Lecture 2. The Kauffman bracket and the Jones polynomial

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Definition 1.1

Regular moves are the Reidemeister moves Ω_2, Ω_3 and their inverses. Then, if we can obtain a knot diagram D' from D by applying regular moves, then we say D and D' are *regularly equivalent*.

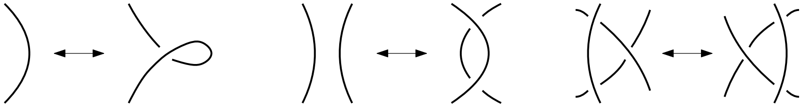


Figure 1: Reidemeister moves Ω_1, Ω_2, Ω_3

Definition 1.2

Let D be an oriented link diagram.

There are two types of classical crossings and we assign $+1$ or -1 for each classical crossing as described in Fig. 2.

We call it *the sign of the crossing c* and denote by sign(c).

Then $\mathrm{w(D)} = \Sigma_{\mathrm{c\,:\,crossings\ in\ D}}\mathrm{sign(c)}$ is called *the writhe of D*.

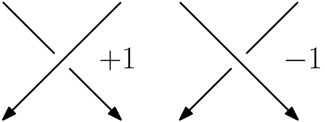


Figure 2: Positive and negative crossings

Definition 1.3

Let K be an unoriented knot (or link) and D is a link diagram of K. *Splice* each crossing point c of D in the two ways as shown in Fig. 3.

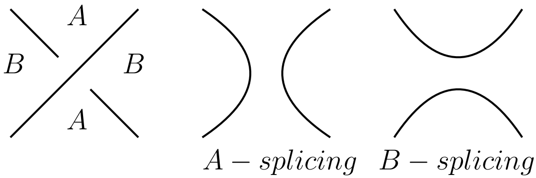
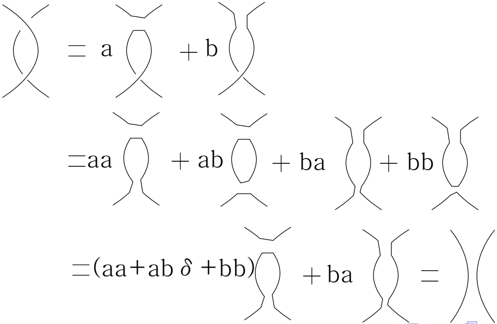


Figure 3: Splicings of a crossing c

Let us try to construct an invariant $\langle \cdot \rangle$ by using splicing of crossings, which satisfies the following conditions:

- $\langle \mathrm{D_c} \rangle = \mathrm{a} \langle \mathrm{D_A} \rangle + \mathrm{b} \langle \mathrm{D_B} \rangle$,
- $\langle \mathrm{L} \sqcup \bigcirc \rangle = \delta \cdot \langle \mathrm{L} \rangle$.

For $\langle \cdot \rangle$ to be an invariant under the second Reidemeister move, $\mathrm{ab} = 1$ and $\mathrm{aa} + \mathrm{bb} + \mathrm{ab}\delta = 0$, see Fig. 6, that is $\mathrm{b} = \mathrm{a}^{-1}$ and $\langle \mathrm{L} \sqcup \bigcirc \rangle = (-\mathrm{a}^2 - \mathrm{a}^{-2}) \langle \mathrm{L} \rangle$. It follows that $\langle \cdot \rangle$ is invariant under RM3.



Kauffman trick

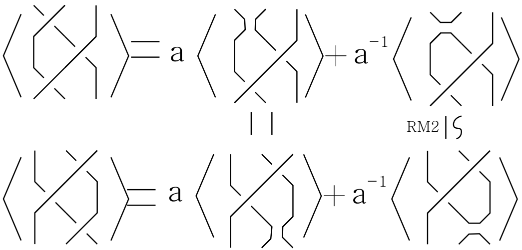


Figure 4: Invariant under RM3 : Kauffman trick

Theorem 1.4 ([4])

Let D be an unoriented knot diagram of a knot or link K. Then there exists a unique one-variable integer polynomial $\langle D \rangle$ satisfying the following conditions:

1. $\langle A \rangle$ is invariant under regular moves.
2. If D is the diagram without crossings, then $\langle D \rangle = 1$
3. If D consists of two split link diagrams D_1 and D_2, i.e. $D = D_1 \sqcup D_2$, then
$$\langle D \rangle = -(a^2 + a^{-2}) \langle D_1 \rangle \langle D_2 \rangle$$
4. Let D_A and D_B be knot diagrams obtained from D by A- and B-splicings for a classical crossing c, respectively. Then the following equation holds:
$$\langle D \rangle = a \langle D_A \rangle + a^{-1} \langle D_B \rangle .$$

Example

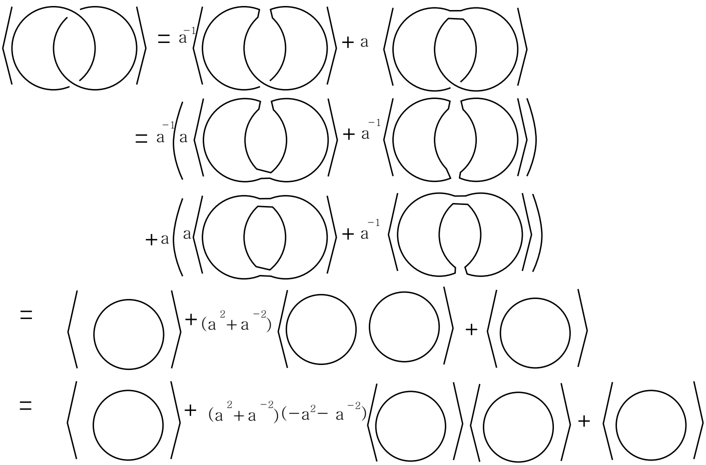


Figure 5: $\langle \text{Hopf link} \rangle = -a^4 - a^{-4}$

Remark 1.5

Note that $\langle |D| \rangle$ is not invariant under the first Reidemeister move. But $\langle |D| \rangle$ can be an invariant by normalization of $\langle |D| \rangle$ as the following theorem.

Theorem 1.6

Let D be an oriented knot diagram of a knot or link K. Let $|D|$ be an unoriented knot diagram forgetting the orientation of D. Then the polynomial $X(D) = (-a)^{-3w(D)} \langle |D| \rangle$, where $w(D)$ is the writhe of D, is an invariant of oriented links. This invariant is called the Jones polynomial.

This is called the Jones polynomial.

Definition 2.1

Let D be an unoriented non-split knot diagram.

$$span(\langle D\rangle) = \mathrm{maxdeg_a} \langle \mathrm{D}\rangle - \mathrm{mindeg_a} \langle \mathrm{D}\rangle .$$

Lemma 2.2

Let D be a connected knot diagram with n crossings. Then

$$\mathrm{span}(\langle \mathrm{D}\rangle) \leq 4\mathrm{n}.$$

Theorem 2.3

Let D be an alternating irreducible non-split knot diagram with n crossings. Then

$$\mathrm{span}(\langle \mathrm{D}\rangle) = 4\mathrm{n}.$$

Tait's conjectures

Theorem 2.4 (First Tait's conjecture)

Let K be a link. Let D be an alternating irreducible non-split knot diagram with n crossings. Then the crossing number of K is n. That is, D is the minimal diagram of K.

Proof.

Let D be an alternating irreducible non-split knot diagram with n crossings. From Theorem 2.3 it follows that $\mathrm{span}(\langle D\rangle) = 4n$. We will show that every diagram, which is obtained from D by RMs, has more than or equal to n crossings. Suppose a diagram D' with n' crossings is obtained from D by RMs. Since $\mathrm{span}(\langle\cdot\rangle)$ is an invariant under RMs, we obtain that

$$\mathrm{span}(\langle D\rangle) = 4n = \mathrm{span}(\langle D'\rangle) \leq 4n',$$

and the proof is completed. □

Tait's conjectures : continue

Definition 2.1 (Alternating Knot (link) diagram)

A link diagram is called *alternating* if while moving along each component, one passes overcrossings and undercrossings alternately.

Tait's conjectures : continue

Theorem 2.5 (Second Tait's conjecture)

Suppose D_1 and D_2 are two reduced alternating diagrams of an alternating knot (or link) K, then $w(D_1) = w(D_2)$.

Indeed, the Tait' s first conjecture are proved by L.H. Kauffman ([3]), K. Murasugi ([5],[6]) and M.B. Thistlethwaite ([7],[1]).

Theorem 2.6 (Third Tait's conjecture, [2],[3])

Suppose D_1 and D_2 are two reduced alternating diagrams of an alternating knot K. Then we can change D_1 into D_2 by performing a finite number of flypes, shown in Fig. 6.

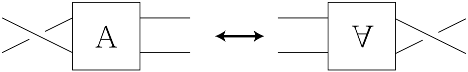


Figure 6: Flypes

Definition 2.7

Let s(D) be the diagram obtained from D by the state s. Denote the state of *A-splicing (B-splicing) f*or all crossing by s_a (s_b). Let $|s(D)|$ be the number of components of s(D).

Definition 2.8 (Adequate)

A link diagram D is called adequate if $|s_a(D)| > s(D)$ and $|s_b(D)| > s(D)$ for all state $s \neq s_a, s_b$.

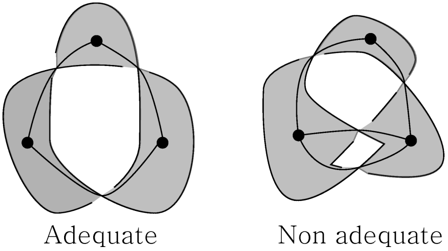


Figure 7: Adequate and non-adequate knot diagrams

Lemma 2.9

For a diagram D,

$$\operatorname{maxdeg}_a \langle D \rangle \leq c(D) + 2|s_a(D)| - 2,$$

$$\operatorname{mindeg}_a \langle D \rangle \geq -c(D) - 2|s_b(D)| + 2.$$

In particular, if D is adequate, then the equalities hold.

Corollary 2.10

Let K be a link and D a diagram of K. If D is adequate, then it is the minimal diagram of K.

Lemma 2.11

Every alternating irreducible non-split knot diagram is adequate.

We leave the proof of Lemma 2.11 as an exercise.

Mutation

Definition 2.12

Suppose that a knot K can be decomposed as in Fig. 8. Then K′ in Fig. 8 is called a mutation of K.

Lemma 2.13

The Kauffman bracket cannot distinguish mutations of a knot.

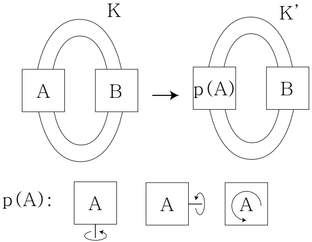


Figure 8: K′ is a mutant knot of K

Connected sum

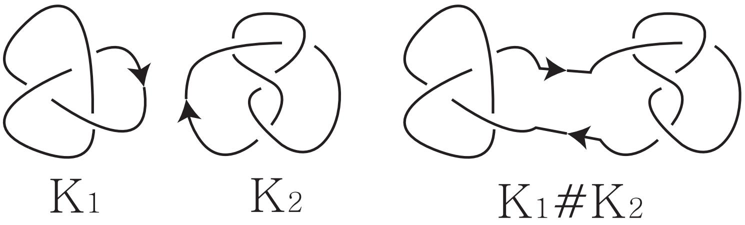


Figure 9: The connected sum $K_1\sharp K_2$ of two oriented knot diagrams K_1 and K_2

Satellite knot

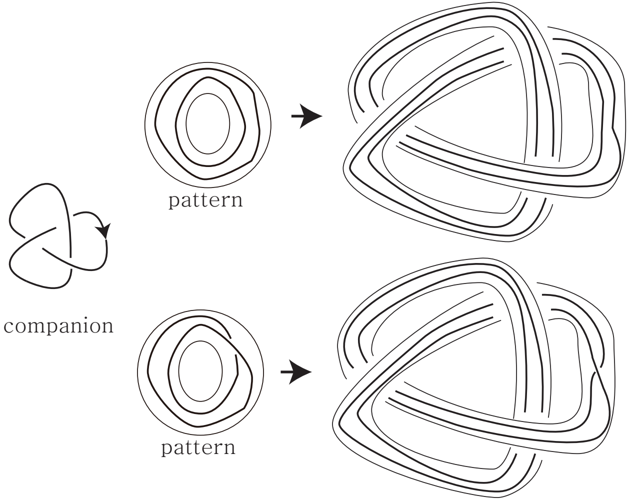


Figure 10: The satellite knot obtained from the given companion and pattern

Exercises

1. Calculate the writhe of following links:
   - left-handed and right-handed Hopf links
   - left-handed and right-handed trefoils
   - figure-eight knot
   - Whitehead link
   - Borromean link
2. Calculate the Kauffman bracket and the Jones polynomial for the following links:
   - left-handed and right-handed Hopf links
   - left-handed and right-handed trefoils
   - figure-eight knot
   - Whitehead link
   - Borromean link
3. Let D' be an oriented diagram obtained from D by applying an increasing RM1 once. Verify that $\langle D' \rangle = (-a)^{\pm 3} \langle D \rangle$.

4. Verify that the value of Kauffman bracket of the mirror image of a link L is obtained from the value of Kauffman bracket of L by replacing $a \to a^{-1}$.
5. Prove that $X(-K) = X(K)$.
6. Prove that if $K_1 \sqcup K_2$ is a split link, then $X(K_1 \sharp K_2) = (-a^2 - a^{-2}) X(K_1) \cdot X(K_2)$.
7. Prove that $X(K_1 \sharp K_2) = X(K_1) \cdot X(K_2)$, where $\sharp$ is a connected sum of knots.
8. Prove that values of Kauffman bracket are not equal to zero.

9. Show that the Jones polynomial satisfies the following relation:

$$a^{-4}X(L_+) - a^4X(L_-) = (a^2 - a^{-2})X(L_0),$$

where L_+,L_- and L_0 are three oriented link diagrams shown in Fig. 11.

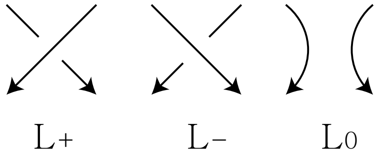


Figure 11: The skein triple L_+, L_-, L_0

10. Show that the Kauffman bracket cannot distinguish mutations of knots.
11. (Difficult) Prove that adequate link diagrams are minimal by using the Jones polynomial only. (Hint. Try to use satellite links and estimate the span of the Jones polynomial of satellite links.)

A research problem

Consider a colouring of a knot K by finitely many colours (say, p colours with respect to the rule c = 2a − b). Now, let us try to find invariants of coloured knots by associating some values to coloure knot diagrams as follows. For each knot diagram K where all arcs are coloured as above, we consider elements A_{ij} of some ring as follows:

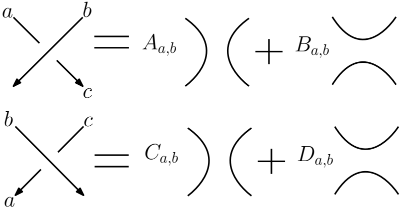


Figure 12: Coloured bracket

Question: How can we make a bracket polynomial stronger in the above manner?

Unsolved problem

- *Can the Jones polynomial distinguish trivial links?*
- *Does the Jones polynomial detect the unknot?*

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