§2 A sketch of Vojta's proof of Faltings' Theorem:

Recall that A is an abelian variety defined over a number field K. Mordell-Weil Theorem states that $A(K)$ is a finitely generated abelian group. Set $A(K)_{tors} = A_{tors} \cap A(K)$.

<u>Lemma 1.18</u>: Let r denote the rank of $A(K)$. Then there is $P_1, \dots, P_r \in A(K)$ such that $A(K) = A(K)_{tors} \oplus \langle P_1, \dots, P_r \rangle$.

<u>Proof</u>: It is well-known that every abelian group G with rank r is the direct sum of its torsion part (the subgroup of G containing all its elements of finite order), which is finite, and its torsion-free part, which is a group generated by r elements. Thus, $A(K) = (A(K) \cap A_{tors}) \oplus \langle P_1, \dots, P_r \rangle$ for some $P_1, \dots, P_r \in A(K)$. □

The goal of this paragraph is to provide a sketchy proof of Faltings' Theorem.

<u>Theorem 1.19 (Faltings, 1983)</u>: Let C/K be a smooth, geometrically irreducible, projective curve of genus at least 2. Then $C(K)$ is finite.

Let C be a smooth and projective curve defined over a number field F. Suppose that $C(F) \neq \emptyset$ and choose $P_0 \in F$. Then we can embed C in its Jacobian J_C (In addition, we can find an embedding which maps P_0 to O). In particular, $C(F) \subset J_C(F)$.

<u>Remark 1.20</u>: In 2021, Dimitrov, Gao and Habegger proved a uniform version of Faltings' Theorem: Let $g \geq 2$ and $d \geq 1$ be integers. Then there is a constant $c(g,d)$, depending only on g and d, with the following property: If C is a smooth, geometrically irreducible, projective curve of genus g defined over a number field F of degree at most d over $\mathbb{Q}$, then $\# C(F) \leq c(d,g)^{1+r}$, where r is the rank of $J_C(F)$. For a proof, see "Uniformity in Mordell-Lang for curves".

We now fix a smooth, geometrically irreducible, projective curve C defined over K. If $C(K) = \emptyset$, then there is nothing to do. So assume that $C(K) \neq \emptyset$ and fix $P_0 \in C(K)$. We then fix a K-embedding $C \hookrightarrow J_C$, which maps P_0 to O.

It allows us to see $C(K)$ as a subset of $J_C(K) = J_C(K)_{tors} \oplus \langle P_1, \dots, P_r \rangle$ for some $P_1, \dots, P_r \in J_C(K)$ according to Lemma 1.18. Recall that $J_C(K)_{tors}$ is a finite group and that $\langle P_1,\dots,P_r\rangle$ is torsion-free of rank r.

Basically, the first part of Vojta's proof consists in reducing it to the case that $C(K)\subset\langle P_1,\dots,P_r\rangle$. For an arbitrary integer $M\geq 1$ and a subset $S\subset J_C(K)$, we set $[M]S$ to be the set $\{[M]P, P\in S\}$. We now put N to be the cardinality of the group $J_C(K)_{tors}$. Note that $[N]C(K)\subset[N]J_C(K)=[N](J_C(K)_{tors}\oplus\langle P_1,\dots,P_r\rangle)=[N]\langle P_1,\dots,P_r\rangle=\langle [N]P_1,\dots,[N]P_r\rangle$. Moreover, we have:

Lemma 1.21: $C(K)$ is finite if and only if $[N]C(K)$ is finite.

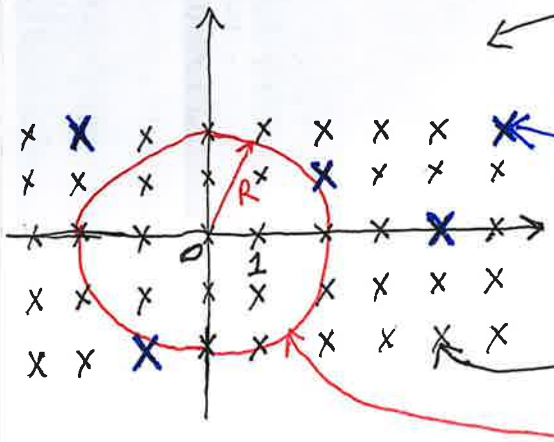
Proof: The direction $\Rightarrow$ is trivial. Let us show $\Leftarrow$. Write $[N]C(K)=\{[N]Q_1,\dots,[N]Q_r\}$ with $Q_1,\dots,Q_r\in C(K)$ pairwise distinct. Now, put $C(K)=\{Q_1,\dots,Q_r,Q_{r+1},\dots,Q_m,\dots\}$. By construction, for all $i\geq r+1$, there is $j\in\{1,\dots,r\}$ such that $[N]Q_i=[N]Q_j$, leading to $Q_i=Q_j+T_{ij}$ for some $T_{ij}\in(J_C)_{tors}$. As $Q_i,Q_j\in C(K)$, which is a subset of $J_C(K)$, we get $T_{ij}\in(J_C)_{tors}\cap J_C(K)=J_C(K)_{tors}$. Thus $C(K)$ is a subset of $\{Q_j+T, j\in\{1,\dots,r\}, T\in J_C(K)_{tors}\}$, which is finite (and with cardinality at most r^M). □.

For brevity, put $Q_i=[N]P_i$ for all $i\in\{1,\dots,r\}$. It is obvious that $\langle Q_1,\dots,Q_r\rangle=[N]\langle P_1,\dots,P_r\rangle$ is torsion-free since $\langle P_1,\dots,P_r\rangle$ is. Moreover, it has rank r, that is, $\langle Q_1,\dots,Q_r\rangle\simeq\mathbb{Z}^r$, which implies $\langle Q_1,\dots,Q_r\rangle\otimes_{\mathbb{Z}}\mathbb{R}\simeq\mathbb{R}^r$. On the other one hand, $J_C(K)\otimes\mathbb{R}=J_C(K)/J_C(K)_{tors}\otimes_{\mathbb{Z}}\mathbb{R}$ according to the arguments in Remark 1.15. As $J_C(K)=J_C(K)_{tors}\oplus\langle P_1,\dots,P_r\rangle$, we infer that

$$J_C(K)\otimes_{\mathbb{Z}}\mathbb{R}=\langle P_1,\dots,P_r\rangle\otimes_{\mathbb{Z}}\mathbb{R}\simeq\mathbb{Z}^r\otimes_{\mathbb{Z}}\mathbb{R}=\mathbb{R}^r\simeq\langle Q_1,\dots,Q_r\rangle\otimes_{\mathbb{Z}}\mathbb{R}.$$

We now fix a symmetric very ample line bundle $\mathcal{L}$ over J_C. Theorem 1.14 claims that $\sqrt{\hat{h}_{\mathcal{L}}}: J_C(\bar{K})\to\mathbb{R}_{\geq 0}$ uniquely extends to a norm on $J_C(\bar{K})\otimes_{\mathbb{Z}}\mathbb{R}$. By restriction, it induces a norm on $J_C(K)\otimes_{\mathbb{Z}}\mathbb{R}$, that is, on $\langle Q_1,\dots,Q_r\rangle\otimes_{\mathbb{Z}}\mathbb{R}=\mathbb{R}^r$.

We can summarize all these informations into a single drawing.



$\mathbb{R}^n = \mathcal{J}_C(K) \otimes_{\mathbb{Z}} \mathbb{R} = \langle Q_1,\dots,Q_n\rangle \otimes_{\mathbb{Z}} \mathbb{R}$

$[N]C(K) \subset \langle Q_1,\dots,Q_n\rangle \subset \mathcal{J}_C(K)$

$\mathbb{Z}^n = \langle Q_1,\dots,Q_n\rangle = \langle Q_1,\dots,Q_n\rangle \otimes_{\mathbb{Z}} \mathbb{Z}$

$D(0,R) = \{P \in \mathbb{R}^n,\ \sqrt{\hat{h}_{\mathcal{L}}(P)} \leq R\}.$

Faltings' Theorem can be rephrased as follows: The number of blue crosses is finite. Let us start with a trivial observation.

Lemma 1.22: For all $R>0$, the intersection $[N]C(K) \cap D(0,R)$ is finite.

Proof: Trivial from the drawing since in an Euclidian space, the intersection of a disc with a lattice is finite. But a drawing is not a proof. Obviously, $[N]C(K) \subset \mathcal{J}_C(K)$, so $[K(P):K]=1$ for all $P \in [N]C(K)$. Then, by definition, $D(0,R) = \{P \in \mathcal{J}_C(K) \otimes_{\mathbb{Z}} \mathbb{R},\ \hat{h}_{\mathcal{L}}(P) \leq R^2\}$. Consequently, $[N]C(K) \cap D(0,R) \subseteq \{P \in \mathcal{J}_C(K),\ [K(P):K] \leq 1 \text{ and } \hat{h}_{\mathcal{L}}(P) \leq R^2\}$, which is finite by Northcott's Theorem. $\square$

Remark 1.23: This proof shows that Northcott's Theorem is a broad generalization of the fact that in a finite-dimensional vector space, the intersection of a lattice with a compact set is finite.

To get Faltings' Theorem, it remains to prove the existence of a real $R_0>0$ such that $[N]C(K) \setminus D(0,R_0)$ is finite, that is, the number of blue crosses outside $D(0,R_0)$ is finite. The next idea comes from Mumford in 1965: splitting up $\mathbb{R}^n$ into a lot of small cones. Recall that $\langle \cdot,\cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ denotes the inner product induced by the norm $\sqrt{\hat{h}_{\mathcal{L}}} : \mathcal{J}_C(K) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathbb{R}$.

Definition 1.24: Two points $P, Q \in \mathcal{J}_C(K) \otimes_{\mathbb{Z}} \mathbb{R}$ are said to have angle θ if $\cos\theta = \dfrac{\langle P, Q\rangle}{\sqrt{\hat{h}_{\mathcal{L}}(P)\,\hat{h}_{\mathcal{L}}(Q)}}$.

Definition 1.25: A set of the form $\left\{x \in \mathcal{J}_C(K) \otimes_{\mathbb{Z}} \mathbb{R},\ \langle x, a\rangle \geq \cos(\theta) \sqrt{\hat{h}_{\mathcal{L}}(x)\,\hat{h}_{\mathcal{L}}(a)}\right\}$ is called a cone with center O, angle θ and axis through $a \in E$, and we denote it by $\mathcal{C}_{\theta,a}$.

Remark 1.26: If $x \in \mathcal{C}_{\theta,a}$, then $\lambda x \in \mathcal{C}_{\theta,a}$ for all real numbers $\lambda \geq 0$. This is obvious since
$\langle \lambda x, a\rangle = \lambda\langle x,a\rangle \geq \lambda\cos(\theta)\sqrt{\hat{h}_{\mathcal{L}}(x)\,\hat{h}_{\mathcal{L}}(a)} = \cos(\theta)\sqrt{\hat{h}_{\mathcal{L}}(\lambda x)\,\hat{h}_{\mathcal{L}}(a)}$ since $\sqrt{\hat{h}_{\mathcal{L}}}$ is a norm on $J_C(K)\otimes_{\mathbb{Z}}\mathbb{R}$.

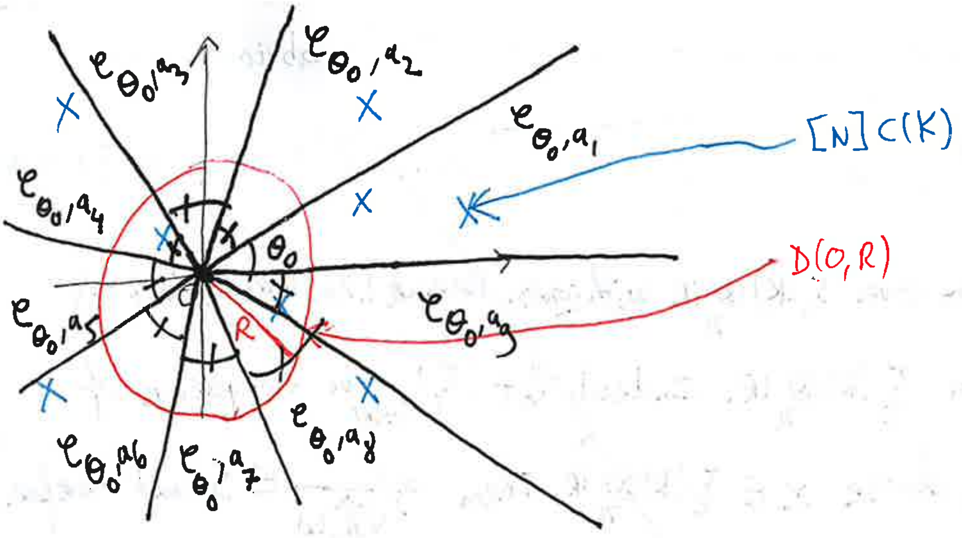
As said previously, Mumford's idea is to cover $J_C(K)\otimes_{\mathbb{Z}}\mathbb{R}$ with cones. Remark 1.26 tells us that it's enough to cover the unit sphere $\mathcal{S}$ in $J_C(K)\otimes_{\mathbb{Z}}\mathbb{R}$. Indeed, let $(\mathcal{C}_i)_i$ be a sequence of cones with center O covering $\mathcal{S}$. Now, choose $x \in J_C(K)\otimes_{\mathbb{Z}}\mathbb{R}$. Then $\dfrac{x}{\sqrt{\hat{h}_{\mathcal{L}}(x)}} \in \mathcal{S}$, and therefore lies in $\mathcal{C}_i$ for some i. Now, Remark 1.26 claims that $x \in \mathcal{C}_i$, and so $(\mathcal{C}_i)_i$ covers $J_C(K)\otimes_{\mathbb{Z}}\mathbb{R}$.

However, to get the finitness of $[N]C(K)$, it is relevant to cover $\mathcal{S}$ with finitely many cones. Let us see how to do it: Let $\theta \in]0;\pi[$ and $a \in J_C(K)\otimes_{\mathbb{Z}}\mathbb{R}$. Then the volume of the cone $\mathcal{C}_{\theta,a}$, intersected with the unit ball $\mathcal{B}$ of $J_C(K)\otimes_{\mathbb{Z}}\mathbb{R}$, depends only on θ (its formula is

$$\frac{\mathrm{vol}(\text{unit ball in } \mathbb{R}^{r-1})}{2}\cdot\left[\frac{2}{r-1}\sin^{r-2}(\theta/2)\cos(\theta/2) + B\left(\tfrac{1}{2},\tfrac{r}{2}\right) - B\left(\cos^2(\theta/2);\tfrac{1}{2},\tfrac{r}{2}\right)\right],$$

where $B(a,b)$ and $B(x;a,b)$ denote the beta function and the incomplete beta function). Thus, given $\theta \in]0;\pi[$, it is always possible to cover $\mathcal{S}$ with at most $N_\theta := \left\lfloor \dfrac{\mathrm{vol}(\mathcal{B})}{\mathrm{vol}(\mathcal{C}_{\theta,a}\cap\mathcal{B})}\right\rfloor + 1$ cones with center O and angle θ.

We now fix $\theta_0 \in]0;\frac{\pi}{2}[$ such that $\cos(\theta_0) \in]\frac{1}{g};1[$ (Recall that by assumption, C is a curve of genus $g \geq 2$). We also fix N_{θ_0} cones $\mathcal{C}_{\theta_0,a_1},\dots,\mathcal{C}_{\theta_0,a_{N_\theta}}$ covering $\mathcal{S}$. Recall that the set $[N]C(K)\cap D(0,R)$ is finite for all $R>0$. Thus, to get Faltings' Theorem, it suffices to prove the existence of a real number $R_0>0$ such that the intersection $[N]C(K)\cap\left(\mathcal{C}_{\theta_0,a_i}\setminus D(0,R_0)\right)$ is finite for all $i \in \{1,\dots,N_\theta\}$.



We can now state, without proof, what showed Mumford and Vojta from this drawing.

Theorem 1.27 (Mumford, 1965, and Vojta '90): Let $i \in \{1,\dots,N_\theta\}$. Then there are constants $R_i > 0$, $\lambda_i > 1$ and $X_i > 1$ such that the following holds: For all $P, Q \in [N]C(K) \cap \left(\mathcal{C}_{\Theta_0,a_i} \setminus D(0,R_i)\right)$ distincts with $\sqrt{\hat{h}_{\mathcal{L}}(Q)} \geq \sqrt{\hat{h}_{\mathcal{L}}(P)}$, we have:

① (Mumford's gap principle) $\quad \lambda_i \sqrt{\hat{h}_{\mathcal{L}}(P)} \leq \sqrt{\hat{h}_{\mathcal{L}}(Q)}$

② (Vojta) $\quad \sqrt{\hat{h}_{\mathcal{L}}(Q)} \leq X_i \cdot \sqrt{\hat{h}_{\mathcal{L}}(P)}$.

Proof: For ①, resp. ②, see Bombieri-Gubler "Heights in Diophantine Geometry", Section 5.4, resp. Chapter 11. □

Mumford's gap principle means that the set $[N]C(K)$ is sparse, that is, two distinct points of $[N]C(K)$ cannot be arbitrary close from each other. A contrario, Vojta proved that two distinct points in $[N]C(K)$ cannot be arbitrary far from each other. This is possible only if $[N]C(K)$ is finite.

Proof of Faltings' Theorem: Let $i \in \{1,\dots,N_\theta\}$ and set $[N]C(K) \cap \left(\mathcal{C}_{\Theta_0,a_i} \setminus D(0,R_i)\right) = \{R_1,\dots,R_n,\dots\}$. We order them so that $\sqrt{\hat{h}_{\mathcal{L}}(R_n)} \leq \sqrt{\hat{h}_{\mathcal{L}}(R_{n+1})}$ for all $n \geq 1$. By Mumford's gap principle, we have $\sqrt{\hat{h}_{\mathcal{L}}(R_{n+1})} \geq \lambda_i \sqrt{\hat{h}_{\mathcal{L}}(R_n)}$ and an easy induction leads to $\sqrt{\hat{h}_{\mathcal{L}}(R_{n+1})} \geq \lambda_i^n \sqrt{\hat{h}_{\mathcal{L}}(R_1)}$. Then Vojta's Theorem claims that $\sqrt{\hat{h}_{\mathcal{L}}(R_{n+1})} \leq X_i \sqrt{\hat{h}_{\mathcal{L}}(R_1)}$ for all $n \geq 1$. In conclusion,

$\frac{x_i}{\lambda_i^n} \geq 1$ for all n. As $\lambda_i > 1$, we infer that $n \leq \frac{\log x_i}{\log \lambda_i}$ and Faltings' Theorem follows since the set

$$[N]C(K) \cap \left(\mathcal{C}_{\theta_0, a_i} \setminus D(0, R_i) \right) = \{Q_1, \dots, Q_2, \dots\} \text{ is finite.}$$

$\blacksquare$

§3. A very short story.

The original proof of Faltings' Theorem is entirely different from that of Vojta. Nonetheless, to show it, Faltings invented a whole new height (the Faltings height). Inspired by Vojta's proof, Faltings proved the following generalization of the Mordell Conjecture.

Theorem 1.28 (Faltings' big Theorem, 1991): Let X be a geometrically irreducible closed subvariety of A, which is not a translate of an abelian subvariety over $\bar{K}$. Then $X \cap A(K)$ is not Zariski dense in X. $\square$.

Proof: See Faltings, "Diophantine approximation on abelian varieties".

Remark 1.29: This story shows how important is it to give several different proofs of a same result. On the one hand, Faltings proved Shafarevich conjecture in order to deduce the first proof of Mordell Conjecture. On the other one hand, thanks to Vojta's proof of Mordell Conjecture, Faltings was able to prove another important theorem in Diophantine geometry.

§4. A proof of Mordell-Weil Theorem:

In this paragraph, we will prove that $A(K)$ is a finitely generated abelian group. It is obvious that it's an abelian group, so we only show the "finitely generated" part.

The first step of the proof, and the only one involving heights, is to reduce the proof to that of the so-called weak Mordell-Weil Theorem: The quotient group $A(K)/[2]A(K)$ is finite.

Theorem 1.30: The group $A(K)$ is finitely generated if and only if $A(K)/[2]A(K)$ is finite.

Proof: $\Rightarrow$