Lecture 10: Index theorem I

Motivation: Understand $\Omega_6^{\mathrm{Spin}\times U(1)} = \mathbb{Z}^2$ and $\Omega_6^{\mathrm{Spin}^c} = \mathbb{Z}^2$ (AS index)

and $\Omega_4^{\mathrm{Pin}^+} = \mathbb{Z}_{16}$ (APS index)

Examples of differential operators:

Laplacian $\Delta = g^{\mu\nu}\partial_\nu\partial_\mu$ $\quad g^{\mu\nu} = \begin{pmatrix} + & & \\ & + & \\ & & \ddots \\ & & & + \end{pmatrix}$

d'Alembertian $\Box = \eta^{\mu\nu}\partial_\nu\partial_\mu$ $\quad \eta^{\mu\nu} = \begin{pmatrix} + & & & \\ & + & & \\ & & \ddots & \\ & & & + \\ & & & & - \end{pmatrix}$

Dirac operator $D = i\gamma^\mu\partial_\mu + m$ $\quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

These operators are regarded as maps of sections

$$D : \Gamma(M, E) \to \Gamma(M, F)$$

E, F: vector bundles over M.

Eg: Dirac operator is a map $\Gamma(M, E) \to \Gamma(M, E)$, E: a spin bundle over M. If inner products are defined on E and F, define the adjoint of D,

$$D^\dagger : \Gamma(M, F) \to \Gamma(M, E)$$

$$\ker D \equiv \{ s \in \Gamma(M, E) \mid Ds = 0 \}$$

$$\ker D^\dagger \equiv \{ s \in \Gamma(M, F) \mid D^\dagger s = 0 \}$$

The <u>analytical index</u> is defined by

$$\mathrm{ind}\, D = \dim \ker D - \dim \ker D^\dagger .$$

This analytical quantity is a topological invariant expressed in terms of an integral of an appropriate characteristic class over M.

<u>Elliptic operators</u>

E, F : complex vector bundles over a manifold M.

A differential operator D is a linear map

$$D : \Gamma(M, E) \to \Gamma(M, F)$$

Take a chart U of M over which E and F are trivial.

Denote the local coordinates of U as x^μ.

Multiindex notation

$$\alpha \equiv (\mu_1, \mu_2, \cdots, \mu_m) \qquad \mu_j \in \mathbb{Z}^{\geq 0}$$

$$|\alpha| = \mu_1 + \mu_2 + \cdots + \mu_m$$

$$D_\alpha = \frac{\partial^{|\alpha|}}{\partial x^\alpha} = \frac{\partial^{\mu_1 + \cdots + \mu_m}}{\partial (x^1)^{\mu_1} \cdots \partial (x^m)^{\mu_m}}$$

If $\dim E = k$ and $\dim F = k'$, the most general form of D is

$$[D s(x)]^b = \sum_{\substack{|\alpha| \leq N \\ 1 \leq a \leq k}} (A^\alpha)^b{}_a (x)\, D_\alpha s^a(x) \qquad 1 \leq b \leq k' .$$

where $s(x)$ is a section of E, x denotes a point whose coordinates are x^{μ}. $(A^{\alpha}) \equiv (A^{\alpha})^{b}_{\;a}$ is a $k \times k'$ matrix which may depend on the position x. The positive integer N is called the <u>order</u> of D.

Dirac operator : $N = 1$

Laplacian : $N = 2$

Eg : If E is a spin bundle over M, the Dirac operator

$D = i\gamma^{\mu}\partial_{\mu} + m : \Gamma(M, E) \to \Gamma(M, E)$ acts on a section $\psi(x)$ of E as

$$[D\psi(x)]^{b} = i(\gamma^{\mu})^{b}_{\;a}\,\partial_{\mu}\psi^{a}(x) + m\psi^{a}(x)$$

The <u>symbol</u> of D is a $k \times k'$ matrix

$$\sigma(D, \xi) \equiv \sum_{|\alpha| = N} (A^{\alpha})^{b}_{\;a}(x)\,\xi_{\alpha}$$

where ξ is a real m-tuple $\xi = (\xi_1, \cdots, \xi_m)$, $\xi_{\alpha} = \xi_1^{\mu_1} \cdots \xi_m^{\mu_m}$

If the matrix $\sigma(D, \xi)$ is invertible for each $x \in M$ and each $\xi \in \mathbb{R}^m - \{0\}$, the operator D is said to be <u>elliptic</u>.

Note that for $D = D_1 \circ D_2$, $\sigma(D, \xi) = \sigma(D_1, \xi)\,\sigma(D_2, \xi)$

$\therefore$ Powers and roots of elliptic operators are elliptic

Eg: x^μ : the natural coordinates in $\mathbb{R}^m$

If E and F are real line bundles over $\mathbb{R}^m$

The Laplacian $\Delta : \Gamma(\mathbb{R}^m, E) \to \Gamma(\mathbb{R}^m, F)$ is defined by

$$\Delta \equiv \frac{\partial^2}{\partial (x^1)^2} + \dots + \frac{\partial^2}{\partial (x^m)^2}$$

the symbol is $\sigma(\Delta, \xi) = \sum_\mu (\xi_\mu)^2$ invertible for $\xi \neq 0$, hence Δ is elliptic

The d'Alembertian $\Box \equiv \frac{\partial^2}{\partial (x^1)^2} + \dots + \frac{\partial^2}{\partial (x^{m-1})^2} - \frac{\partial^2}{\partial (x^m)^2}$

is not elliptic since the symbol

$$\sigma(\Box, \xi) = (\xi^1)^2 + \dots + (\xi^{m-1})^2 - (\xi^m)^2$$

vanishes everywhere on the light cone

$$(\xi^m)^2 = (\xi^1)^2 + \dots + (\xi^{m-1})^2 .$$

Fredholm operators

$D : \Gamma(M, E) \to \Gamma(M, F)$: an elliptic operator.

The <u>kernel</u> of D is

$$\ker D \equiv \{ s \in \Gamma(M, E) \mid Ds = 0 \}$$

Suppose E and F are endowed with fibre metrics, denoted $\langle \, , \, \rangle_E$

and $\langle\, ,\, \rangle_F$, respectively. The adjoint $D^\dagger : \Gamma(M,F) \to \Gamma(M,E)$ of D is defined by

$$\langle s', Ds \rangle_F \equiv \langle D^\dagger s', s \rangle_E$$

where $s \in \Gamma(M,E)$ and $s' \in \Gamma(M,F)$. Define the cokernel of D by

$$\operatorname{coker} D \equiv \Gamma(M,F) / \operatorname{Im} D$$

If the kernel and cokernel of an elliptic operator are finite dimensional, then it is called a Fredholm operator. The analytical index

$$\operatorname{ind} D \equiv \dim \ker D - \dim \operatorname{coker} D \qquad (1)$$

is well-defined for a Fredholm operator. It is known that elliptic operators on a compact manifold are Fredholm operators.

Thm Let $D : \Gamma(M,E) \to \Gamma(M,F)$ be a Fredholm operator. Then

$$\operatorname{coker} D \cong \ker D^\dagger \equiv \{ s \in \Gamma(M,F) \mid D^\dagger s = 0 \}$$

Pf: First, we show that $\operatorname{coker} D = \Gamma(M,F)/\operatorname{Im} D \cong (\operatorname{Im} D)^\perp$

Since D is a Fredholm operator, $\operatorname{Im} D$ is closed.

$\therefore \Gamma(M,F) = \operatorname{Im} D \oplus (\operatorname{Im} D)^\perp \quad \therefore \Gamma(M,F)/\operatorname{Im} D \cong (\operatorname{Im} D)^\perp.$

Next, we show that $(\operatorname{Im} D)^\perp \cong \ker D^\dagger$

$\forall s \in \ker D^\dagger$, $\forall s' \in \Gamma(M,E)$, $0 = \langle D^\dagger s, s' \rangle = \langle s, Ds' \rangle$

$\therefore s \in (\operatorname{Im} D)^\perp$.

$\forall s \in (\operatorname{Im} D)^\perp$, $\langle D^\dagger s, D^\dagger s \rangle = \langle s, DD^\dagger s \rangle = 0 \Rightarrow$

$D^\dagger s = 0 \quad \therefore s \in \ker D^\dagger. \quad \therefore \operatorname{coker} D \cong \ker D^\dagger.$ $\square$

This thm shows that

$$\operatorname{ind} D = \dim \ker D - \dim \ker D^\dagger.$$

Elliptic complexes

Consider a sequence of Fredholm operators

$$\cdots \to \Gamma(M, E_{i-1}) \xrightarrow{D_{i-1}} \Gamma(M, E_i) \xrightarrow{D_i} \Gamma(M, E_{i+1}) \xrightarrow{D_{i+1}} \cdots$$

where $\{E_i\}$ is a sequence of vector bundles over a compact manifold.

The sequence (E_i, D_i) is called an elliptic complex if $D_i \circ D_{i-1} = 0$ for any i. Eg: $\Gamma(M, E_i) = \Omega_i(M)$ and $D_i = d$ (exterior derivative).

The adjoint of $D_i : \Gamma(M, E_i) \to \Gamma(M, E_{i+1})$ is denoted by

$$D_i^\dagger : \Gamma(M, E_{i+1}) \to \Gamma(M, E_i).$$

The Laplacian $\Delta_i : \Gamma(M, E_i) \to \Gamma(M, E_i)$ is

$$\Delta_i \equiv D_{i-1} D_{i-1}^\dagger + D_i^\dagger D_i$$

The Hodge decomposition also applies to this case

$$s_i = D_{i-1} s_{i-1} + D_i^\dagger s_{i+1} + h_i \, .$$

where $s_{i+1} \in \Gamma(M, E_{i+1})$ and $\Delta_i h_i = 0$

Analogously to the de Rham cohomology groups, define

$$H^i(E, D) = \ker D_i / \operatorname{Im} D_{i-1}$$

As in the case of the de Rham theory, $H^i(E, D) \cong \ker \Delta_i$

$$\dim H^i(E, D) = \dim \operatorname{Harm}^i(E, D)$$

where $\operatorname{Harm}^i(E, D)$ is a vector space spanned by $\{h_i\}$. The <u>index</u> of this elliptic complex is defined by

$$\operatorname{ind} D \equiv \sum_{i=0}^{m} (-1)^i \dim H^i(E, D) = \sum_{i=0}^{m} (-1)^i \dim \ker \Delta_i \qquad (2)$$

This index generalizes the Euler characteristic.

Related to (1)? Consider the complex $\Gamma(M, E) \xrightarrow{D} \Gamma(M, F)$

$$0 \xhookrightarrow{i} \Gamma(M, E) \xrightarrow{D} \Gamma(M, F) \xrightarrow{\varphi} 0$$

The index according to (2) is

$$\dim \ker D - \{ \dim \Gamma(M, F) - \dim \operatorname{Im} D \} = \dim \ker D - \dim \operatorname{coker} D$$

agrees with (1).

Define $E_+ \equiv \bigoplus_r E_{2r}$, $E_- = \bigoplus_r E_{2r+1}$

<u>even bundle</u> <u>odd bundle</u>

Consider the operators

$$A \equiv \bigoplus_r (D_{2r} + D^\dagger_{2r-1}), \quad A^\dagger \equiv \bigoplus_r (D_{2r+1} + D^\dagger_{2r})$$

Then $A : \Gamma(M, E_+) \to \Gamma(M, E_-)$ and $A^\dagger : \Gamma(M, E_-) \to \Gamma(M, E_+)$

Construct two Laplacians:

$$\Delta_+ \equiv A^\dagger A = \bigoplus_{r,s} (D_{2r+1} + D^\dagger_{2r})(D_{2s} + D^\dagger_{2s-1})$$

$$= \bigoplus_r (D_{2r-1} D^\dagger_{2r-1} + D^\dagger_{2r} D_{2r}) = \bigoplus_r \Delta_{2r}$$

$$\Delta_- \equiv AA^\dagger = \bigoplus_r \Delta_{2r+1}$$

$\ker A^\dagger A = \ker A$, $\ker AA^\dagger = \ker A^\dagger$

Then $\operatorname{ind}(E_\pm, A) = \dim \ker A - \dim \ker A^\dagger$

$$= \dim \ker \Delta_+ - \dim \ker \Delta_-$$

$$= \sum (-1)^r \dim \ker \Delta_r = \operatorname{ind}(E, D)$$

Eg: Consider the de Rham complex $\Omega(M)$ over a compact manifold M without boundary.

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(M) \to 0$$

where $m = \dim M$ and $d \equiv d_r : \Omega^r(M) \to \Omega^{r+1}(M)$

$H^r(E, D) =$ de Rham cohomology group $H^r(M, \mathbb{R})$.

The index

$$\mathrm{ind}\,(\Omega^*(M), d) = \sum_{r=0}^{m} (-1)^r \dim H^r(M, \mathbb{R}) = \chi(M).$$

By Hodge theory, $\dim H^r(M, \mathbb{R}) = \dim \mathrm{Harm}^r(M) = \dim \ker \Delta_r$

where Δ_r is the Laplacian

$$\Delta_r = (d + d^\dagger)^2 = d_{r-1} d_{r-1}^\dagger + d_r^\dagger d_r$$

$d_r^\dagger : \Omega^{r+1}(M) \to \Omega^r(M)$: adjoint of d_r.

$$\chi(M) = \sum_{r=0}^{m} (-1)^r \dim \ker \Delta_r \qquad (3)$$

Note that $\chi(M) = \int_M e(TM)$

$$(3) \Rightarrow \quad \sum_{r=0}^{m} (-1)^r \dim \ker \Delta_r = \int_M e(TM) \qquad (4)$$

"index theorem"

LHS : analytical index RHS: topological index

The two term complex is given by

$$\Omega^+(M) \equiv \bigoplus_r \Omega^{2r}(M), \quad \Omega^-(M) \equiv \bigoplus_r \Omega^{2r+1}(M).$$

The corresponding operator are

$$A \equiv \bigoplus_r (d_{2r} + d_{2r-1}^\dagger) \qquad A^\dagger \equiv \bigoplus_r (d_{2r-1} + d_{2r}^\dagger).$$

Then $\operatorname{ind}(\Omega^*(M, A)) = \dim\ker A - \dim\ker A^\dagger$

$= \dim\ker A^\dagger A - \dim\ker A A^\dagger$

$= \dim\ker \Delta_+ - \dim\ker \Delta_-$

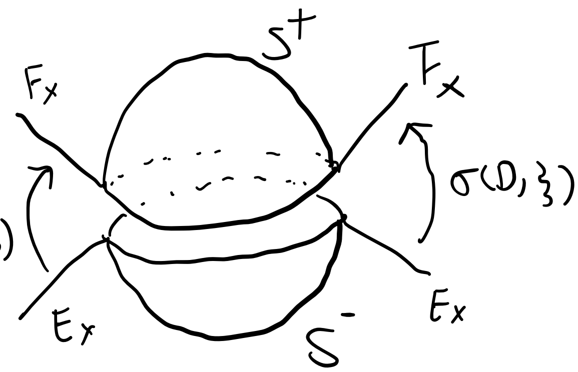
$= \operatorname{ind}(\Omega^*(M), d) = \chi(M)$

The Atiyah-Singer index thm

E, F: complex vector bundles over M, $\dim M = m$

$D: \Gamma(M, E) \to \Gamma(M, F)$ elliptic

locally, $D = \sum_{|\alpha| \leq N} A^\alpha(x) D_\alpha$



The symbol
$$\sigma(D, \xi) = \sum_{|\alpha| = N} A^\alpha(x) \xi_\alpha$$

S^+, S^-: the upper and lower hemispheres of S^m.

Attach $S^- \times E_x$ to $S^+ \times F_x$ by identifying each pair (ξ, v) with the pair $(\xi, \sigma(D, \xi)(v))$, where $v \in E_x$ and $\xi \in S^+ \cap S^-$ is regarded as a unit vector in $\mathbb{R}^m$.

This "clutching construction" results in a bundle over S^m.

As x varies, we obtain a complex vector bundle Σ over the sphere bundle $S^m M$ associated to TM.

$S^m M$: the fiberwise one-point compactification of TM.

Actually, we want a bundle over $B^m M / S^{m-1} M$, where $B^m M$ is the unit disk bundle over M.

$$B^m M = \{ v \in TM \mid \|v\| \leq 1 \}$$

$$S^{m-1} M = \{ v \in TM \mid \|v\| = 1 \}$$

The bundle constructed above is in general nontrivial over $S^+ M$ and hence does not define a bundle over $S^m M / S^+ M = B^m M / S^{m-1} M$. However, the formal difference $\Sigma - F$ is, trivial over $S^+ M$ and hence does define a "virtual" bundle over $B^m M / S^{m-1} M$. This object is called the <u>symbol bundle</u> of D, denoted $\sigma(D)$. Clearly,

$$\sigma(D)\big|_M = E - F.$$

Now consider an elliptic complex $(E, D) = \{ (E_i, D_i) \}$

$$\cdots \xrightarrow{D_{i-1}} \Gamma(M, E_i) \xrightarrow{D_i} \Gamma(M, E_{i+1}) \xrightarrow{D_{i+1}} \cdots$$

$$D_{i+1} D_i = 0$$

The symbol associated to the operators D_i give rise to a sequence

$$\cdots \xrightarrow{\sigma(D_{i-1}, \xi)} E_{i,x} \xrightarrow{\sigma(D_i, \xi)} E_{i+1,x} \xrightarrow{\sigma(D_{i+1}, \xi)} \cdots$$

Since D_i is elliptic, the construction described above gives rise to a virtual bundle on $B^m M / S^{m-1} M$, called the symbol bundle of D and denoted $\sigma(D)$. One then has

$$\sigma(D)|_M = \sum_i (-1)^i E_i .$$

The non compact manifold TM has coordinate nbhds of the form $U \times \mathbb{R}^m$, and hence its tangent bundle has at each point a natural decomposition as $\mathbb{R}^m + \mathbb{R}^m$. A local orientation on M is given by an oriented frame $b_1, \dots, b_m$ in $\mathbb{R}^m$. The usual method to orienting TM is to take the frame $b_1, \dots, b_m, b_1', \dots, b_m'$ (where the $'$ indicates that a vector belongs to the 2nd term of the direct sum),

We will use the orientation given by the frame $b_1, b_1', \dots, b_m, b_m'$.

In any case, since an arbitrary permutation of $b_1, \dots, b_m$ produces an even permutation of $b_1, b_1', \dots, b_m, b_m'$, these local orientations of M give rise to a global orientation of TM.

Each compactly supported cohomology class $a \in H_c^*(TM, \mathbb{Q})$ has a well-defined value $\int_{TM} a \in \mathbb{Q}$.

Chern characters

The total Chern character of a complex vector bundle F is

$$ch(F) = \sum_{j=1}^{k} \exp(x_j)$$

where x_j are the Chern roots of F.

In terms of the elementary symmetric functions $S_r(x_j)$, the total Chern character becomes

$$\sum_{j=1}^{k} \exp(x_j) = \sum_{j=1}^{k} \left(1 + x_j + \frac{x_j^2}{2!} + \frac{x_j^3}{3!} + \cdots\right)$$

$$= k + S_1(x_j) + \frac{1}{2!}\left[S_1(x_j)^2 - 2S_2(x_j)\right] + \cdots$$

Accordingly, each Chern character is expressed in terms of the Chern classes as

$$ch_0(F) = k$$

$$ch_1(F) = c_1(F)$$

$$ch_2(F) = \frac{1}{2}\left[c_1(F)^2 - 2c_2(F)\right]$$

$$\vdots$$

where k is the fibre dimension of the bundle.

Properties of the chern characters

Thm (a) (Naturality) Let $E \xrightarrow{\pi} M$ be a vector bundle with fibre $\mathbb{C}^k$.

Let $f: N \to M$ be a map. Then

$$ch(f^*E) = f^* ch(E)$$

(b) Let E and F be vector bundles over a manifold M. The chern characters of $E \otimes F$ and $E \oplus F$ are given by

$$ch(E \otimes F) = ch(E) \wedge ch(F)$$
$$ch(E \oplus F) = ch(E) + ch(F).$$

The homomorphism ch gives $ch(\sigma(D)) \in H^*(B^m M, S^{m-1}M; \mathbb{Q})$

$= H_c^*(TM, \mathbb{Q})$; compactly supported cohomology.

Todd classes

For a complex vector bundle F,

$$Td(F) = \prod_j \frac{x_j}{1-e^{-x_j}}$$

where x_j are the Chern roots of F.
Expand it in powers of x_j,

$$Td(F) = \prod_j \left(1 + \frac{1}{2} x_j + \sum_{k \geq 1} (-1)^{k-1} \frac{B_k}{(2k)!} x_j^{2k}\right)$$

$$= 1 + \frac{1}{2}\sum_j x_j + \frac{1}{12}\sum_j x_j^2 + \frac{1}{4}\sum_{j<k} x_j x_k + \cdots$$

$$= 1 + \frac{1}{2}c_1(F) + \frac{1}{12}\left[c_1(F)^2 + c_2(F)\right] + \cdots$$

where the B_k are the Bernoulli numbers

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad B_5 = \frac{5}{66}, \ \ldots$$

$$Td_0(F) = 1$$

$$Td_1(F) = \frac{1}{2}c_1$$

$$Td_2(F) = \frac{1}{12}(c_1^2 + c_2)$$

$$\vdots$$

Hirzebruch's motivation of the Todd class

X : compact complex manifold

$Td(X) \in H^{even}(X, \mathbb{Q})$

In general, E : complex vector bundle over X

$Td(E) \in H^{even}(X, \mathbb{Q})$ $\quad (Td(X) = Td(TX))$ satisfies

(1) $Td(E \oplus F) = Td(E) \wedge Td(F)$

(2) $Td(\varepsilon) = 1$ if ε is the trivial line bundle

(3) $\langle Td(\mathbb{C}P^n), [\mathbb{C}P^n] \rangle = 1$ for all n $\Leftarrow$ Hirzebruch-Riemann-Roch thm.

Since $T\mathbb{C}P^n \oplus \varepsilon = F^{\oplus n+1}$ where $F = \bar{L} = Hom(L, \varepsilon)$

ε: trivial line bundle L: canonical line bundle.

$\therefore$ (1) & (2) $\Rightarrow$ $Td(\mathbb{C}P^n) = (Td(F))^{n+1}$

where $Td(F) = 1 + b_1 x + b_2 x^2 + \cdots = f(x)$ $b_i \in \mathbb{Q}$

Since $[\mathbb{C}P^n]$ is Poincaré dual to x^n where $x = c_1(F) = -c_1(L)$ is the so-called positive generator of $H^2(\mathbb{C}P^n, \mathbb{Z})$

(3) $\Rightarrow$ the coefficient of x^n in $f(x)^{n+1}$ is 1 for all n ($\star$)

Thm ($\star$) $\Rightarrow$ $f(x) = \dfrac{x}{1-e^{-x}}$.

Thm (Atiyah-Singer) Let M be a compact manifold without boundary, $\dim M = m$, let (E, D) be an elliptic complex over M, and let $\sigma(D)$ be the symbol bundle of D. Then

$$\operatorname{ind} D = (-1)^m \int_{TM} ch(\sigma(D))\, Td(TM^{\mathbb{C}})$$

Rmk: ① Even when the bundles E_i are trivial, so that the D_i operate on functions, the bundle $\sigma(D)$ will generally be nontrivial.

② When the RHS is positive and D consists of a single operator, the thm $\Rightarrow$ the PDE $Ds = 0$ has at least one solution.

(3) The LHS is always an integer, It is by no means evident a priori that the RHS is an integer.

Let $\psi: H^i(M) \to H_c^{m+i}(TM)$ be the Thom isomorphism (if M is non orientable, twisted coefficients must be used).

To express ind D as the value of a cohomology class on [M]. One may use the formula $\int_{TM} b = (-1)^{\frac{m(m-1)}{2}} \int_M \psi^{-1} b$

(The sign arises from the difference between the present orientation convention for TM and the usual one). To calculate $\psi^{-1}(b)$ note that

$$b = \psi(a) = aU \quad (U: \text{Thom class})$$

implies that

$$i^* b = ae$$

where i^* denotes the restriction to M and $e = e(TM)$. Thus when the solution of the equation $i^* b = ae$ is unique we may write

$$a = \frac{i^* b}{e}$$

It follows that if $e(TM)$ is neither zero nor a zero division, one has

$$\operatorname{ind} D = (-1)^{\frac{m(m+1)}{2}} \int_M \frac{i^* ch(\sigma(D))\, Td(TM^{\mathbb{C}})}{e(TM)}$$

Moreover, since $i^* ch(\sigma(D)) = ch(\sigma(D))\big|_M = ch\left(\sum_i (-1)^i E_i\right)$, we obtain

$$\operatorname{ind} D = (-1)^{\frac{m(m+1)}{2}} \int_M \frac{ch\left(\sum_i (-1)^i E_i\right) Td(TM^{\mathbb{C}})}{e(TM)}.$$

Cor Let $\Gamma(M, E) \xrightarrow{D} \Gamma(M, F)$ be a two-term elliptic complex,

The index of D is given by

$$\begin{aligned}\operatorname{ind} D &= \dim \ker D - \dim \ker D^{+} \\ &= (-1)^{\frac{m(m+1)}{2}} \int_M (ch E - ch F) \frac{Td(TM^{\mathbb{C}})}{e(TM)}.\end{aligned}$$