [W, M] \rangle = 0 \quad \square$$

Step 3:

Prop 6: If $a, b \in H^1(X, \mathbb{Z}/2)$, then

$$P(ab) = \Theta(ab^3 + a^3 b)$$

where $\Theta: H^*(X, \mathbb{Z}/2) \to H^*(X, \mathbb{Z}/4)$ is induced by the multiplication by 2 map $(\cdot 2): \mathbb{Z}/2 \to \mathbb{Z}/4$

Now (1) is easily verified. $\square$