STRING THEORY I

BIMSA, Year 2022-2023 – First Semester

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2d Fermionic Path Integrals

We wish to check modular invariance for the closed oriented Type IIA & Type IIB superstrings by direct computation of their partition functions on the torus

Before going to that, we learn how to compute fermionic path integrals on world-sheets of various topologies

Weyl Fermion on the Torus

We compute the partition function on a torus of periods $(2\pi, 2\pi\tau)$ of a free Weyl fermion $\boldsymbol{\lambda}(z)$ with action

$$\frac{1}{2\pi}\int \bar{\boldsymbol{\lambda}}\bar{\partial}\boldsymbol{\lambda}$$

subjected to the general periodicity condition

$$\boldsymbol{\lambda}(w+2\pi) = e^{\pi i(1-\alpha)}\boldsymbol{\lambda}(w),$$
$$\boldsymbol{\lambda}(w+2\pi\tau) = e^{\pi i(1-\beta)}\boldsymbol{\lambda}(w)$$

where α, β are real numbers with $-1 < \alpha, \beta \leq 1$

The left-moving Fermi-field pair $\boldsymbol{\lambda}(z)$, $\bar{\boldsymbol{\lambda}}(z)$ form a fermionic b, c system with $\lambda = (1-\lambda) = \frac{1}{2}$ and "non-standard" Hermitian structure

We see the A-cycle of the torus as spatial circle,
B-cycle as periodic Euclidean time
Path integral $\equiv$ trace in the Hilbert space $\mathbf{H}_\alpha$ of Weyl fermions quantized in the circle with b.c. $\boldsymbol{\lambda}(w+2\pi) = e^{\pi i(1-\alpha)}\boldsymbol{\lambda}(w)$
Twisted b.c. shifts the modes of $\boldsymbol{\lambda}$, $\bar{\boldsymbol{\lambda}}$ as

$$\boldsymbol{\lambda}(w) = \sum_{m\in\mathbb{Z}} \boldsymbol{\lambda}_{m+(1-\alpha)/2}\, e^{i[m+(1-\alpha)/2]w}$$

the raising operators now are

$$\boldsymbol{\lambda}_{-m+(1-\alpha)/2} \;\; \text{and} \;\; \bar{\boldsymbol{\lambda}}_{-m+(1+\alpha)/2}, \quad m = 1, 2, \dots$$

that is, the appropriate "twisted b.c. vacuum" $|\mathrm{tws}\rangle$ satisfies

$$\begin{cases} \boldsymbol{\lambda}_n|\mathrm{tws}\rangle = 0 & \text{for } n \geq -\frac{\alpha}{2} + \frac{1}{2} \\ \bar{\boldsymbol{\lambda}}_n|\mathrm{tws}\rangle = 0 & \text{for } n > \quad \frac{\alpha}{2} - \frac{1}{2} \end{cases}$$

Conclusion

$|\mathrm{tws}\rangle$ is the Fermi sea $|\alpha/2\rangle$ for $\lambda = 1/2$ (sea level $\alpha/2$)

From the general theory of Fermi/Bose sea states, we know that the chiral sea state $|\alpha/2\rangle$ has energy $H = H_L \equiv L_0 - c/24$

$$h - \frac{c}{24} \equiv \frac{1}{2} q(q+Q) - \frac{c}{24},$$

where in the present case $Q = 0$, $q = \alpha/2$, and $c = 1$

That is, the H_L-eigenvalue of $|\alpha/2\rangle$ is

$$E_{\alpha/2} \equiv \frac{3\alpha^2 - 1}{24}$$

The complex fermion has a $U(1)$ current $:\bar{\boldsymbol{\lambda}}\boldsymbol{\lambda}(z):$ and an associated conserved (non-anomalous!) charge J

The Fermi sea $|\alpha/2\rangle$ has charge $J = \alpha/2$,
while $\boldsymbol{\lambda}(z)$ has charge $+1$ and $\bar{\boldsymbol{\lambda}}(z)$ charge -1

We define a partition function in the Hilbert space $\mathbf{H}_\alpha$ valued in the $U(1)$ characters as

$$Z^\alpha_\beta(\tau) \stackrel{\text{def}}{=} \mathrm{Tr}_\alpha\big[q^{L_0-c/24}e^{\pi i\beta J}\big] =$$
$$= q^{(3\alpha^2-1)/24}\, e^{\pi i\alpha\beta/2}\prod_{m=1}^{\infty}\big[1+e^{\pi i\beta}q^{m-(1-\alpha)/2}\big]\big[1+e^{-\pi i\beta}q^{m-(1+\alpha)/2}\big]$$
$$= \frac{1}{\eta(q)}\,\vartheta\begin{bmatrix}\alpha/2\\ \beta/2\end{bmatrix}(\tau),$$

where in the last line we write the answer in terms of ϑ-function with characteristics by using Jacobi's triple product identity

We need some facts about ϑ functions

Interlude: ϑ-functions with characteristics

The θ-function with characteristics (a, b) on a torus of modulus $\tau \in \mathcal{H}$ is the sum

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} \exp\big[\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)\big]$$

It satisfies the identities (here $m, n \in \mathbb{Z}$)

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(z+m+n\tau|\tau) = e^{2\pi i m a - i\pi n^2\tau - 2\pi i n(z+b)}\, \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(z|\tau)$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = e^{i\pi a^2\tau + 2\pi i a(z+b)}\, \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(z+a\tau+b|\tau)$$

$$\vartheta \begin{bmatrix} a+m \\ b+n \end{bmatrix}(z|\tau) = e^{2\pi i n a}\, \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) \qquad \vartheta \begin{bmatrix} -a \\ -b \end{bmatrix}(z|\tau) = \vartheta \begin{bmatrix} a \\ b \end{bmatrix}(-z|\tau)$$

The dependence on the first argument, z, may be absorbed in the characteristics

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) = \vartheta \begin{bmatrix} a \\ b+z \end{bmatrix}(0|\tau)$$

and we write $\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](\tau)$ for $\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0|\tau)$ (the so-called *theta-constants*)

The *Jacobi triple-product identity* allows to rewrite the function as an infinite product

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau) = \eta(\tau)\, e^{2\pi i ab} q^{a^2/2-1/24} \prod_{n=1}^{\infty} \left(1 + q^{n+a-1/2} e^{2\pi i b}\right)\left(1 + q^{n-a-1/2} e^{-2\pi i b}\right),$$

Its modular transformations are (for $|\arg\sqrt{-i\tau}| < \pi/2$)

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau+1) = e^{-\pi i(a^2-a)}\, \vartheta\begin{bmatrix} a \\ a+b-\frac{1}{2} \end{bmatrix}(\tau), \qquad \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(-1/\tau) = \sqrt{-i\tau}\, e^{2\pi i ab}\, \vartheta\begin{bmatrix} -b \\ a \end{bmatrix}(\tau),$$

ϑ's with characteristics $a, b = 0, \frac{1}{2}$ related to *spin structures* on T^2. Special names

$$\vartheta\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \vartheta_1, \quad \vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \vartheta_2 \quad \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vartheta_3, \quad \vartheta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix} = \vartheta_4$$

$$\text{other notation} \qquad \vartheta_{ab}(\tau) = \vartheta\begin{bmatrix} a/2 \\ b/2 \end{bmatrix}, \qquad a, b = 0, 1$$

From the definition $\vartheta_{11}(\tau) = 0$, and we have Jacobi's "abstruse identity"

$$\vartheta_{00}(\tau)^4 - \vartheta_{01}(\tau)^4 - \vartheta_{10}(\tau)^4 = 0$$

A useful identity is $\partial_z \vartheta_{11}(z|\tau)\Big|_{z=0} = -2\pi\, \eta(\tau)^3$ where $\eta(\tau)$ is the Dedekind function

Back to the computation of fermionic path integrals

$$Z^{\alpha}_{\beta}(\tau) \equiv \mathrm{Tr}_{\alpha}\left[q^{L_0 - c/24} e^{\pi i \beta J}\right] = \frac{1}{\eta(q)}\, \vartheta\begin{bmatrix}\alpha/2\\ \beta/2\end{bmatrix}(\tau),$$

The world-sheet fermion number is $F \equiv J \bmod 2$, so that $(-1)^F \equiv e^{\pi i J}$

Thus if we have a pair ψ^a $(a = 1, 2)$ of MW fermions, which we combine in a complex Weyl fermion $\boldsymbol{\lambda} = (\psi^1 + i\psi^2)/\sqrt{2}$, their torus partition functions with the 4 possible spin-structures are

$$\begin{array}{ll}
(A, A) & Z^{0}{}_{0}(\tau) = \mathrm{Tr}_{\mathsf{NS}}[q^{L_0 - c/24}] \\
(A, P) & Z^{0}{}_{1}(\tau) = \mathrm{Tr}_{\mathsf{NS}}[(-1)^F\, q^{L_0 - c/24}] \\
(P, A) & Z^{1}{}_{0}(\tau) = \mathrm{Tr}_{\mathsf{R}}[q^{L_0 - c/24}] \\
(P, P) & Z^{1}{}_{1}(\tau) = \mathrm{Tr}_{\mathsf{R}}[(-1)^F\, q^{L_0 - c/24}]
\end{array}$$

(P, P) periodic for left- and right-movers;
(P, A) periodic for left-, anti-periodic for right-movers. etc

Modular Properties

The transformation under the modular group of the Fermi partition functions with given spin-structure follow from the modular properties of the ϑ_{ab} and η functions

$$\frac{1}{\eta(\tau+1)}\,\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\tau+1) = \frac{e^{-i\pi/12}}{\eta(\tau)}\; e^{-\pi i(a^2-a)}\,\vartheta\begin{bmatrix} a \\ a+b-\frac{1}{2} \end{bmatrix}(\tau)$$

$$\frac{1}{\eta(-1/\tau)}\,\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(-1/\tau) = \frac{1}{\sqrt{-i\tau}\,\eta(\tau)}\;\sqrt{-i\tau}\,e^{2\pi iab}\,\vartheta\begin{bmatrix} -b \\ a \end{bmatrix}(\tau) \equiv \frac{e^{2\pi iab}}{\eta(\tau)}\,\vartheta\begin{bmatrix} -b \\ a \end{bmatrix}(\tau)$$

We get

$$\begin{aligned}
&Z^0{}_0(\tau+1) = e^{-i\pi/12} Z^0{}_1(\tau), && Z^1{}_0(\tau+1) = e^{i\pi/6} Z^1{}_0(\tau),\\
&Z^0{}_1(\tau+1) = e^{-i\pi/12} Z^0{}_0(\tau), && Z^1{}_1(\tau+1) = e^{i\pi/6}\, Z^1{}_1(\tau),\\
&Z^0{}_0(-1/\tau) = Z^0{}_0(\tau), && Z^0{}_1(-1/\tau) = Z^{-1}{}_0(\tau) \equiv Z^1{}_0(\tau)\\
&Z^1{}_0(-1/\tau) = Z^0{}_1(\tau), && Z^1{}_1(-1/\tau) = e^{i\pi/2}\, Z^1{}_1(\tau),
\end{aligned} \qquad (\clubsuit)$$

Note that $Z^1{}_1$ (the **odd** spin-structure) transforms into itself while the 3 **even** $Z^0{}_0$, $Z^0{}_1$, $Z^1{}_0$ transform into each other

Dirac Fermion on the Cylinder

Cylinder $[0, \pi] \times \mathbb{R}/2\pi t\mathbb{Z}$ where $t > 0$ is the real modulus of its complex structure. We see the interval $[0, \pi]$ as space and the circle as periodic Euclidean time, i.e. path integral $\equiv$ thermal partition function for the Dirac fermion quantized in the interval. With impose standard strip b.c.

$$\tilde{\boldsymbol{\lambda}}(\pi, y) = \boldsymbol{\lambda}(\pi, y) \qquad \boldsymbol{\lambda}(0, y) = e^{\pi i(1-\alpha)}\tilde{\boldsymbol{\lambda}}(0, y)$$

Doubling trick $\Rightarrow$ Hilbert space $\simeq$ to $\mathbf{H}_\alpha$ for 1 Weyl fermion on S^1

$$\mathrm{Tr}_{\mathbf{H}_\alpha}\Big[e^{\beta J} q^{-2\pi t(L_0 - c/24)}\Big] = Z^\alpha_\beta(it).$$

Crossed-Channel Viewpoint In the crossed channel the path integral is seen as a closed string of length 2π which propagates for an Euclidean time π/t between suitable boundary states $|B\rangle$, $|B'\rangle$ of the fermionic CFT

$$\langle B'|\exp[-\pi(L_0 + \tilde{L}_0 - c/12)/t|B\rangle$$

Dirac Fermion on the Klein bottle

The Klein bottle of real modulus $t > 0$ is obtained from a cylinder of circumference 2π and length $2\pi t$ by identifying the two boundaries with a Ω-twist

The Klein bottle partition function then differs from the torus one with $\tau = it$ just by the insertion of Ω in the Hilbert space trace

$$\mathrm{Tr}_{\alpha,\tilde{\alpha}}\Big[\Omega\, e^{\pi i(\beta F+\tilde{\beta}\tilde{F})}\, q^{L_0-c/24}\,\bar{q}^{\tilde{L}_0-c/24}\Big], \qquad q = e^{-2\pi t},$$

where $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} = 0, 1$

Since Ω interchanges left- and right-movers, the amplitude vanishes if $\alpha \neq \tilde{\alpha}$ while the states that contribute have $F \equiv \tilde{F}$ and $L_0 \equiv \tilde{L}_0$ the expression reduces to

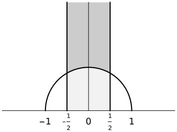
$$\mathrm{Tr}_{\alpha,\alpha}\Big[\Omega\, e^{\pi i(\beta+\tilde{\beta})F}\, e^{-4\pi t(L_0-c/24)}\Big] = Z^{\alpha}_{\beta+\tilde{\beta}}(2it)$$

Dirac fermion on the Möbius strip: Exercise

Modular Invariance in Type II

We check modular invariance of the closed oriented Type II superstrings – **already established last time from first principles** – by direct computation of their one-loop amplitude

As for the bosonic string, the torus amplitude Z_{T^2} is given, in terms of the physical particle spectrum, by the same Coleman-Weinberg formula as in QFT **except that the region of integration over the Schwinger parameters τ_1, τ_2 is restricted from the strip region $R \subset \mathcal{H}$ in the upper half-plane to the fundamental domain $F_0 \simeq \mathcal{H}/PSL(2,\mathbb{Z})$ of the moduli space of tori**



Fundamental domain $\boldsymbol{F_0}$ for the action on $\boldsymbol{SL(2,\mathbb{Z})}$ on the upper half plane $\mathcal{H}$ is the **darker gray region**
Half strip $\boldsymbol{R}$ is the **union of dark and light gray regions**

Torus vacuum amplitude

$$Z_{T^2} = V_{10} \int_{F_0} \frac{d^2\tau}{\tau_2} \int \frac{d^{10}k}{(2\pi)^{10}} \sum_i (-1)^{\mathbf{F}_i}\, q^{(k^2+m_i^2)/2}\, \bar{q}^{(k^2+\tilde{m}_i^2)/2} \qquad (\star)$$

where V_{10} is the volume of 10d spacetime, $q = \exp(2\pi i\tau)$, and:

1. $\sum_i$ stands for the trace over physical states at fixed momentum k_μ, $\equiv$ to the trace over the Hilbert space $\mathbf{H}_\perp$ of *transverse* oscillators. The trace includes a sum over the sectors $(\alpha, F; \tilde{\alpha}, \tilde{F})$ of $\mathbf{H}_\perp$;
2. spacetime fermions have a ***minus sign*** in the CW formula. $\mathbf{F}$ is the **spacetime** fermion number (not confuse with F);
3. masses given by the transverse oscillator numbers, $N_\perp$ and $\tilde{N}_\perp$:

$$m^2 = 2(N_\perp - \nu), \qquad \tilde{m}^2 = 2(\tilde{N}_\perp - \tilde{\nu})$$

with $\nu \equiv \frac{1}{2}(1-\alpha)$, $\tilde{\nu} \equiv \frac{1}{2}(1-\tilde{\alpha})$ equal to 0, $\frac{1}{2}$ for R resp. NS sector

In each sector $(\alpha, F; \tilde{\alpha}, \tilde{F})$ the trace over the transverse oscillators, including the integral over the transverse bosonic zero-mode $k_\perp$, decouples into the product of independent traces over the Hilbert space of each transverse field X^i, ψ^i and $\tilde{\psi}^i$, that is, in the product of the corresponding free-field torus partition functions

We have already computed all the relevant path integrals

The Partition Function of X The path integral for a single non-compact scalar field X was computed 2 lectures ago. The total contribution from the oscillators of X, together with the integral over its zero-mode (momentum integral), is

$$Z_X(\tau) = (8\pi^2\tau_2)^{-1/2}|\eta(q)|^{-2}$$

where $\eta(q)$ it the **Dedekind function**

In $(\bigstar)$ there is no contribution from the $X^\pm$ oscillators (cancelled by ghosts!) but their zero-modes (k_+, k_-) contribute, giving an additional factor

$$(8\pi^2\tau_2)^{-1}$$

The Partition Function of ψ

The partition function on the left-moving fermions depends on their spatial periodicity specified by $\alpha \in \{0,1\}$ ($\alpha = 0$ NS, $\alpha = 1$ R sector), and includes inside the trace the GSO projection operator

$$P_{\pm} = \frac{1}{2}\big[1 \pm (-1)^{F_{\mathsf{GSO}}}\,\big]$$

on the appropriate chirality selected by the GSO$\pm$ projection

We replace the 8 transverse MW fermions ψ^i ($i = 1, \cdots, 8$) by 4 complex Weyl fermions $\boldsymbol{\lambda}^j$ ($j = 1, \cdots, 4$). The partition functions of a single free Weyl fermion $\boldsymbol{\lambda}$ for the various spin structures is

$$Z^{\alpha}_{\beta}(\tau) \equiv \mathrm{Tr}_{\alpha}\big[q^{L_0 - c/24} e^{\pi i \beta J}\big] = \frac{1}{\eta(q)}\,\vartheta\begin{bmatrix}\alpha/2\\ \beta/2\end{bmatrix}(\tau),$$

GSO Projection

We compute the chiral partition function $Z^{\pm}_{\psi}(\tau)$ of the superstring 8 real transverse fermions ψ^i subjected to the GSO projection which keeps the sectors NS+ and R$\pm$

We know that the Fermi number F_{GSO} relevant for the GSO projection differs from F by the Fermi numbers (mod 2) of the longitudinal $\psi^{\pm}$ zero-modes and of the spinor ghosts β,γ. Using the standard (-1) picture for the NS sector we see that in the NS sector

$$(-1)^{F_{\rm GSO}} = -(-1)^F,$$

while (by definition) in the R sector, $(-1)^{F_{\mathbf{GSO}}} = \pm(-1)^F$ if R$\pm$ survives

$$\begin{aligned} Z^{\pm}_{\psi} &= \mathrm{Tr}_{\mathsf{NS}}\Big[\tfrac{1+(-1)^{F_{\mathbf{GSO}}}}{2} q^{L_0 - c/24}\Big] - \mathrm{Tr}_{\mathsf{R}}\Big[\tfrac{1\pm(-1)^{F_{\mathbf{GSO}}}}{2} q^{L_0 - c/24}\Big] = \\ &= \frac{1}{2}\Big(Z^0{}_0(\tau)^4 - Z^0{}_1(\tau)^4 - Z^1{}_0(\tau)^4 \mp Z^1{}_1(\tau)^4\Big), \end{aligned}$$

where **the minus in second term of the first line arises from the space-time fermion number sign factor $(-1)^{\mathsf{F}_i}$ in the CW formula**
Partition functions for right-movers $\tilde{\psi}$ complex conjugate $(Z^{\pm}_{\psi})^*$.

Modular properties integrand

Putting everything together, the closed superstring one-loop vacuum amplitude is

$$Z_{T^2} = V_{10} \int_{F_0} \frac{d^2\tau}{32\pi^2\tau_2^2}\, Z_X(\tau,\bar{\tau})^8\, Z_\psi^+(\tau)\, Z_\psi^\pm(\tau)^*, \qquad \begin{cases} + & \text{for IIB} \\ - & \text{for IIA} \end{cases}$$

As in the bosonic string, modular invariance of the integrand is a consistency condition

$d^2\tau/\tau_2^2$ is the $SL(2,\mathbb{R})$-invariant Poincaré volume form which is obviously modular invariant, as it is $Z_X(\tau)$

It remains to discuss the modular properties of the GSO-projected fermionic traces $Z_\psi^\pm$ and $(Z_\psi^\pm)^*$

Modular Properties of Fermi Partitions Functions

The modular transformations of the functions $Z^{\alpha}_{\beta}(\tau)$ were tabled a few slides ago in table (♣)

There is a subtlety which requires a comment. The partition function $Z^1_1(\tau)$ *vanishes identically:* in the path integral formalism this is due to the presence of a Fermi zero-mode for periodic b.c. Since $Z^1_1(\tau)$ is zero, it is modular invariant for all choices of overall phases in its transformation; we *declare* these phases to be as in (♣). The physical reason for this will be explained later

From table (♣) we see that $S\colon \tau \mapsto -1/\tau$ acts on the set

$$\left\{(Z^0_0)^4, \quad (Z^1_0)^4, \quad (Z^0_1)^4, \quad (Z^1_1)^4\right\}$$

by permuting 2nd and 3rd elements, leaving invariant the expression

$$2\, Z^{\pm}_{\psi} \equiv (Z^0_0)^4 - (Z^1_0)^4 - (Z^0_1)^4 \mp (Z^1_1)^4$$

so the partition functions are S-invariant

Since $(e^{-i\pi/12})^4 \equiv -e^{2\pi i/3}$, under $T\colon \tau \to \tau+1$ we have

$$\begin{aligned} 2\, Z_\psi^{\pm}(\tau+1) &= Z^0{}_0(\tau+1)^4 - Z^0{}_1(\tau+1)^4 - Z^1{}_0(\tau+1)^4 \mp Z^1{}_1(\tau+1)^4 \\ &= -e^{2\pi i/3} Z^0{}_1(\tau)^4 + e^{2\pi i/3} Z^0{}_0(\tau)^4 - e^{2\pi i/3} Z^1{}_0(\tau)^4 \mp e^{2\pi i/3} Z^1{}_1(\tau)^4 \\ &= 2\, e^{2\pi i/3}\, Z_\psi^{\pm}(\tau). \end{aligned}$$

The two combinations $Z_\psi^{+}(\tau)\, Z_\psi^{\pm}(\tau)^*$ are thus fully modular-invariant, and hence the integrand of the torus partition function ($\bigstar$) is ***modular invariant***, and we should restrict the integral to the fundamental domain F_0 to avoid multiple counting

This implies that there are no UV divergences

As we shall see, NO IR divergence either

Discussion & important issues

(A) The one-loop vacuum amplitude in a supersymmetric theory is expected to vanish by cancellations between fermions and bosons. This holds because of ${Z^1}_1(\tau)=0$ and the identity found by Jacobi and called by him *"aequatio identica satis abstrusa"* (the "abstruse identity")

$$\vartheta_3(q)^4=\vartheta_4(q)^4+\vartheta_2(q)^4 \quad\Rightarrow\quad ({Z^0}_0)^4-({Z^0}_1)^4-({Z^1}_0)^4=0$$

In String Theory II we shall prove a more general version of this identity

(B) Last time we stated that global $\textbf{Diff}^+$ anomalies cancel in all perturbative amplitudes provided *all* genus 1 BRST-invariant amplitudes are modular invariant. We have shown modular invariance of 1-loop amplitude *without* insertions. We must check that modular invariance is not spoiled by insertions of GSO-allowed BRST-invariant operators. Easy: 2d fields are free, path integrals are Gaussian. Amplitudes for each spin structure (α,β) are given by a product of functional determinants times a contraction of propagators (by Wick's thm). Anomalous phases under $\mathrm{MCG}(T^2)$ from the determinants, are the same as in absence of insertions

(C) A general one-loop amplitude has the form

$$\mathcal{A} = \int_{F_0} \frac{d^2\tau}{\tau_2^2}\, F(\tau,\bar{\tau}),$$

for some real-analytic function $F(\tau,\bar{\tau})$ such that

$$F\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = F(\tau,\bar{\tau}) \quad \text{for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$$

The fundamental domain F_0 has finite Poincaré volume

$$\int_{F_0} \frac{d^2\tau}{\tau_2^2} = \frac{2\pi}{3}.$$

thus if $F(\tau,\bar{\tau})$ is bounded, the amplitude $\mathcal{A}$ is finite. F_0 is biholomorphic to a punctured sphere, where the puncture is at $\tau = i\infty$, i.e. $q \to 0$. The integrand $F(\tau,\bar{\tau})$ may possibly diverge only in the $q \to 0$ limit which is controlled by the *lightest* state $F = O((q\bar{q})^{m^2_{\text{lightest}}})$. In absence of tachyons, the integrand is bounded and then the amplitude *finite*

Divergences and Tadpoles in Type I Theories

We claimed that Type I is consistent only when the Chan-Paton gauge group is $SO(32)$

There are various ways of seeing this

The main tool to detect inconsistencies is to require absence of divergences in the open string one-loop amplitudes, or, equivalently, require absence of tree-level tadpoles which cannot be consistently shifted away, i.e. cancelled by a redefinition of the vacuum on which we define the theory

This criterion plays the same role for open strings as modular invariance in the closed case

Cylinder Amplitude

Let us pretend for a moment that there is an *oriented* open superstring theory. We already know that such a model is inconsistent from argument based on spacetime low-energy physics

Now we wish to see how the inconsistency is reflected at the full superstring level in terms of one-loop divergences/disk tadpoles

The open-string 1-loop processes given by cylinder amplitudes

The open-string sectors are labelled by $\alpha = 0, 1$ (NS vs. R) and by the Chan-Paton labels (a, b), $a, b = 1, 2, \ldots, N$, subjected to the GSO projection

$$P_{\mathsf{GSO}} = \frac{1}{2} \sum_{\beta=0}^{1} (-1)^{\beta F_{\mathsf{GSO}}}$$

Then the open one-loop vacuum amplitude takes the form

$$
\begin{aligned}
Z_{\mathrm{Cy}} &= \frac{N^2}{2} \sum_{\alpha,\beta=0}^{1} (-1)^{\alpha} \int_0^{\infty} \frac{dt}{4t} \int \frac{d^{10}k}{(2\pi)^{10}} e^{-4\pi t k^2}\, \mathrm{Tr}'_{\alpha}\Big[(-1)^{\beta F} e^{-2\pi t (L_0 - c/24)}\Big] \\
&= N^2 \int_0^{\infty} \frac{dt}{8t} (16\pi^2 t)^{-5} \eta(it)^{-8} \big[Z^0{}_0(it)^4 - Z^1{}_0(it)^4 - Z^0{}_1(it)^4 - Z^1{}_1(it)^4\big]
\end{aligned}
$$

where t is the real modulus of the cylinder, i.e. the radius of the circle (the length of the cylinder is fixed to π). Of course, Z_{Cy} vanishes by space-time supersymmetry: the space-time fermionic contribution, $\alpha = 1$, exactly cancel the bosonic one, $\alpha = 0$

Indeed, Z_{Cy} vanishes by $Z^1{}_1 = 0$ and the abstruse identity

Z_{Cy} may be interpreted in the crossed channel as a *tree-level* closed-string process: a closed string propagates between the two boundary states $|B\rangle$, $|B'\rangle$ associated with the open string boundary condition. In the closed-string channel, $\beta = 0$ corresponds to the NS-NS sector and $\beta = 1$ to the R-R one. We write

$$
\begin{aligned}
Z_{\mathrm{Cy}} &= Z_0 - Z_1 \\
Z_0 &= N^2 \int_0^\infty \frac{dt}{8t} (16\pi^2 t)^{-5} \eta(it)^{-8} \big[Z^0{}_0(it)^4 - Z^1{}_0(it)^4 \big] \\
Z_1 &= N^2 \int_0^\infty \frac{dt}{8t} (16\pi^2 t)^{-5} \eta(it)^{-8} \big[Z^0{}_1(it)^4 + Z^1{}_1(it)^4 \big]
\end{aligned}
$$

Of course $Z_0 = Z_1$ since the total amplitude Z_{Cy} vanishes

However if we wish the amplitude to be finite for arbitrary *planar* insertions (i.e. insertions on one boundary component only) each amplitude Z_0, Z_1 should be ***separately finite***

Physically, this issue may be understood as follows: divergences in the tree-level closed string amplitude Z_0 arises from infinitely long cylinders,

$$\ell \equiv 1/t \to \infty,$$

which corresponds to propagation of zero-momentum NS-NS states in the crossed channel. The divergent part of the amplitude Z_0 is then proportional to the square of the disk amplitude with the zero-momentum NS-NS vertex inserted, i.e. to the square of the tadpole amplitude for the dilaton, just as in the bosonic string

Using the modular properties

$$\eta(it) = t^{-1/2}\,\eta(i/t), \qquad Z^{\alpha}_{\ \beta}(it)^4 = Z^{\beta}_{\ \alpha}(i/t)^4$$

we rewrite

$$Z_0 = \frac{N^2}{8(16\pi^2)^5}\int\limits_0^\infty d\ell\;\eta(i\ell)^{-8}\big[Z^0_{\ 0}(i\ell)^4 - Z^0_{\ 1}(i\ell)^4\big]$$

The asymptotics of the function $\eta(\tau)$ as $\ell \to \infty$ is

$$\eta(i\ell) = e^{-\pi\ell/12}\prod_{n=1}^{\infty}\big(1-e^{-2\pi n\ell}\big) = e^{-\pi\ell/12}\big(1+O(e^{-2\pi\ell})\big),$$

while, using the "abstruse" identity

$$Z^0_{\ 0}(i\ell)^4 - Z^0_{\ 1}(i\ell)^4 \equiv Z^1_{\ 0}(i\ell)^4 = \left(\frac{1}{\eta(i\ell)}\cdot\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(0|i\ell)\right)^4 =$$

$$= \Big(e^{\pi\ell/12}\cdot 2\,e^{-\pi\ell/4}\big(1+O(e^{-2\pi\ell})\big)\Big)^4 = 16\,e^{-2\pi\ell/3}\big(1+O(e^{-2\pi\ell})\big),$$

Then the NS-NS cylindric amplitude

$$Z_0 = \frac{N^2}{8(16\pi^2)^5} \int_0^\infty d\ell \, \left[16 + O(e^{-2\pi\ell})\right],$$

has a linear divergence which is proportional to the square of a NS-NS tadpole on the disk, analogous to the one we have in the open bosonic string

The NS-NS tadpole is given by the disk amplitude with one insertion of the dilaton vertex at zero-momentum

$$\left\langle \eta_{\mu\nu}\, \psi^\mu \tilde{\psi}^\nu e^{-\phi-\tilde{\phi}} \right\rangle_{\text{disk}} \neq 0.$$

The R-R amplitude Z_1 has an **identical** linear divergence which should be interpreted in terms of a R-R tadpole on the disk.

The R-R Tadpole: a (apparent) Paradox)

No *propagating* R-R 10d field which can be responsible for this tadpole

Indeed, in picture $(-\frac{1}{2}, -\frac{1}{2})$ the *propagating* R-R states have vertices proportional to k_μ which vanish in the zero-momentum limit. Equivalently, a non-zero tadpole for a gauge field form, $\langle A^{(k)} \rangle \neq 0$, breaks the spacetime gauge symmetry (i.e. BRST-invariance) and this is *not* allowed in a consistent theory

We arrived at an ***apparent*** paradox. Consistency with BRST quantization *requires* the R-R tadpole to be the disk amplitude with one R-R-sector *BRST-invariant* operator inserted in the bulk. By CFT state-operator isomorphism, this BRST-invarint operator must correspond to a BRST-invariant *state*. **No state visible in light-cone or OCQ will do since their vertices vanish at zero-momentum**, so – if the states visible in OCQ were the *only* physical states – we would get a contradiction.

Our careful discussion of BRST quantization solves this tricky conundrum in a very transparent way:

FUNDAMENTAL FACT

There are more zero-momentum BRST-invariant states than ***naively*** *expected on the basis of analysis in the light-cone approach (or OCQ)*

The R-R tadpole arises from such subtle BRST-invariant states which ***are invisible*** in the light-cone gauge or OCQ (quasi-topological modes)

The relevant "subtle" state is easily understood: in Type IIB the BRST-invariant R-R vertex in the *appropriate* $(-\frac{1}{2}, -\frac{3}{2})$ picture.

The R-R vertex in picture $(-\frac{1}{2}, -\frac{3}{2})$ has the form

$$\sum_{k\ \text{even}} A^{(k)}_{\mu_1\cdots\mu_k}(X)\big(S\gamma^{\mu_1\cdots\mu_k}\tilde{S}\big)\, e^{-\phi/2-3\tilde{\phi}/2}$$

and the *even* degree form $A = \sum_k A^{(k)}(x)$ satisfies the Kähler-Dirac equation in $\mathbb{R}^{9,1}$

$$\text{vertex is BRST-invariant} \iff (d-\delta)A = 0.$$

We know that A can be chosen to be self-dual, $*A = iA$. Suppose that $A \equiv A^{(10)}$ has *pure degree 10*. The BRST condition becomes

$$0 = (d-\delta)A^{(10)} = *d(*A^{(10)}) \quad \Rightarrow \quad *A^{(10)} = \mathrm{const},$$

so that taking $A \equiv A^{(10)}$ to be a *constant* 10-form produces a BRST-invariant vertex which is not BRST-trivial, and hence it must lead to ***observable physical effects***

We stress that this physical vertex does not correspond to any *propagating* 10d degree of freedom, since its momentum is frozen to zero by BRST-invariance

The disk tadpole of the R-R zero-momentum vertex is non-zero

$$\Big\langle C^{\alpha\dot\beta}\, S_\alpha \tilde{S}_{\dot\beta}\, e^{-\phi/2-3\tilde\phi/2}\Big\rangle_{\text{disk}} = \kappa N \neq 0,$$

see NOTES for more details

The tadpole has a factor of N from the trace of the CP labels on the boundary of the disk

Spacetime Interpretation

Being BRST-invariant in the 2d sense, the zero-momentum R-R vertex corresponds to a *gauge-invariant interaction in 10d target space*. In Fourier analysis zero-momentum means integration over the full $\mathbb{R}^{9,1}$ space, so that the 10d interaction must have the form

$$\mu \int_{\mathbb{R}^{9,1}} A^{(10)} \qquad \text{"topological coupling"}$$

for some non-zero constant μ. This space-time coupling is both **Diff**$^+$-invariant (being topological) as well as gauge invariant

$$\mu \int_{\mathbb{R}^{9,1}} A^{(10)} \equiv \mu \int_{\mathbb{R}^{9,1}} \Big(A^{(10)} + d\lambda^{(9)}\Big),$$

as expected from its 2d BRST-invariance. The only possible R-R tadpole is then a 10-form tadpole, and its value is given by the coupling. The cylinder amplitude in the R-R channel shows that the theory of the oriented open superstring necessarily has a non-zero 10-form tadpole μ proportional to N.

The 10d effective action contains $A^{(10)}$ only trough the topological coupling since the effective Lagrangian $\mathcal{L}_{\text{eff}}$ is local and gauge invariant, and hence – a part for the topological term – $A^{(10)}$ may enter in $\mathcal{L}_{\text{eff}}$ only through its gauge-invariant field-strength

$$F^{(11)} = dA^{(10)} \equiv 0,$$

which however is *identically zero* in $\mathbb{R}^{9,1}$. The topological coupling is the only one consistent with gauge invariance, and we see that gauge invariance freezes $A^{(10)}$ to zero momentum, exactly as expected from the world-sheet BRST cohomology.
The equation of motion of $A^{(10)}$ takes the form

$$0 = \frac{\delta S_{\text{eff}}}{\delta A^{(10)}(x)} \equiv \mu.$$

In oriented open superstring theory this equation is inconsistent since μ is a non-zero constant $\mu = \kappa N$. It is pretty clear that this tadpole cannot be shifted away: oriented open superstrings are really inconsistent. We already knew that this model is *inconsistent*: we just confirmed our previous conclusion from a different "stringy" perspective

Conclusion

The theory of open oriented superstrings is inconsistent at the quantum level

How to cure this?

We sketch the story. Full details in the notes

Notations

we write $|B\rangle$ for the boundary state corresponding to the Neumann b.c. on S^1 with CP taking values $1, \cdots, N$ i.e. $\langle 0|\mathcal{O}_1 \cdots \mathcal{O}_s|B\rangle$ is the disk amplitude with insertions $\mathcal{O}_j$

$|C\rangle$ stands for the crosscap state, i.e. $\langle 0|\mathcal{O}_1 \cdots \mathcal{O}_s|C\rangle$ is the $\mathbf{R}\mathbb{P}^2$ amplitude with insertions $\mathcal{O}_j$

For open + closed non-oriented superstrings 4 topologies at 1-loop:

1. torus: **finite**
2. cylinder: 2 boundaries, amplitude factor N^2 from trace on CP d.o.f.

$$\langle B|\exp[-\pi(L_0+\tilde{L}_0-c/12)s]|B\rangle = \sum_{\psi} e^{-2\pi(h-c/24)s}\langle B|\psi\rangle\langle\psi|B\rangle$$

   divergence proportional to $|\langle A^{(10)}|B\rangle|^2 = c_1^2 N^2$

3. Klein bottle: 0 boundaries, amplitude independent on N

$$\langle C|\exp[-\pi(L_0+\tilde{L}_0-c/12)s]|C\rangle = \sum_{\psi} e^{-2\pi(h-c/24)s}\langle C|\psi\rangle\langle\psi|C\rangle$$

   divergence proportional to $|\langle A^{(10)}|C\rangle|^2 = c_2^2$

4. Möbius strip: 1 boundary, amplitude $\propto N$, two possible signs $\mp$ for $SO(N)$, $USp(N)$ resp.
   divergence proportional to $2\langle B|A^{(10)}\rangle\langle A^{(10)}|C\rangle = \mp 2\, c_1 c_2 N$

Summing all contributions: divergence proportional to

$$c_1^2 N^2 \mp 2\, c_1 c_2 N + c_2^2 = (c_1 N \mp c_2)^2$$

i.e. proportional to the square of the dangerous tadpole at tree-level (sum of the disk and $\mathbf{R}\mathbb{P}^2$ tadpoles)

if c_2/c_1 is an integer, we have a choice of gauge group without tadpoles/divergences $SO(N)$ if $c_2/c_1 > 0$ or $USp(N)$ if $c_2/c_1 < 0$

explicit computation (see Notes) the good gauge group is $SO(32)$

We shall give a deeper proof/interpretation of this result in String Theory II