Lecture 7. The Markov theorem

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Markov's theorem. Formulation

Let's recall the formulation.

Theorem 1.1 (Markov's theorem)

The closures of braids A and B are isotopic if and only if B can be obtained from A by a sequence of the following moves (Markov moves):

1. *conjugation $b \to a^{-1}ba$ by an arbitrary braid a with the same number of strands as b,*
2. *the move $b \to b\sigma_n^{\pm 1}$, where b is a braid on n strands and the obtained braid has $n+1$ strands,*
3. *the inverse transformation of 2.*

Markov moves

The necessity of these two moves is evident. The isotopies between corresponding pairs of braid closures are shown in Fig. 1.
In Fig. 1.b the first Reidemeister move comes into play. This move did not take part in braid isotopies, so this kind of knot isotopy appears here.

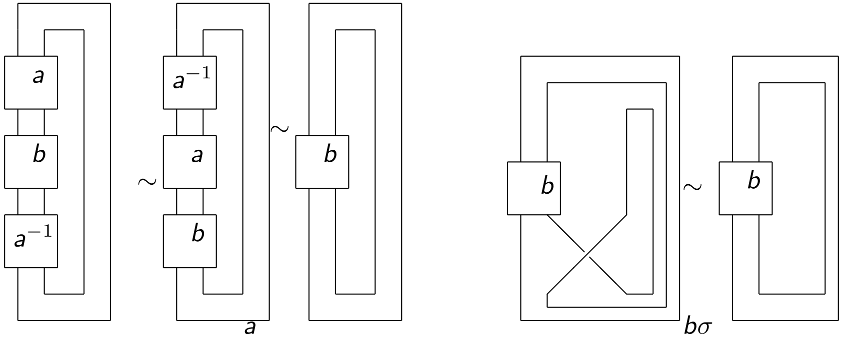

Figure 1: Two Markov moves represent isotopy.

Introduction

In the present talk, we shall give a proof of Markov's theorem by using L-moves.

Originally L-moves appeared in the paper by Lambropoulou and Rourke [LR], and then they were very intensively explored for virtual braids and many other braid groups (Lambropoulou, Kauffman and others).

Main idea

The main idea of the L-move method of the proof of Markov's theorem is as follows. We try to make a braid with infinite ends.
Good news is that one can construct a braid diagram out of link hence proving the Alexander theorem and then, when analysing the ambiguity in the construction and the effect of Reidemeister's moves on the initial braid diagram, one gets the Markov moves.

L-moves

Definition 2.1

An *L-move* on a braid consists of cutting one arc of the braid open and splicing into the broken strand new strands to the top and bottom, both either *under* or *over* the rest of the braid.

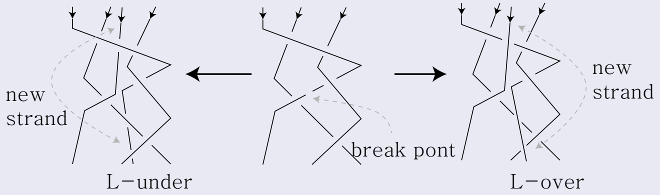

Figure 2: L-moves.

L-moves and isotopy generate an equivalence on braids called *L-equivalence*. Our goal is to show that L-equivalence classes of braids are in bijective correspondence with isotopy classes of oriented links in S^3 where the bijection is induced by "closing" the braid to form a link. As a consequence, L-equivalence is the same as the usual Markov equivalence, and thus we get the *classical Markov's theorem*.

The proofs are based on a canonical process for turning a combinatorial oriented link diagram in the plane (with a little extra structure) into an open braid. Our braiding as well as the *L-moves* are based on building blocks of *combinatorial isotopy*, the *triangle moves* or Δ*-moves*.
We use the point at the infinity as the reference point for braiding. The change of reference point to infinity makes the proof of the Markov theorem easy.

The starting point

There are two combinatorial moves on diagrams which we shall consider:

1. Δ-*move*: An arc is replaced by two arcs forming a triangle (and its inverse) respecting orientation and crossings. "Respecting crossings" mean that, if we lift the diagram to an embedding in 3-space, then the Δ-move lifts to an elementary isotopy move.
2. Subdivision: A vertex is introduced/deleted in an arc of the diagram.

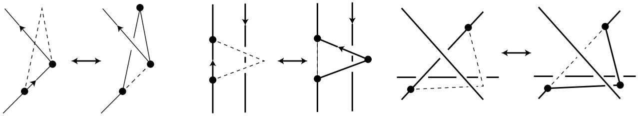

Figure 3: Reidemeister moves corresponding to a Δ-move.

Subdivision

Subdivision moves can be viewed as special cases of Δ-moves. We shall call the equivalence relation generated by these two moves a *combinatorial isotopy* or just an *isotopy*.
As we know, this notion of isotopy is equivalent to the standard one when we pass to the smooth category. We left the verification of this statement as an exercise.

Definition of L-moves

Definition 2.2 (L-moves.)

Let D be a link diagram/braid and P a point of an arc of D such that P is not vertically aligned with any of the crossings or (other) vertices of D (note that P itself may be a vertex). Then we can perform the following operation: Cut the arc at P, bend the two resulting smaller arcs apart slightly by a small isotopy and introduce two new vertical arcs to new top and bottom end-points in the same vertical line as P. The new arcs are both oriented downwards and they run either both *under* or both *over* all other arcs of the diagram. Thus there are two types of L-moves, an *under L-move or L_u-move* and an *over L-move or L_o-move*.

This gives the following algebraic expression for an L_o-move and an L_u-move respectively.

$$\alpha = \alpha_1\alpha_2 \sim \sigma_i^{-1} \dots \sigma_n^{-1}\tilde{\alpha_1}\sigma_{i-1}^{-1} \dots \sigma_{n-1}^{-1} \dots \sigma_i\tilde{\alpha_2}\sigma_n \dots \sigma_i$$

$$\alpha = \alpha_1\alpha_2 \sim \sigma_i \dots \sigma_n\tilde{\alpha_1}\sigma_{i-1} \dots \sigma_{n-1} \dots \sigma_i^{-1}\tilde{\alpha_2}\sigma_n^{-1} \dots \sigma_i^{-1}$$

where α_1, α_2 are elements of B_n and $\tilde{\alpha_1}, \tilde{\alpha_2} \in B_{n+1}$ are obtained from α_1, α_2 by replacing each σ_j by σ_{j+1} for $j = i, \dots, n-1$.

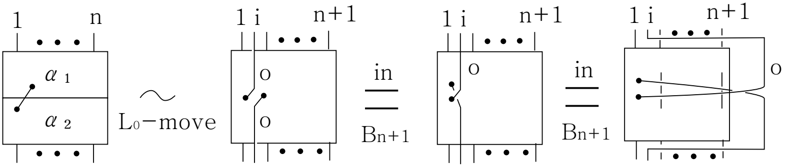

Figure 4: L-moves in terms of braid generators.

One-move Markov's theorem

Theorem 2.3

The closure of braids induces a bijection between the set of L-equivalence classes of bradis and the set of isotopy types of (oriented) link diagrams.

The braiding process

We shall define an inverse bijection to $\mathcal{C}$ by means of a canonical *braiding process* which turns an oriented link diagram (with little extra structure) into a braid. Note that we only work with oriented diagrams. Let D be a link diagram with no horisontal arcs, and consider the arcs in D which slope upwards with respect to their orientations; call these arcs *opposite arcs*. In order to obtain a braid from that diagram we want:

1. to keep the arcs that go downwards;
2. to eliminate the opposite arcs and produce braid strands instead.

The braiding process: continuation

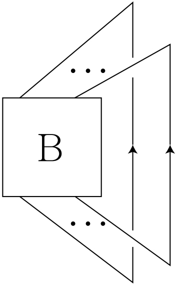

Figure 5: The closure of B.

Triangle condition

Triangle condition

Non-adjacent sliding triangles are only allowed to meet if they are of opposite types.

Lemma 2.4

Given a link diagram D, there is a subdivision D' of D such that (for appropriate choices of under/over for free up-arcs) the triangle condition is satisfied.

Generic diagram and generic Δ-move

Definition 2.5

A generic diagram is a link diagram with subdividing points and sliding triangles put in general position with respect to the height function, such that the following conditions hold:

1. there are no horisontal arcs,
2. no two disjoint subdividing points are in vertical alignment, where by 'disjoint' we mean subdividing points that do not share a common edge.
3. any two non-adjacent sliding triangles satisfy the triangle condition and if they intersect, this should be along a common interior (and not a single point).

Definition 2.6

A generic Δ-move is a Δ-move between generic diagrams.

Lemma 2.7

An isotopy between generic link diagrams can be realised using only generic Δ-moves.

Proof. If after some Δ-move during the isotopy appears a horisontal arc or vertical alignment of vertices we remove it by replacing one of the participating vertices by a point arbitrarily close to it, so that the new point will not cause such a singularity or violation of condition 3 in all the diagrams of the isotopy chain. If two non-adjacent sliding triangles touch so that condition 3 is violated (Fig. 6) we argue as above.

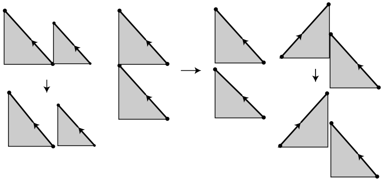

Figure 6:

Proof of Lemma 2.7 (continued)

The remaining possibilities are the ones illustrated in Fig. 7 and they can be removed. The proof is completed.

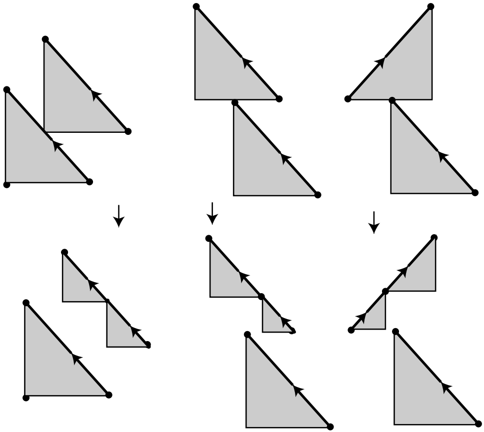

Figure 7:Making Δ-moves generic for the braid.

Alexander's theorem

Corollary 2.8 (Alexander's theorem)

Any (oriented) link diagram is isotopic to the closure of a braid.

By the local nature of the Δ-moves we may assume that for a given link diagram we have done the braiding for all up-arcs except for the ones that we are interested in every time; these will be lying in the "magnified" region placed inside the braid

Now, two braids that differ by a finite sequence of L-moves have isotopic closures. Therefore the function C from L-equivalence classes of braids to isotopy types of link diagrams is well-defined. To show that C is a bijection we shall use our braiding process to define an inverse function B. Namely, for a diagram D let $B(D)$ be the braid resulting from the braiding algorithm applied to it. We have to show that B is a well-defined function from link-diagram types to L-equivalence classes of braids, therefore we have to check that $B(D)$ does not depend up to L-equivalence on the choices made before the braiding and on Δ-moves between link diagrams.

Lemma 3.1

If we add on an up-arc, α, an extra subdividing point P and label the two new up-arcs, α_1 and α_2, the same as α, the corresponding braids are L-equivalent.

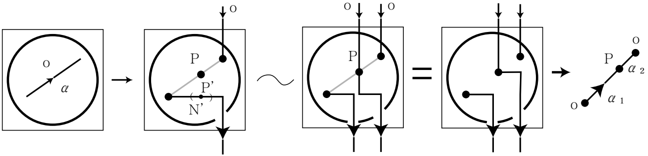

Figure 8:

Lemma 3.2

When we meet a free up-arc, which we have the choice of labelling 'u' or 'o', the resulting braid does not depend – up to L-equivalence – on this choice.

Corollary 3.3

If we have a chain of overlapping sliding triangles of free up–arcs so that we have a free choice of labelling for the whole chain then, by Lemmas 3.1 and 3.2, this choice does not affect – up to L-equivalence – the final braid.

Corollary 3.4

If by adding a subdividing point on an up–arc we have a choice for relabelling the resulting new up–arcs so that the triangle condition is still satisfied then, by Lemmas 3.1 and 3.2, the resulting braids are L-equivalent.

Corollary 3.5

Given any two subdivisions, S_1 and S_2, of a diagram which will satisfy the triangle condition with appropriate labellings, the resulting braids are L-equivalent.

Contents

Markov's theorem after Morton

In the previous section we studied the proof of Markov's theorem, which is suggested by S. Lambropoulou and C. Rourke.
In the present section let us study another proof of Markov's theorem suggested by H. Morton in [Mor1]. To prove Markov's theorem Morton uses the original idea of *threading* — an alternative way of representing a link as a closure of a braid.

Remark 4.1

We shall consider each closure of braids as a set of curves inside the cylinder not intersecting its axis; the *axis* of the braid is the closure of the curve coinciding with the axis of the cylinder inside the cylinder.

Definition 4.2

Let K be an oriented link in $\mathbb{R}^3$. Let L be an unknotted curve. We say that K is *braided with respect to* L or $K \cup L$ is a *braid–link* , if K and L represent the closure of some braid and the axis of this braid, respectively (i.e., K lies inside the full torus $S^1 \times D$, where the coordinate of S^1 is increasing, and L is the axis of the full torus); see Fig. 9.

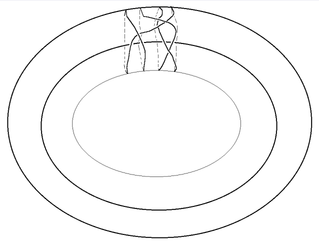

Figure 9: Representing a braid in a full torus.

Having a planar diagram of some braid closure, the corresponding braid–link can be obtained from it by threading this diagram by a circle; see Fig. 10.

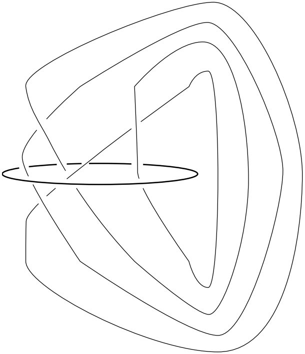

Figure 10: A threaded braid closure

Let K be a planar diagram of some oriented link. Consider some curve L on the projection plane P of the link K such that the curve L intersects the projection of the link K transversely and does not pass through crossings of K.

Definition 4.3

A *choice of overpasses* for a link diagram K is a union of two sets $S = \{s_1, \dots, s_k\}, F = \{f_1, \dots, f_k\}$ of points at the edges of K (points should not coincide with crossings) such that while passing along the orientation of K, the points from S alternate with points from F; besides, each interval $[s, f]$ does not contain undercrossings and each $[f, s]$ does not contain overcrossings; i.e. $[s, f]$ are arcs and $[f, s]$ are lower arcs.

Definition 4.4

We say that a curve L whose projection on the plane of the link K is a simple curve *threads* K *according to a given choice* (S, F) *of overpasses* if the interval of K goes over L when it starts in a domain containing points from S, and it passes under L if it starts in a domain containing points from S, as it is shown in Fig. 11.

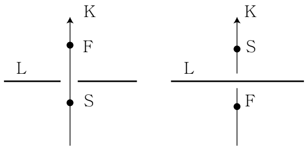

Figure 11: Crossings with L.

Remark 4.5

We do not require that this interval of L contain elements of the set S or F.

For a given link diagram K and a curve L on the projection plane, such that L separates points from S from points from F then we can arrange over– and undercrossings at intersection points between K and L in such a way that the curve L' obtained from L threads the link K.

Let us now prove the following theorem.

Theorem 4.6

If L threads the link K then K is a braid with respect to L.

Proof. Let us choose the overpasses (S, F) for the diagram of the link K (in an arbitrary way) and some curve L on the projection plane P of the diagram K such that L separates points from S from points belonging to F. Let us straighten the curve L in the plane P by a homeomorphism of P onto itself: we require that the transformed L is a straight line inside a domain D, containing K; L should be closed outside D (say, by a large half–circle). Consequently, points of S lie on one side of this line, and points of F lie on the other side. Such a transformation is shown in Fig. 12.

Proof of Theorem 4.6 (continued)

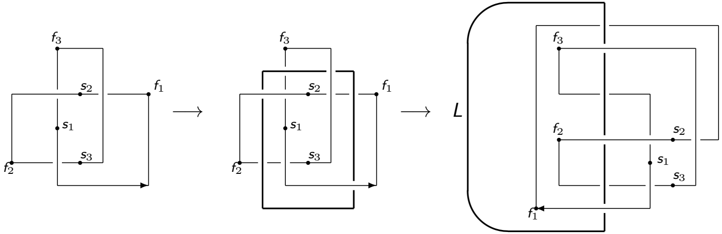

Figure 12: Straightening the curve L.

Without loss of generality, we can suppose that all under– and overcrossings of the diagram L lie in two planes parallel to P (just over and under the images of the corresponding projections).

Proof of Theorem 4.6 (continued)

Now, let us change the point of view and think of P as the plane Oxz and L as the axis Oz that is closed far away from the origin of coordinates.

Let us consider the line L (without its "infinite" circular part) as the axis of cylindrical coordinates. Then the plane P is divided into two half–planes; one of them is given by the equation $\{\theta = 0\}$ and the other one satisfies the equation $\{\theta = \pi\}$. Here the half–plane $z = 0, x > 0$ is thought to have coordinate $\theta = 0$; points over this half-plane are thought to have positive coordinates.

Let us construct a link isotopic to K as follows. Place all lower arcs of K (i.e., all intervals $[f, s]$) on the half planes $\{\theta = -\varepsilon\}$ and $\{\theta = \pi + \varepsilon\}$, and all arcs on the half–planes $\{\theta = \varepsilon\}$ and $\{\theta = \pi - \varepsilon\}$, where ε is small enough. Herewith, we shall add small intervals over all points belonging to S or F such that each interval is projected to one point on Oxz.

Proof of Theorem 4.6 (continued)

Let us represent the arcs where K intersects with L by vertical arcs. Thus, we have made the polar coordinate θ of the link K to be always constant or increasing.

In Figs. 13 and 14 we show how to construct a knot with non-decreasing polar coordinate. This knot is isotopic to the knot shown in Fig. 12.

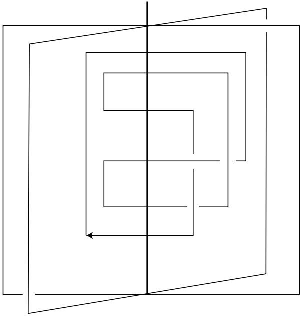

Figure 13: Link diagram intersecting L.

Proof of Theorem 4.6 (continued)

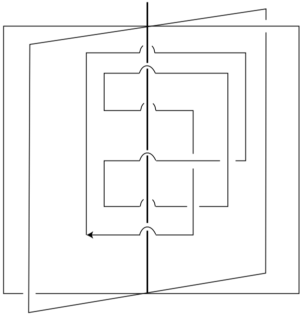

Figure 14: The polar coordinate is increasing while moving along the link.

After a small deformation of the obtained link, we can make this coordinate strictly monotonic.
Thus, the transformed link (which we shall also denote by K) will represent a braid with respect to L. $\square$

Theorem 4.7

Each closure K of any braid B admits a threading by some curve L in such a way that K is a braid with respect to L.

Proof. Let $D^2 \times I$ be a cylinder. Consider B as a braid connecting points lying on the upper base of the cylinder with points on the lower base of the same cylinder. Now, let us close the braid as follows. Connect the lower points with the upper ones by lines, going horisontally along the bases and vertically at some discrete moments, as shown in Fig. 15.

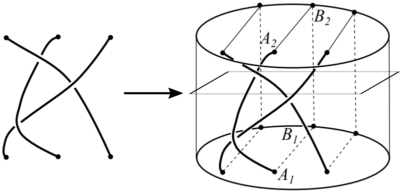

Figure 15: A braid and a braided link.

Proof of Theorem 4.7 (continued)

Let us apply the isotopy that straightens the strands and changes homeomorphically the upper base of the cylinder. Thus we obtain a link that admits a simple threading that can be constructed as follows. Let h_0 be the height level of the lower base and h_1 be the level of the upper base. Denote the set of lower ends of the braid B by A_1 and the set of upper ones by A_2. One can assume that the levels h_0 and h_1 contain some additional sets of vertices B_1 and B_2 by means of which we are going to construct the closure of the braid. More precisely, the points from A_1 are connected by parallel lines with points from A_2, and points from B_1 are connected by parallel lines with points from B_2. Now, let us consider the circle lying on the plane at the level $h = \frac{h_1+h_2}{2}$ and separating sets of lines A_1A_2 and B_1B_2.
Let us project the diagram on the base of the cylinder and take the set A_1 as S and A_2 as F.
It is easy to see that in this case the projection of the circle is really a threading of the link. □

Markov's theorem

Theorems 4.6 and 4.7 imply the Alexander theorem; the proofs of these theorems give us a concrete algorithm (different from Alexander's and Vogel's methods) to represent any link as a closure of a braid.
Let us remind Markov's theorem.

Theorem 4.8 (Markov's theorem)

The closures of braids A and B are isotopic if and only if B can be obtained from A by a sequence of the following moves (Markov moves):

1. *conjugation $b \to a^{-1}ba$ by an arbitrary braid a with the same number of strands as b,*
2. *the move $b \to b\sigma_n^{\pm 1}$, where b is a braid on n strands and the obtained braid has $n+1$ strands,*
3. *the inverse transformation of 2.*

Let us now reformulate the difficult part of the Markov's theorem.
To do it, we shall need some definitions.

Definition 4.9

We say that two braided links $K \cup L$ and $K' \cup L'$ are *simply Markov equivalent* if there exists an isotopy of the second one, taking L' to L and K' to the link coinciding with K everywhere except one arc. The link K contains an arc α and K' contains a link α' with the same ends.

1. The polar coordinate is constant on the arc α and monotonically increasing on α'.
2. The arcs α, α' bound a disc intersecting L transversely at a unique point.

Definition 4.10

Two braided links are *Markov equivalent* if one of them can be transformed to the other by a sequence of isotopies and simple Markov equivalences.

Lemma 4.11

If links $K \cup L$ and $K' \cup L'$ are simply Markov equivalent then they represent threaded closures of braids, which are isotopic to some braids $\beta \in Br(n)$ and $\beta\sigma_n^{\pm 1} \in Br(n+1)$.

Proof. Suppose the polar coordinate evaluated at points of the curve α equals θ_0. Consider the arc α_0. Without loss of generality, one can assume that the coordinate is almost everywhere equal to θ_0 and in some small neighbourhood of L' the arc α_0 makes a loop and this loop corresponds to the n–th (last) strand of the braid α. We can isotope the braids K and K' in the neighbourhood $\{\alpha = \theta_0 \pm \varepsilon\}$ in such a way that the final points of the arcs α and α_0 lie in a small neighbourhood of L'. The remaining part of the Lemma is now evident. □

Exercise 4.12

Show that closures of two n–strand braids are isotopic in the class of closures of n–strand braids if and only if these two braids are conjugated.

Lemma 4.11 together with Exercise 4.12 allows us to reformulate the difficult part of the Markov's theorem as follows:

Theorem 4.13 (Markov's theorem)

Let β and γ be two braids whose closures B and Γ are isotopic as oriented links. Let us thread these closures and obtain some braided links B' and Γ'. Then B' and Γ' are Markov equivalent.

Lemma 4.14

Consider an oriented link diagram K on the plane P and fix the choice of overpasses (S, F). Then the threadings of K by different curves L and L' separating the sets S and F are Markov equivalent.

Proof. The main idea of the proof is the following. First we consider the case when the curves L and L' are isotopic in the complement $P \backslash (S \cup F)$. In this case, one of them can be transformed to the other by means of moves in such a way that each of these moves is a Markov equivalence.
In the common case we shall use one extra move when two branches of the line L pass through some point from S (or F). Such a move is a Markov equivalence as well (this will be clear from the definition).

Proof of Lemma 4.14 (continued)

Let us give the proof in more detail.
The case a).
Suppose L and L' are isotopic in $P\backslash(S \cup F)$. Then $K \cup L$ and $K \cup L'$ can be obtained from each other by a sequence of transformations of the first and the second type, shown in Fig. 16.

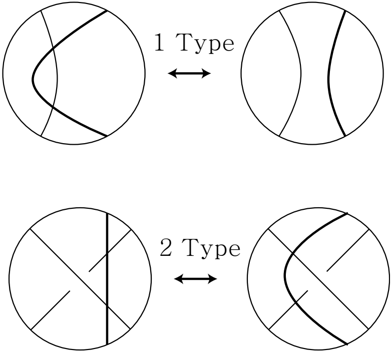

Figure 16: Types of transformations.

Proof of Lemma 4.14 (continued)

The first type is represented either by the second Reidemeister move or by the "hooking" move, that adds two crossings in alternating order. It will be shown below that this move is a simple Markov equivalence.
The second type of transformation is an isotopy in all cases except that shown in Fig. 17.

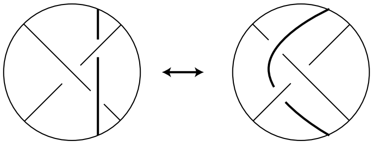

Figure 17: Nonisotopic transformation of type II.

Proof of Lemma 4.14 (continued)

In this case, the first threading can be transformed to the second one by a sequence of moves of the first type and isotopies; see Fig. 18.

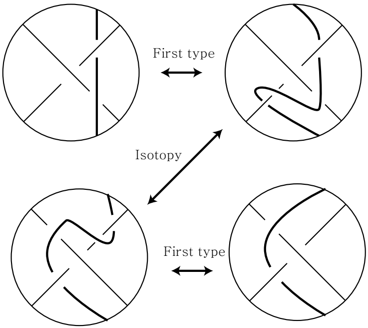

Figure 18: Transformation of type II reduces to transformations of type I.

Proof of Lemma 4.14 (continued)

Here one should note that the passes of the diagram of K under the line L are alternating with passes of K over L while going along the link K. It remains to show that the two threadings obtained from each other by a transformation of the first type are simply Markov equivalent.

Thus, the part of the link K shown in Fig. 18 belongs either to an upper branch or to the lower branch. In the threading construction we can assume that both parts of the link K on the same side of the line L lie on one and the same level p_L (it might be either $\{\theta = \pi \pm \epsilon\}$ or $\{\theta = \pm\epsilon\}$ depending on the side of overcrossing).

Proof of Lemma 4.14 (continued)

Let us connect them by an arc as shown in Fig. 19.

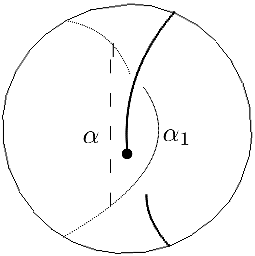

Figure 19: Transformation of type I is a simple Markov equivalence.

Now, the arcs α and α' bound a disc. Thus we obtain a simple Markov equivalence of the two threadings.

Proof of Lemma 4.14 (continued)

The case b).

In the general case, note that if the curve M' of P that separates the set S from the set F can be isotoped to the curve M by means of moving the two arcs of the link K through some point of S or F (as shown in Fig. 20) then M and M' represent isotopic threadings.

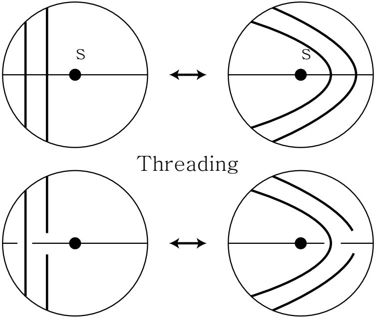

Figure 20: Moving two arcs.

Proof of Lemma 4.14 (continued)

Such an isotopy of the "curvilinear" line L (or, equivalently, motions of points from the sets S and F) is divided into several steps. Between these steps, we apply discrete moves changing the combinatorial type of the disposition for L with respect to the sets S and F. Thus, one can consider the discrete set of such dispositions, between two of each some elementary transformation takes place.

Proof of Lemma 4.14 (continued)

Without loss of generality, we may assume the following.
Let S and F consist of points $\{(-1, a_i)\}$ and $\{(1, a_i)\}$ for some $a_1, \ldots a_k$, respectively. Let L be a part of Oy closed by a large half–circle in such a way that the interior of L contains F. We may assume that K is parallel to Ox near each of the points $s_1, \ldots, s_k$.
Let L' be any simple closed curve separating S from F and restricting a domain that contains the set F. Without loss of generality, we can suppose that the curve L' intersects the rays $y = a_i, x < -1$ transversely; see Fig.21.

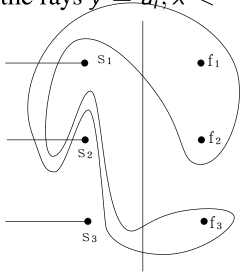

Figure 21: Curve L'

Proof of Lemma 4.14 (continued)

Let us enumerate these intersections according to the decreasing of the abscissa. For each of the rays, the number of intersections is even. For each ray, let us group them pairwise: $(1, 2), (3, 4), \ldots$. To do this, we first isotope L' inside $P \backslash (S \bigcup F)$ in such a way that between each pair of points there are no crossings of the diagram K.

Proof of Lemma 4.14 (continued)

Now, let us move the curve line L' to the right in such a way that after performing the operation all points lie on the left side of the curve L. Let us divide such a transformation into stages when L' does not contain points from S and moments when L' does. In the first case, such a transformation is a Markov equivalence as in the case a). In the second case, let us assume that the intersection points of L' with each ray disappear pairwise; i.e., the curve L' consequently passes two crossings with the same point s_i. This move is a Markov equivalence as well; see Fig. 20.

Thus, the threading by means of L' is Markov equivalent to the threading by the curve lying on the right hand related from all s_i. This curve is isotopic to L inside the set $P \backslash (S \bigcup F)$; i.e., the threading by means of such curve is Markov equivalent to the threading by means of L, see a). Consequently, the threading by L' is Markov equivalent to the threading by L. This completes the proof of the lemma.

Lemma 4.15

Given a choice (S, F) of overpasses for a link diagram K, and a point s in K, not belonging to F, then there exists a choice of overpasses $(\bar{S}, \bar{F})$ such that $s \in \bar{S}, S \subset \bar{S}, F \subset \bar{F}$.

Proof. The idea of the proof is pretty simple: we add elements of S or F where we want compensating them by corresponding elements of F or S. If s lies on an upper arc of (S, F) then one can choose f just before s with respect to the orientation of K; thus the interval $[f, s] \subset K$ contains no overcrossings. In the case when s lies on a lower arc, we can add f just after s; in this case $[s, f] \subset K$ contains no undercrossings.

Theorem 4.16

Each two threadings $K \cup L$ and $K \cup L'$ of the same diagram K are Markov equivalent.

Proof. Let us choose some overpasses (S, F) for the threading $K \cup L$ and (S', F') for the threading $K \cup L'$. According to Lemma 4.15, there exists a choice of overpasses $(S^{''}, F^{''})$ such that $(S, F), (S', F') \subset (S^{''}, F^{''})$, and the two threadings with this choice of overpasses, the first of which is Markov equivalent to $K \cup L$ and the second is Markov equivalent to $K \cup L'$. By Lemma 4.14, these two threadings are Markov equivalent. Thus, the initial two threadings are Markov equivalent and this completes the proof.

Theorem 4.17

Any two planar diagrams of isotopic links have Markov-equivalent threadings.

Proof. To prove this theorem, we have to show how to construct Markov equivalent threadings for diagrams obtained from each other by using Reidemeister moves.
By Theorem 4.16, we can take any choice of overpasses for each of these diagrams. The idea is to be able to reconstruct the choice of overpasses together with L after each Reidemeister move.

Proof of Theorem 4.17 (continued)

Without loss of generality, for the first two Reidemeister moves we can choose the separating curve outside the small disc of the move (in the case of Ω_1 we choose one vertex s inside the disc; in the case of Ω_2 all vertices from (S, F) are outside the disc).
We have shown that the link diagrams obtained from each other by Ω_1 or Ω_2 obtain Markov equivalent threadings.
For Ω_3 we can take one vertex s and one vertex f inside the disc and all the other vertices outside the disc; see Fig. 22.

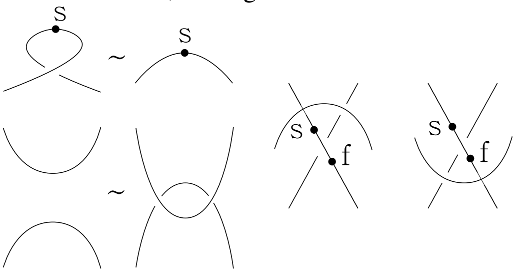

Figure 22: Choice of overpasses for Reidemeister moves.

Proof of Theorem 4.17 (continued)

In Fig. 23 we show that the threading corresponding to diagrams obtained from each other by Ω_3 are isotopic and hence Markov equivalent (we show only one case, the other cases of Ω_3, with orientation and disposition of L and K in the left picture are quite analogous).

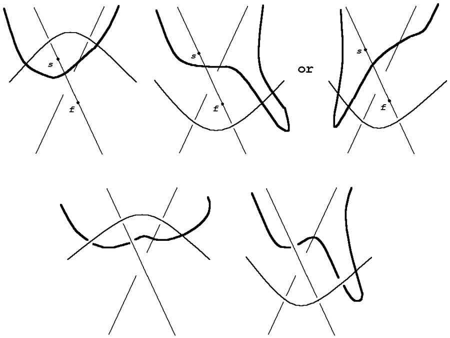

Figure 23: Isotopic threadings of diagrams differ by Ω_3.

Proof of Theorem 4.17 (continued)

In the upper part of Fig. 23, we show how the line L can be transformed with respect to this move Ω_3. In the lower part, we show how one concrete transformation is realised with over– and undercrossings between L and K. □

Proof of Markov's theorem

Now, we are ready to prove the difficult part of the Markov theorem.

Proof of Markov's theorem.

Let $K \cup L$, $K' \cup L'$ be two braided links whence K and K' are isotopic as links.
By Theorem 4.7, the link $K \cup L$ is a threading of some diagram K and the link $K' \cup L'$ is a threading of some diagram K'. By Theorem 4.17, one can choose Markov–equivalent threadings for the first and the second diagram.
By Theorem 4.16, the first one is Markov equivalent to $K \cup L$ and the second one is Markov equivalent to $K' \cup L'$.
Consequently, the threading $K \cup L$ is Markov equivalent to $K' \cup L'$, which completes the proof of Markov's theorem.

Let us now present an example of how to use the Markov's theorem.
As we have proved before, for each two coprime numbers p and q, the toric knots of types (p, q) and (q, p) are isotopic. Let us demonstrate the Markov moves for the braids, whose closures represent trefoils: $(2, 3)$, $(3, 2)$.

Example 4.18

Actually, the first braid has two strands and is given by σ_1^{-3}; the second one (which has three strands) is given by $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$. Let us write down a sequence of Markov moves transforming the first braid to the second one:

$$\sigma_1^{-3} \overset{2\ move.}{\longrightarrow} \sigma_1^{-3}\sigma_2^{-1} \overset{conj.}{\longrightarrow} \sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-1}$$

$$= \sigma_1^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}) \overset{braid\ isotopy}{\longrightarrow} \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}.$$

Exercises

1. Show that closures of two n–strand braids are isotopic in the class of closures of n–strand braids if and only if these two braids are conjugated.
2. Perform Markov moves and braid isotopy to show the following two torus knots are equivalent:
   1. $T(2, 2n+1)$ and $T(2n+1, 2)$
   2. $T(3, 4)$, $T(4, 3)$.
   3. $T(p, q)$, $T(q, p)$.

Research problems:

How to construct braids whose plat closure is the given link by using l-moves
What about 3-manifolds (Kurlin)?
Ask Lambropoulou: are there any invariants of 3-manifolds coming from L-moves
How to construct invariants for links by using Morton's construction?
How to construct a map from braids to knots (Rogozhkin)?

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