Lecture 1: Reidemeister moves, colouring invariant, linking number

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今日唐诗

夜宿山寺

李白

危楼高百尺，

手可摘星辰。

不敢高声语，

恐惊天上人。

Definition 1.1

By a *knot* we mean a smooth embedding of the circle S^1 in $\mathbb{R}^3$ (or in the sphere S^3) as well as the image of this embedding.

Definition 1.2

Two knots K_1 and K_2 are *equivalent*, if there exists an orientation preserving diffeomorphism of $\mathbb{R}^3$ (or S^3) to itself which maps K_1 to K_2.

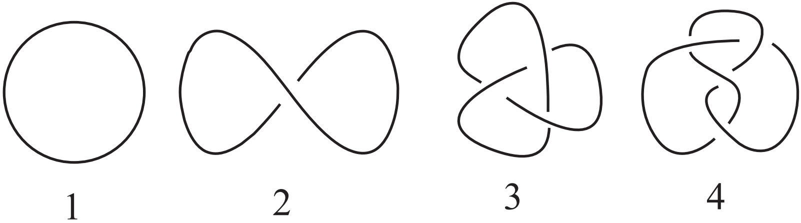


Fig. 1: Examples of knots

Remark 1.3

In knot theory we consider "*smooth*" embeddings of circles. If we consider just embeddings of circles in $\mathbb{R}^3$, then we meet with interesting knots, which is called wild knots, depicted in Fig. 2.

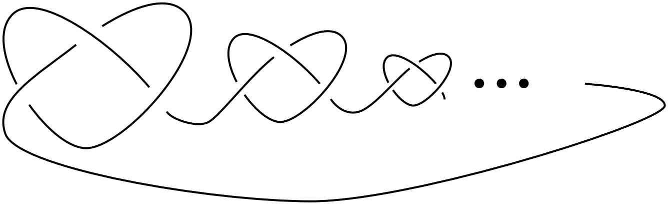


Fig. 2: A wild knot

Definition 1.4

A *link* is a finite ordered collection of knots $K_1 \sqcup \cdots \sqcup K_n$ which do not intersect each other, equivalently, a link is a smooth embedding of $S^1 \sqcup \cdots \sqcup S^1$ in $\mathbb{R}^3$. Each knot K_i is called a component of the link.

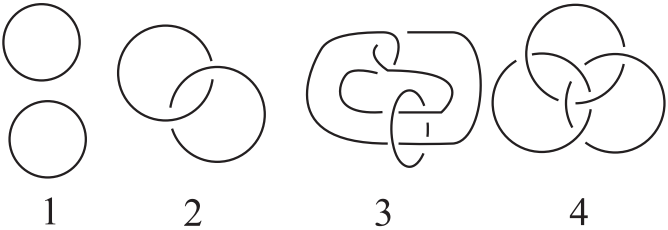


Fig. 3: Example of links

Definition 1.5

Two links $L = K_1 \sqcup \cdots \sqcup K_m$ and $L' = {K'}_1 \sqcup \cdots \sqcup {K'}_m$ are *equivalent* if the following two conditions hold:

1. $m = n$
2. There exists an orientation-preserving diffeomorphism of $\mathbb{R}^3$ that maps K_i to K'_i for each $i \in \{1, \ldots, n\}$.

Definition 1.6

For a link $L \in \mathbb{R}^3$ consider a projection of L on a plane along a vector v in general position. The image of the projection is called a *shadow*. For each double point of the shadow we can give over-under information with respect to v as described in Fig. 4. We call a shadow with over-under structure a knot diagram.

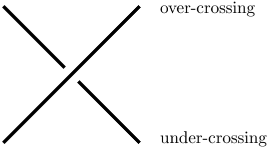


Fig. 4: Classical crossing of a diagram

Theorem 1.7

Two oriented diagrams D_1 and D_2 of smooth links generate *equivalent links* if and only if D_1 can be transformed into D_2 by using a finite sequence of planar isotopy and the three Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$, shown in Fig. 5.

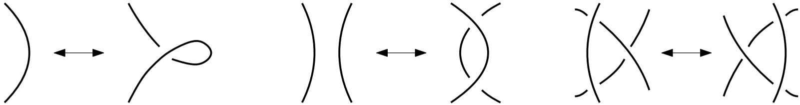


Fig. 5: Reidemeister moves Ω_1, Ω_2, Ω_3

Oriented link

Definition 1.8

An *oriented knot (or link)* is a smooth embedding of an oriented circle (or disjoint union of oriented circles) in $\mathbb{R}^3$. Roughly speaking, each component of a link is oriented.

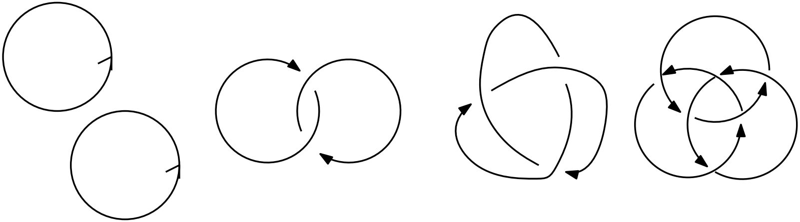


Fig. 6: Oriented links

Theorem 1.9

Two oriented diagrams D_1 and D_2 of smooth links generate *equivalent links* if and only if D_1 can be transformed into D_2 by using a finite sequence of planar isotopy and the three Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$, shown in Fig. 5.

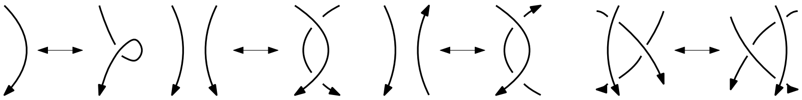
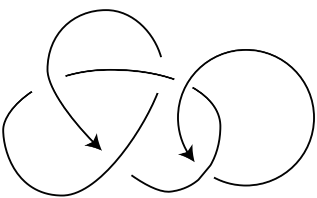


Fig. 7: Oriented Reidemeister moves Ω_1, Ω_2, Ω_2', Ω_3

Orientation reversal



$L_1 \sqcup L_2$ $\qquad$ $L_1 \sqcup -L_2$

Fig . 8: Orientation reverse

Definition 1.10

Let K be an oriented knot. If K is equivalent to $-$K, then we call it *invertible*, otherwise *non-invertible*.

Example 1.11

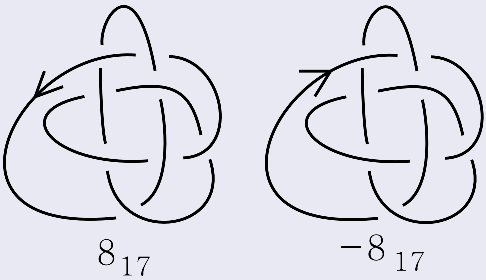


Fig . 9: 8_{17} knot – non-invertible knot

For a knot K many knot invariants cannot recognize the orientation reversal. The existence of (infinitely many) non-invertible knots is proved by H.F. Trotter by using "pretzel knots" and "the knot group".

Definition 1.12

Let $L = L_1 \sqcup \cdots \sqcup L_n$ be an oriented link of n components.
There are two types of classical crossings and we assign $+1$ or -1 for each classical crossing as described in Fig. 10.
We call it the sign of the crossing c and denote by sign(c).
For L_i and L_j, $i \neq j$, the *linking number* $lk(L_i, L_j)$ between i and j-th components is defined by $lk(L_i, L_j) = \frac{1}{2}\Sigma_{c \in L_i \cap L_j} \mathrm{sign}(c)$.

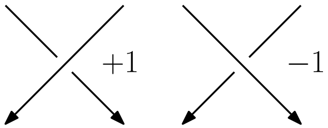


Fig. 10 : Positive and negative crossings

Example 1.13

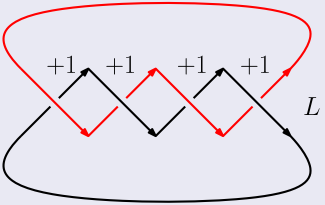


Fig. 11: $\text{lk}\,(L_1, L_2) = \frac{1}{2}\,\Sigma\text{sign}(c) = 2$

Remark 1.14

The linking number is found by C.F. Gauss, which is calculated by an explicit formula as a double line integral, the *Gauss linking integral*:

$$\mathrm{lk}(L_1, L_2) = \frac{1}{4\pi}\oint_{\gamma_1}\oint_{\gamma_2}\frac{r_1 - r_2}{|r_1 - r_2|^3}\cdot(dr_1 \times dr_2)$$
$$= \frac{1}{4\pi}\int_{S^1\times S^1}\frac{\det\dot{\gamma_1}(s), \dot{\gamma_2}(s), \gamma_1(s) - \gamma_2(t))}{|\gamma_1(s) - \gamma_2(t)|^3}dsdt,$$

where $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$. This formula is generalized by M.L. Kontsevich, and it is called the Kontsevich integral.

Theorem 1.15

Let $L = L_1 \sqcup \cdots \sqcup L_n$ be an oriented link of n components. For each pair $i, j \in \{1, \ldots, n\}$ the linking number $lk(L_i, L_j)$ is invariant under Reidemeister moves.

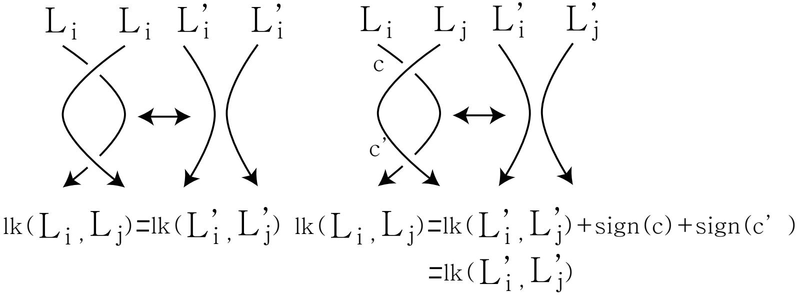


Fig. 12: $lk(L_1, L_2) = lk(L'_1, L'_2)$ under Reidemeister move 2

Definition 1.16

Let D be a diagram and A(D) the set of all arcs. For a prime number p the *Fox p-colouring* (or simply p colouring) is a function $f : A(D) \to \mathbb{Z}_p$ satisfying the condition described in Fig. 13.
If $f(a) = r$ for every $a \in A(D)$ for some $r \in \mathbb{Z}_p$, then it is called a trivial colouring.
If D has non-trivial p colouring, then we call it p colourable.
The number of Fox p-colourings on D is denoted by $c_p(D)$.

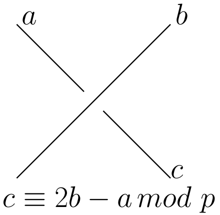


Fig. 13: Colouring condition

Example 1.17

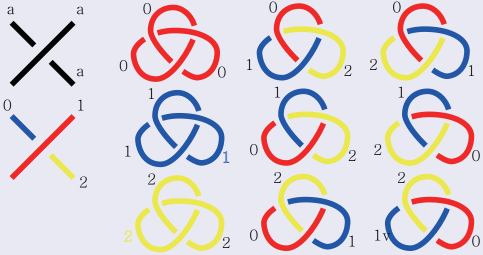


Fig. 14: Proper colorings of trefoil by $\mathbb{Z}_3$

Theorem 1.18

Let D be a link diagram. For each prime number p $c_p(D)$ is an *invariant* under Reidemeister moves.

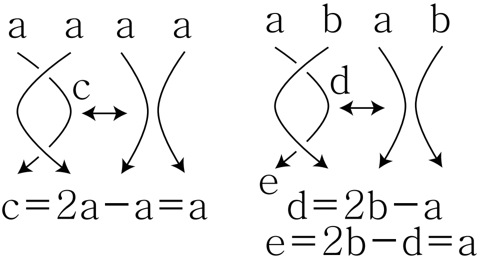


Fig. 15: $c_p(D)$ is an invariant under the second Reidemeister move

Some digressions 1

Remark 1.19

We can think about knots in $\mathbb{R}^4$, i.e. embedded S^1 in $\mathbb{R}^4$. But, it is well-known that every knots in $\mathbb{R}^4$ is isotopic to the trivial knot in $\mathbb{R}^4$, i.e. a boundary of a 2-dimensional disc in $\mathbb{R}^3 \subset \mathbb{R}^4$ (exercise). But embedded "surfaces" in $\mathbb{R}^4$ give a non-trivial theory, so called surface knot theory.

Some digressions 2

Remark 1.20

The link, shown in Fig. 16 is called *Borromean rings*. It has the following property: When we remove one of components, then we obtain a trivial link diagram, but it is non-trivial link. Links satisfying the above property are called Brunnian links. To show non-triviality of Brunnian links is not easy task. This problem will be asserted in the future with link homotopies.

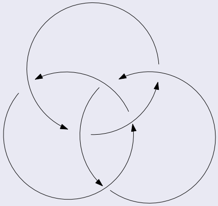
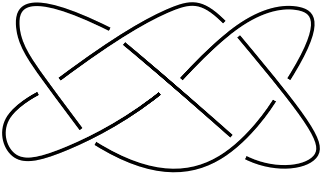


Fig. 16: Borromean rings

Exercises

1. Find an unknot diagram D such that no decreasing Ω_1 and Ω_2 and no Ω_3 can be applied to D.
2. Verify that the left-handed and right-handed figure-eight knots are isotopic.
3. Show that every "knot" diagram with $0, 1$ or 2 crossings is equivalent to a knot diagram with no crossing.

4. Show that a descending knot diagram is equivalent to a knot diagram with no crossings. Show that every link diagram can be changed to **an unknot diagram** by crossing change.
5. Define an unknotting number of a knot by the minimal number of crossing changes(**plural**) for a given knot to be **unknot**.
   - Verify(**just a misprint, twice**) that the unknotting numbers of the trefoil knot and the figure-eight knot are at most 1.
   - Verify that the unknotting numbers of the knot 7_4 described in Fig. 17 is at most 2. Is it true that the unknotting number of 7_4 is 2?



7_4

Fig. 17: A diagram of 7_4 knot

6. Verify that for a link L the linking number $\mathrm{lk}(L_i, L_j)$ is an invariant under Reidemeister move 3.
7. Let $L = L_1 \sqcup L_2$ be a link. Show that $\mathrm{lk}(L_1, L_2) = \mathrm{lk}(L_2, L_1)$.
8. Let $L = L_1 \sqcup L_2$ be a link. Show that $-\mathrm{lk}(L_1, L_2) = \mathrm{lk}(-L_1, L_2)$.
9. Calculate linking numbers for the following links
   - Hopf link
   - Borreomeao rings
   - Whitehead link
10. Calculate the number of 3-colourings of the following knots:
    - Right and left handed trefoil knots
    - Figure-eight knot
    - Borromean rings

11. Show that the number of proper colourings by p colours is equal to p^n for some $n \in \mathbb{N}$. (Hint: Use linear algebra.)
12. Prove that the unknotting number is greater than or equal to $\ln_3(c_3(K)) - 1$.
13. (Problem)
    - Estimate the number of Reidemeister moves needed to transform two isotopic diagrams having n crossings and m crossings.
    - Estimate the number of crossings of diagrams, which appeared when we transform a diagram having n crossings to an equivalent digram and m crossings.

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