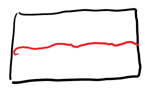
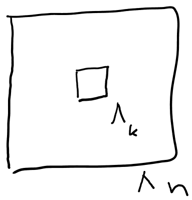
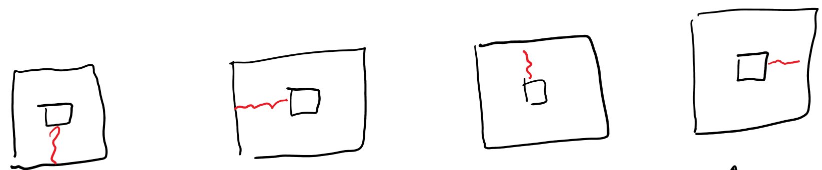
① If $p > p_{cr}$ $\quad \mathbb{P}_p\Big($  $\Big)_{H_n} \longrightarrow 1 \quad n \to \infty$

$n \gg k \gg 1$





$A_1 \quad A_2 \quad A_3 \quad A_4$

$$\mathbb{P}_p(A_1) \geq 1 - \Big[\underbrace{1 - \mathbb{P}_p(A_1 \cup A_2 \cup A_3 \cup A_4)}_{\shortparallel}\Big]^{1/4}$$

$$\mathbb{P}_p(\Lambda_k \not\leftrightarrow \partial\Lambda_n)$$

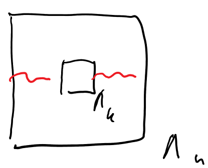
$$\vee\!|$$

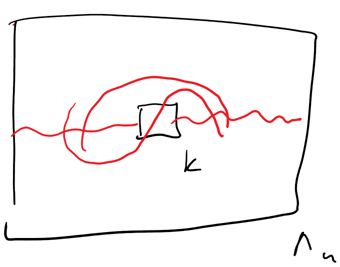
$$\boxed{\mathbb{P}_p(\Lambda_k \not\leftrightarrow \infty) \longrightarrow 0 \quad \text{as } k \to \infty}$$

$$\boxed{\mathbb{P}_p(A_2 \cap A_4)} \geq 1 - 2\,\mathbb{P}_p(\Lambda_k \leftrightarrow \partial\Lambda_n)$$

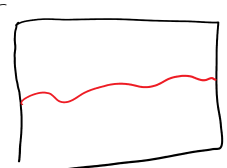
$\mathbb{P}(C) \geq 1 - c \qquad \mathbb{P}(D) \geq 1 - d$

$\Longrightarrow \mathbb{P}(C \cap D) \geq 1 - c - d$

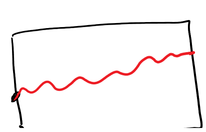




No connection left-right $\Longrightarrow$ two inf clusters. — But this is impossible $\Longrightarrow$ $\exists$ connection:

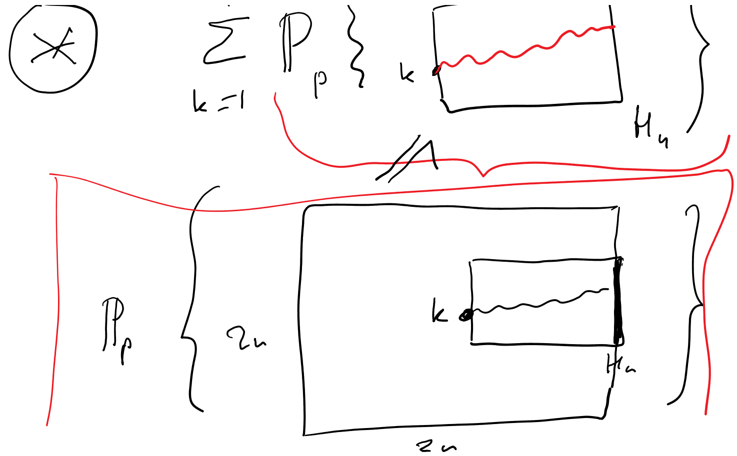
② $p < p_{cr} \Longrightarrow \mathbb{P}_p\Big\{$  $\Big\}_{H_n} \longrightarrow 0 \quad n \to \infty$

$/\!\!|\!|$

(*) $\displaystyle\sum_{k=1}^{n} \mathbb{P}_p\Big\{_k$  $\Big\}$

$C_p > 0$

$$\circledast \quad \sum_{k=1} \mathbb{P}_p \Big\{ \text{crossing from } k \text{ in } H_n \Big\} \le \mathbb{P}_p\Big\{ \dots \Big\} \le \exp\{-c_p n\}$$



$$\circledast \quad \sum_{k}^{n} \text{——}\,''\,\text{——} \le n \exp\{-c_p n\} \longrightarrow 0$$

(exp. fast) ∎

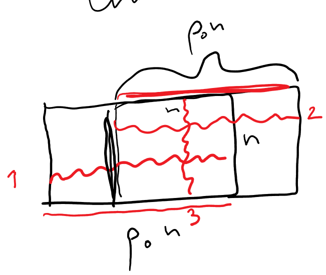
RSW-theory (Russo–Seimour–Welsh)

$$\boxed{\underline{\underline{c(p)}} \le \mathbb{P}_{1/2}\Big\{ \text{crossing of } pn \times n \Big\} \le \underline{\underline{1 - c(p)}}}$$

(*) – do not depend on n,

$$\forall p \quad \exists\, c(p) \in (0, \tfrac12)$$

Proof. 1). If we know (*) for some $\underline{\underline{p_0 > 1}}$, then we know (*) for all p-s.

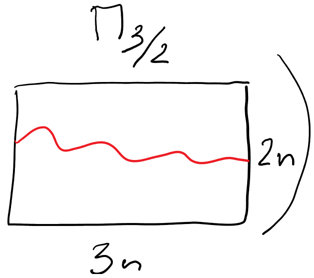


If $1 \cap 2 \cap 3$ – happens $\Longrightarrow$ $\exists$ crossing in $(2p_0 - 1)n$ rectangle.

$$P_{1/2}(1 \cap 2 \cap 3) \ge \underbrace{P_{1/2}(1)\, \underline{\underline{P(2)}}}_{\ge\, c(p_0)^2} P_{1/2}(3)$$

$$\ge c(p_0)^2 \cdot \frac{1}{2}$$

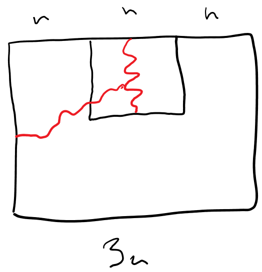
2). $$\mathbb{P}_{1/2}\left(\text{crossing of } 3n \times 2n \right) \ge \frac{1}{128} \; (= 2^7).$$



$\frac{1}{2} \Big|$  $\Big)$ $\qquad (\sim 2^7)$

$3n$

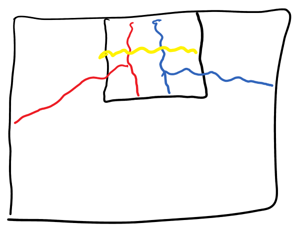
(2a)

$B_n =$ 

If $\mathbb{P}_{1/2}(B_n) \geq c$

– then $\mathbb{P}_{1/2}(\Pi_{3/2}) \geq c^2 \cdot \frac{1}{2}$

Proof.



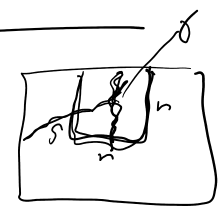
(2b)

$\mathbb{P}_{1/2}(B_n)$

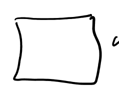
General fact:

if $C_1, \dots, C_k$, $C_i \cap C_j = \emptyset$

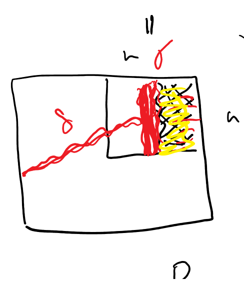
$$P(B) \geq \sum_{i=1}^{k} P(B \cap C_i)$$



$\mathbb{P}_{1/2}(B_n) \geq$

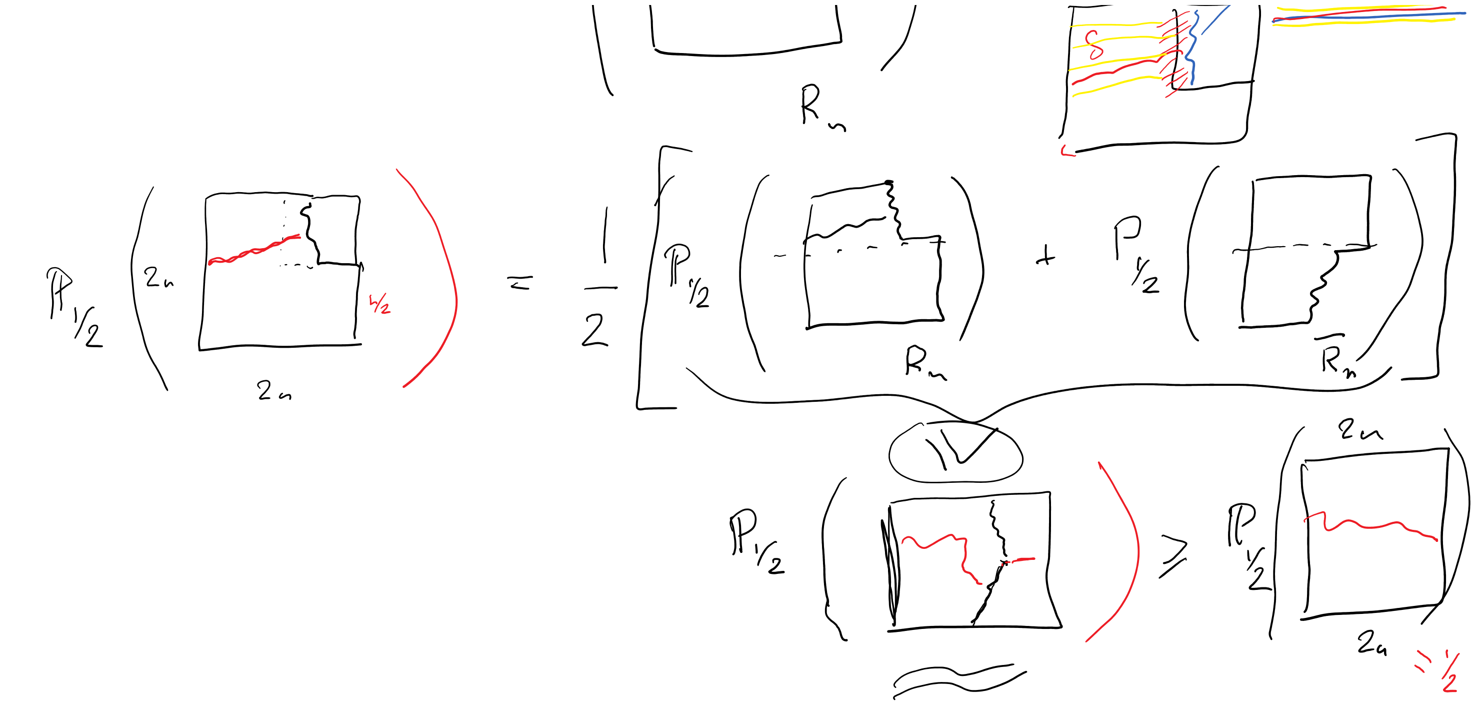
$\geq \sum_\gamma \mathbb{P}_{1/2}\{\gamma;\ \gamma$ – connected ~~by~~ a path δ, γ – is the right most path ($^{top}/_{bottom}$) in  $\}$

$= \sum_\gamma \mathbb{P}_{1/2}(\gamma)\ \underbrace{\mathbb{P}_{1/2}\big(\exists\, \delta \text{ connecting } \gamma \text{ to the left side of } 2n \times 3n \,\big|\, \gamma\big)}$

$\mathbb{P}_{1/2}\Bigg($  $\Bigg)$

because γ – is the right most connection.

left most

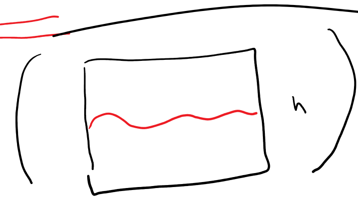


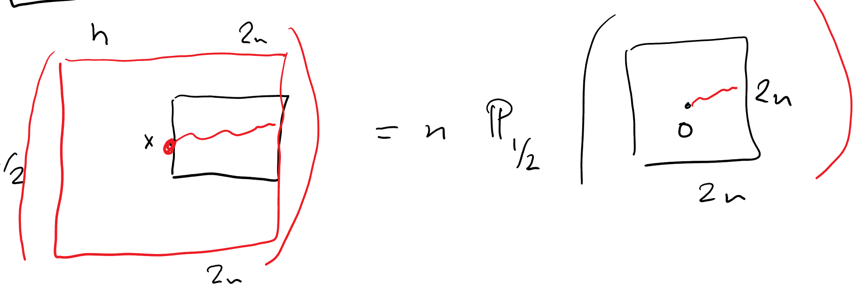
Corollary.

Power-law decay of correlations.

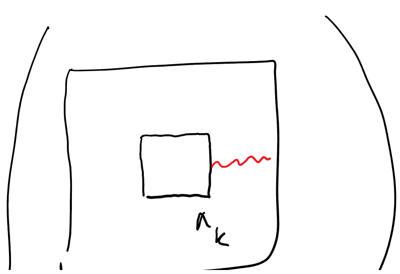
$$\frac{1}{2n} \le \mathbb{P}_{1/2}\left[0 \longleftrightarrow \partial\Lambda_n\right] \le n^{-\alpha}$$

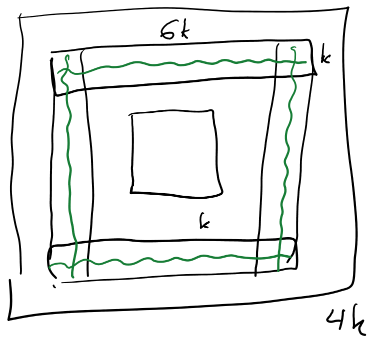
1

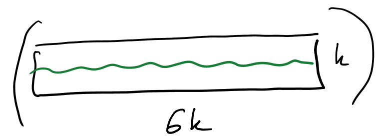
$\frac{1}{2} = \mathbb{P}_{1/2}$  $\le$

$\le \sum_x \mathbb{P}_{1/2}$  $= n\,\mathbb{P}_{1/2}$

2

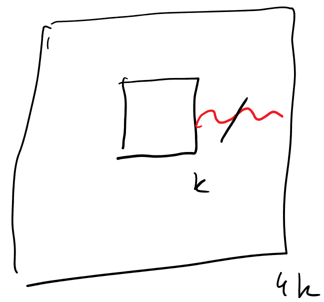
$\mathbb{P}_{1/2}$  $\le \underline{1-c}$ unif. in k

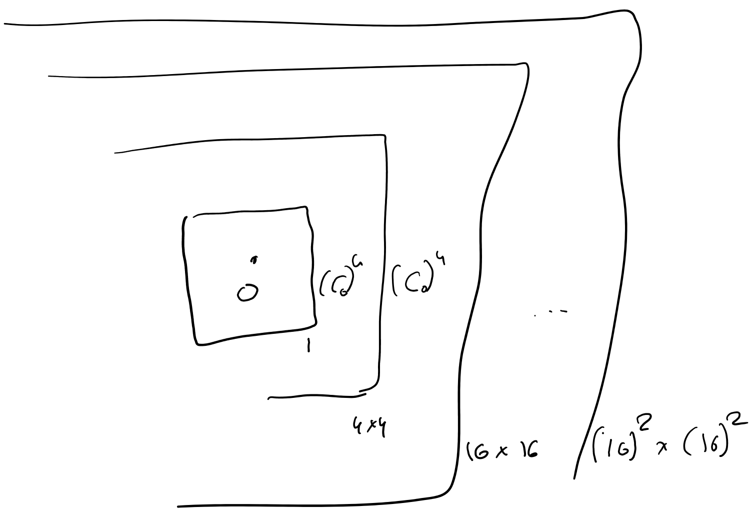


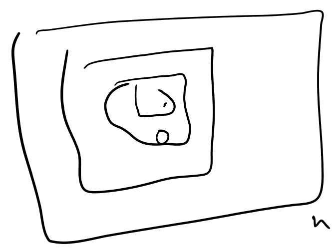
$\mathbb{P}_{1/2}$  $> c_0$

$\Longrightarrow$ no red connection between

$\geq (c_0)^4$







number of layers is $\log_4 n$.

$$\left[(c_0)^4\right]^{\log_4 n} = n^{-\alpha}$$

$\alpha > 0$.