Sharp interface limit for a quasi-linear large deviation rate function

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Outline of the talk

1. Introduction
2. Dynamical large deviations [Landim-T, AIHP, 2018]
3. Motion by mean curvature from Glauber-Kawasaki process
   - 1. [Funaki-T, JSP, 2019]
   - 2. [Funaki-van Meurs-Sethuraman-T, JSP, 2023]
4. Sharp interface limit for LD functional [Kagaya-T, 24+] available at arXiv:2402.12155
5. Summary of the talk

1. Introduction

Glauber-Kawasaki process

- Glauber-Kawasaki process on $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d, N, d \in \mathbb{N}$.

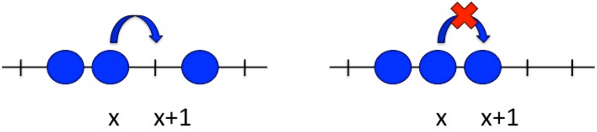
  Exclusion process + Glauber dynamics.

- State space: $\{0,1\}^{\mathbb{T}_N^d} \ni \eta = \{\eta(x)\}_{x \in \mathbb{T}_N^d}$; for $x \in \mathbb{T}_N^d$,

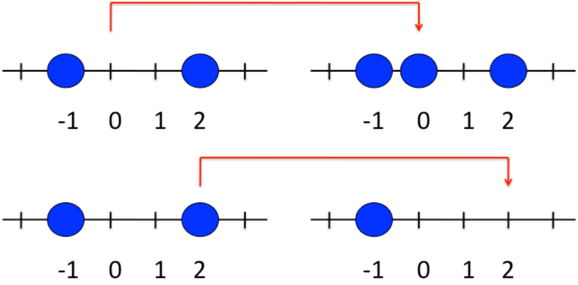
$$\eta(x) = 1 \cdots \text{there is a particle at site } x.$$
$$\eta(x) = 0 \cdots \text{there is no particle at site } x.$$

Model

- Symmetric simple exclusion process, rate N^2:



- Glauber dynamics, rate $c(x, \eta)$:



Generator description

- Let η_t^N be a Markov process on $\{0,1\}^{\mathbb{T}_N^d}$ generated by

$$
\begin{aligned}
L_N f &= N^2 L_E f + L_G f, \\
L_E f(\eta) &= \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_N^d} \{ f(\eta^{x,x+e_j}) - f(\eta) \}, \\
L_G f(\eta) &= \sum_{x \in \mathbb{T}_N^d} c(x,\eta) \{ f(\eta^x) - f(\eta) \}.
\end{aligned}
$$

- In the above formula,

$$
\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise}, \end{cases} \qquad \eta^x(z) = \begin{cases} 1 - \eta(x) & \text{if } z = x, \\ \eta(z) & \text{if } z \neq x. \end{cases}
$$

Glauber jump rate $c(x, \eta)$

- For $0 \le \gamma < 1$,

$$c(x,\eta) = \frac{1}{d}\sum_{j=1}^{d}\Big[1 + \gamma(1-2\eta(x))(\eta(x+e_j)+\eta(x-e_j)-1) + \gamma^2(2\eta(x-e_j)-1)(2\eta(x+e_j)-1)\Big],$$

where $\gamma = \tanh\beta$. (β: inverse temperature)
cf. d-dimensional nearest neighbor Ising model.

Macroscopic behavior

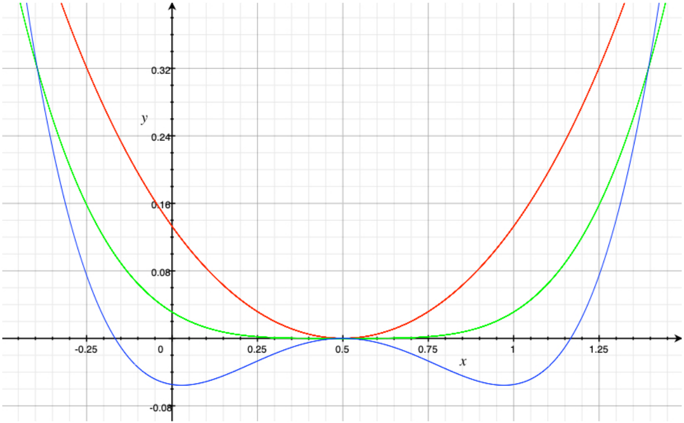
- This model has been introduced by De-Masi, Ferrari and Lebowitz in 1986 to derive the reaction-diffusion equation:

$$\partial_t \rho = \Delta \rho + F(\rho), \quad \rho : [0, \infty) \times \mathbb{T}^d \to [0, 1].$$

- For $\rho \in [0, 1]$, $F(\rho)$ and $W'(\rho) = -F(\rho)$ are given by

$$\begin{aligned} F(\rho) &= B(\rho) - D(\rho) \\ B(\rho) &= E_{\nu_\rho}[c(0, \eta)(1 - \eta(0))] \\ D(\rho) &= E_{\nu_\rho}[c(0, \eta)\eta(0)] \\ W(\rho) &= (1 - 2\gamma)(\rho - 1/2)^2 + 2\gamma^2(\rho - 1/2)^4. \end{aligned}$$

Potential



red: $\gamma = 1/4$, green: $\gamma = 1/2$, blue: $\gamma = 3/4$.

Empirical measure

- For $\eta \in \{0,1\}^{\mathbb{T}_N^d}$,

$$\pi^N(\eta; d\theta) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x) \delta_{x/N}(d\theta).$$

  δ_θ: Dirac measure at $\theta \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d = [0,1)^d$.
- Recall: $L_N = N^2 L_E + L_G$; the diffusive scaling.
- Note: $\pi_t^N(d\theta) = \pi^N(\eta_t^N; d\theta)$ is a random measure on $\mathbb{T}^d$.

Hydrodynamic limit

Theorem 1 (De Masi-Ferrari-Lebowitz, JSP, 1986)

Assume that for some measurable $\rho_0 : \mathbb{T}^d \to [0,1]$

$$\pi_0^N(d\theta) \to \rho_0(\theta)d\theta \text{ in prob. as } N \to \infty.$$

Then for any $t \geq 0$,

$$\pi_t^N(d\theta) \to \rho(t,\theta)d\theta \text{ in prob. as } N \to \infty,$$

where $\rho(t,\theta)$ *is the unique weak sol. to*

$$\begin{cases} \partial_t \rho = \Delta\rho + F(\rho), \\ \rho(0,\cdot) = \rho_0(\cdot). \end{cases}$$

2. Dynamical large deviations

- Fix a trajectory $\pi(t, dx) = \phi(t, x)dx, \phi : [0, T] \times \mathbb{T}^d \to [0, 1]$. For any function $H \in C^{1,2}([0, T] \times \mathbb{T}^d)$, define

$$
\begin{aligned}
J^H(\pi) = & \int_{\mathbb{T}^d} \phi(T, x)H(T, x)\, dx - \int_{\mathbb{T}^d} \phi(0, x)H(0, x)\, dx \\
& - \int_0^T \int_{\mathbb{T}^d} \left\{ \phi \partial_t H + \phi \Delta H + \chi(\phi)|\nabla H|^2 \right\}\, dxdt \\
& - \int_0^T \int_{\mathbb{T}^d} \left\{ B(\phi) \left(e^H - 1\right) + D(\phi_t) \left(e^{-H} - 1\right) \right\}\, dxdt.
\end{aligned}
$$

where $\chi(\alpha) = \alpha(1 - \alpha)$.

- Let $S : D([0, T], \mathcal{M}_+) \to [0, \infty]$ be the function defined by

$$
S(\pi) = \begin{cases} \sup_{H \in C^{1,2}} J^H(\pi) & \text{if } \pi(t, dx) = \phi(t, x)dx\ , \\ \infty & \text{otherwise}\ . \end{cases}
$$

Previous works

- LDP: $\mathbb{P}(\pi_\cdot^N \sim \pi) \sim \exp\{-N^d S(\pi)\}$
- LDP for exclusion process
  [Kipnis-Olla-Varadhan, CPAM, 1989]
- LDP for Glauber-Kawasaki process
  [Jona Lasinio-Landim-Vares, PTRF, 1993]
  [Landim-T, AIHP, 2017]
  cf. [Farfan-Landim-T, PTRF, 2019]
- LDP for Zero-range process
  [Benois-Kipnis-Landim, SPA, 1995]
  [Fehrman-Gess, Inventiones, 2023]
- LDP for non-gradient exclusion process
  [Bertini-Faggionato-Gabrielli, AAP, 2013]

3. Motion by mean curvature

Glauber-Kawasaki process

$$L_N = N^2 L_{Kaw} + L_{Gla}$$

$$\xrightarrow{N \to \infty}$$

Hydrodynamic equation

$$\begin{cases} \partial_t \rho = \Delta \rho + F(\rho) \\ \rho(0, \cdot) = \chi_{\Gamma_0}(\cdot) \end{cases}$$

Glauber-Kawasaki process

$L_N = N^2 L_{Kaw} + K L_{Gla}$

$\xrightarrow{N \to \infty}$

Hydrodynamic equation

$$\begin{cases} \partial_t \rho = \Delta \rho + K F(\rho) \\ \rho(0, \cdot) = \chi_{\Gamma_0}(\cdot) \end{cases}$$

Hydrodynamic equation

$$\begin{cases} \partial_t \rho = \Delta\rho + K F(\rho) \\ \rho(0,\cdot) = \chi_{\Gamma_0}(\cdot) \end{cases}$$

$K \to \infty$

Motion by mean curvature

$\{\Gamma_t : t \in [0,T]\}$: closed & smooth hypersurfaces

$$\begin{cases} \mathrm{V} = \kappa \\ \Gamma_t|_{t=0} = \Gamma_0 \end{cases}$$

V: inword normal velocity, κ: $(d-1)\times$mean curvature

Glauber-Kawasaki process

$L_N = N^2 L_{Kaw} + K_N L_{Gla}$

$$\searrow \quad \begin{aligned} &N \to \infty \\ &K_N \to \infty \end{aligned}$$

Motion by mean curvature

$\{\Gamma_t : t \in [0,T]\}$: closed & smooth hypersurfaces

$$\begin{cases} \mathrm{V} = \kappa \\ \Gamma_t|_{t=0} = \Gamma_0 \end{cases}$$

V: inword normal velocity, κ: $(d-1)\times$mean curvature

Previous works

- [Katsoulakis-Souganidis, ARMA, 1994] $\cdots$ level-set method
- [Bonaventura, Nonl Anal, 1995] $\cdots$ correl. function method
- [Funaki-T, JSP, 2019] $\cdots$ entropy method
- [EFHPS, CMP, 2022] $\cdots$ entropy method
  for Glauber+Zero-range process

$$\partial_t \rho = \Delta P(\rho) + F(\rho)$$

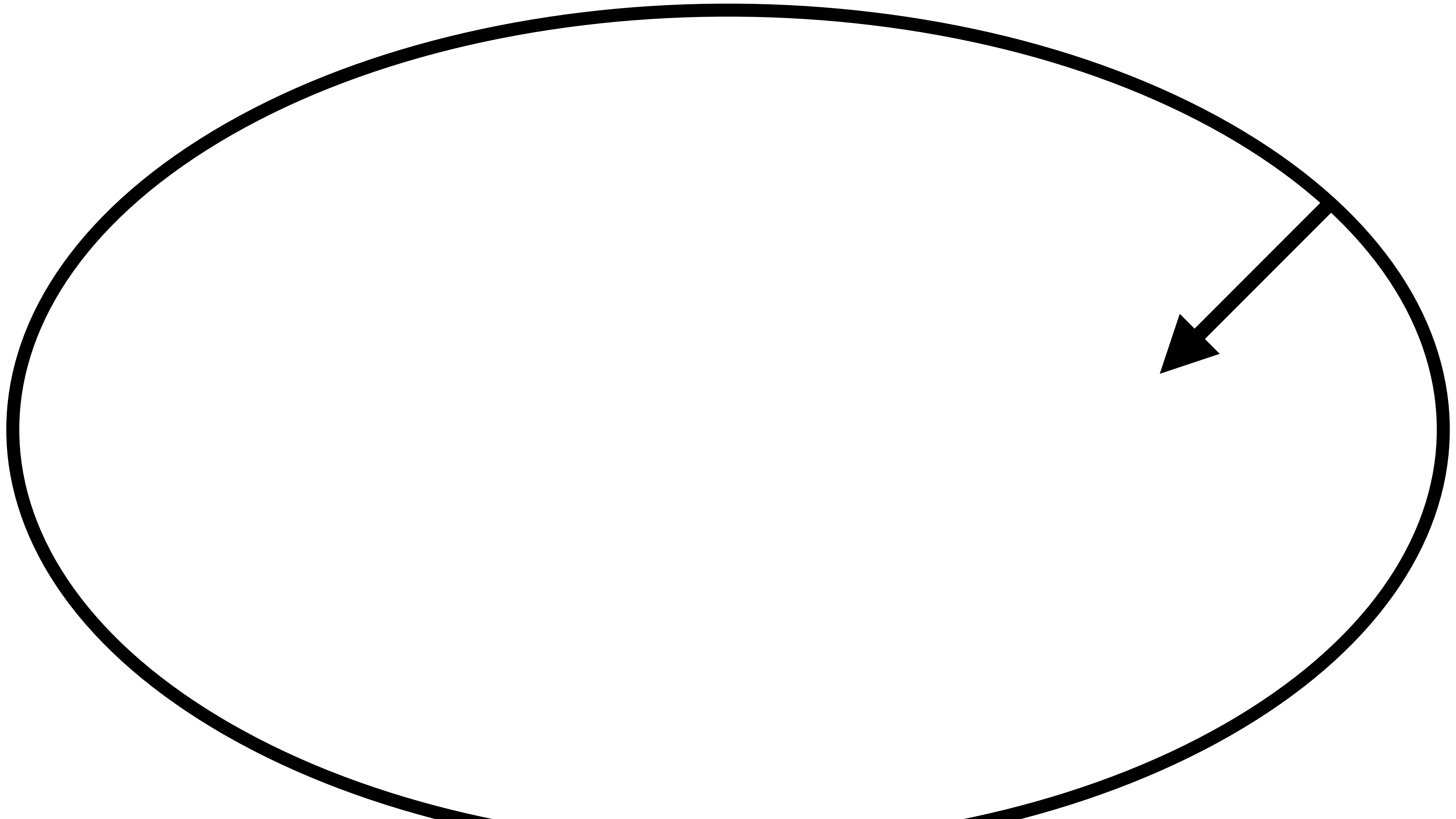
- [F-van Meurs-S-T, JSP, 2023] $\cdots$ entropy method
  for Glauber+gradient Kawasaki process
- [Funaki, in preparation] $\cdots$ Non-gradient case

$\Gamma = \{\Gamma_t\}_{t \in [0,T]}$: hypersurfaces

$d(t,\cdot)$: the signed distance from Γ_t

V_t: the inward normal velocity

κ_t: the mean curvature



4. Sharp interface limit for LD functional

- Fix a density evolution $\phi : [0, T] \times \mathbb{T}^d \to [0, 1]$.
  For any function $H \in C^{1,2}([0, T] \times \mathbb{T}^d)$, define

$$
\begin{aligned}
J^H(\phi) = & \int_{\mathbb{T}^d} \phi(T, x) H(T, x)\, dx - \int_{\mathbb{T}^d} \phi(0, x) H(0, x)\, dx \\
& - \int_0^T \int_{\mathbb{T}^d} \left\{ \phi \partial_t H + P(\phi) \Delta H + \sigma(\phi) |\nabla H|^2 \right\}\, dxdt \\
& - \int_0^T \int_{\mathbb{T}^d} \left\{ B(\phi) \left(e^H - 1\right) + D(\phi_t) \left(e^{-H} - 1\right) \right\}\, dxdt.
\end{aligned}
$$

  where $\sigma(\alpha) = P(\alpha)\alpha(1 - \alpha)$ and $S(\phi) = \sup_{H \in C^{1,2}} J^H(\phi)$.
- S is the (expected) LD rate functional from the HDL of the Glauber+ gradient Kawasaki process.

We consider the smooth functions

$$P, B, D, W : [0,1] \to \mathbb{R}$$

satisfying the following conditions:

(A1) $P(0) = 0$ and $P'(\rho) > 0$ for any $\rho \in [0,1]$.

(A2) $B(\rho) + D(\rho) > 0$ for any $\rho \in [0,1]$ and $B - D = -W'$.

(A3) W is a double-well potential: that is, there exist exactly three critical points $0 < \rho_- < \rho_* < \rho_+ < 1$ such that $W(\rho_\pm) < W(\rho)$ for any $\rho \neq \rho_\pm$ and $W''(\rho_\pm) > 0$.

(A4) P-balance condition:

$$\int_{\rho_-}^{\rho_+} W'(\rho) P'(\rho) \, d\rho = 0.$$

- Let us introduce the scaled LD functional: for each $\varepsilon > 0$,

$$
\begin{aligned}
J_\varepsilon^H(\phi) = &\frac{1}{\varepsilon}\int_{\mathbb{T}^d} \phi(T,x)H(T,x)\,dx - \frac{1}{\varepsilon}\int_{\mathbb{T}^d} \phi(0,x)H(0,x)\,dx \\
&- \frac{1}{\varepsilon}\int_0^T \int_{\mathbb{T}^d} \left\{\phi\partial_t H + P(\phi)\Delta H + \sigma(\phi)|\nabla H|^2\right\}\,dxdt \\
&- \frac{1}{\varepsilon^3}\int_0^T \int_{\mathbb{T}^d} \left\{B(\phi)\left(e^H - 1\right) + D(\phi_t)\left(e^{-H} - 1\right)\right\}\,dxdt,
\end{aligned}
$$

and set $S_\varepsilon(\phi) = \sup_{H\in C^{1,2}} J_\varepsilon^H(\phi)$.

Question What is the limit of $S_\varepsilon(\phi)$ when ϕ creates a separating interface $\{\Gamma_t\}_{t\in[0,T]}$?

- Let $\Gamma = \{\Gamma_t\}_{t\in[0,T]}$ be a family of smooth hypersurfaces.
- V_t: the inward normal velocity of Γ_t
- κ_t: the mean curvature of Γ_t
- $d(t,\cdot)$: the signed distance from Γ_t
- Define the action functional $S_{\mathrm{ac}}(\Gamma)$ by

$$S_{\mathrm{ac}}(\Gamma) = \int_0^T \int_{\Gamma_t} \frac{(V_t - \theta\kappa_t)^2}{4\mu}\, d\mathcal{H}^{d-1} dt,$$

$\mathcal{H}^{d-1}$: the $(d-1)$-dimensional Hausdorff measure of $\mathbb{T}^d$.

- We choose the constants θ and μ later.

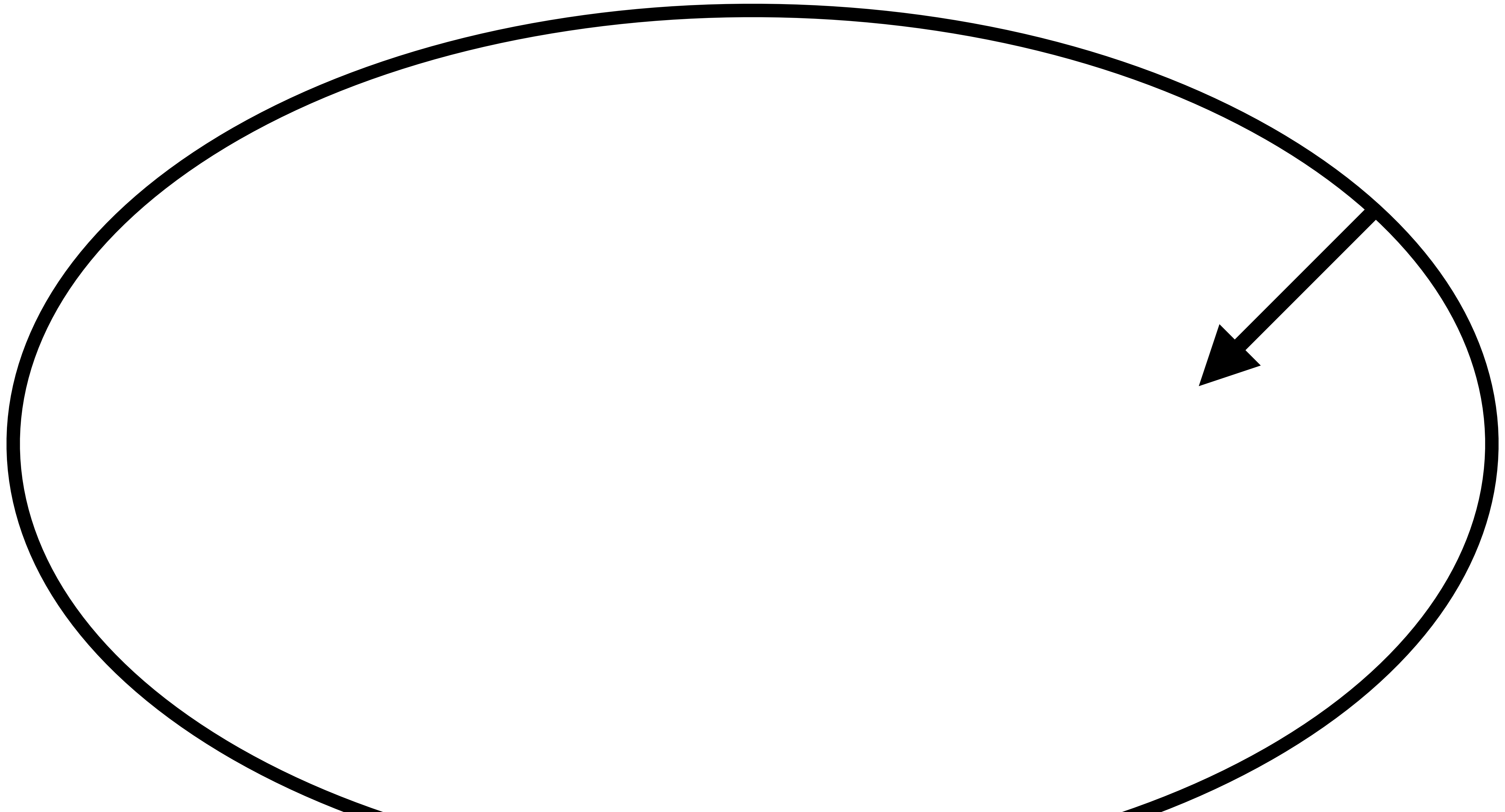
  θ: mobility, μ: transport coefficient

$\Gamma = \{\Gamma_t\}_{t \in [0,T]}$: hypersurfaces

$d(t, \cdot)$: the signed distance from Γ_t

V_t: the inward normal velocity

κ_t: the mean curvature



- Let $\bar{u} : \mathbb{R} \to (0,1)$ solve the following one-dimensional ODE:

$$\begin{cases} (P(\bar{u}))'' + B(\bar{u}) - D(\bar{u}) = 0 \quad \text{in } \mathbb{R}, \\ \bar{u}(\pm\infty) = \rho_\pm, \quad \bar{u}(0) = \dfrac{\rho_+ + \rho_-}{2}. \end{cases}$$

- For each bounded $Q : [0, T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$, set

$$\phi_\varepsilon(t,x) = \bar{u}\left(\frac{d(t,x)}{\varepsilon} + \varepsilon Q\left(t, x, \frac{d(t,x)}{\varepsilon}\right)\right).$$

- Note that for each $t \in [0, T]$,

$$\lim_{\varepsilon \to 0} \phi_\varepsilon(t,\cdot) = \begin{cases} \rho_+ & \text{on the exterior of } \Gamma_t, \\ \rho_- & \text{on the interior of } \Gamma_t. \end{cases}$$

- Define the linear operator $L_{\bar{u}}$ by

$$L_{\bar{u}}\psi(\xi) = [2\sigma(\bar{u}(\xi))\psi'(\xi)]' - [B(\bar{u}(\xi)) + D(\bar{u}(\xi))]\,\psi(\xi).$$

- Let $\bar{v} = P(\bar{u})$ and ν be the constant defined by

$$\nu = \langle \bar{v}', (-L_{\bar{u}})\bar{v}' \rangle_{L^2}/2.$$

- Define the constants θ_1, θ_2 by

$$\theta_1 = \int_{\rho_-}^{\rho_+} \sqrt{2\widetilde{W}(\rho)}\, d\rho, \quad \theta_2 = \int_{\rho_-}^{\rho_+} P'(\rho)\sqrt{2\widetilde{W}(\rho)}\, d\rho, \qquad (1)$$

where the function $\widetilde{W}$ is defined as

$$\widetilde{W}(\rho) = \int_{\rho_-}^{\rho} W'(\widetilde{\rho})P'(\widetilde{\rho})\, d\widetilde{\rho}. \qquad (2)$$

- Then define the mobility μ and the transport coefficient θ by

$$\mu := \nu/\theta_1^2, \quad \theta := \theta_2/\theta_1.$$

Theorem 2 (Kagaya-T, 24+)

Let $\Gamma = \{\Gamma_t\}_{t\in[0,T]}$ *be a family of smooth hypersurfaces. For smooth functions* $Q : [0,T]\times\mathbb{T}^d\times\mathbb{R}\to\mathbb{R}$*, define*

$$\phi_\varepsilon(t,x) = \bar{u}\left(\frac{d(t,x)}{\varepsilon} + \varepsilon Q\left(t, x, \frac{d(t,x)}{\varepsilon}\right)\right).$$

1. *If* Q *is "nice",*

$$\liminf_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon) \geq S_{\mathrm{ac}}(\Gamma).$$

2. *There exists* $\widehat{Q} = \widehat{Q}(\Gamma)$ *such that, choosing* $Q = \widehat{Q}$

$$\lim_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon) = S_{\mathrm{ac}}(\Gamma).$$

Related results

- Stochastic reaction-diffusion equation

$$I_\varepsilon(\rho) = \frac{1}{4}\int_0^T \int_{\mathbb{T}^d} \left(\partial_t \rho - \Delta\rho + \varepsilon^{-2} W'(\rho)\right)^2 \, dxdt.$$

[Kohn-Otto-Reznikoff-Vanden Eijnden, 2007, CPAM]

- Glauber dynamics for Ising systems with Kac potentials
- Glauber+Kawasaki process ($P(u) = u$)
  [Bertini-Buttà-Pisante, 2019, AHP]
- In any cases, the limit functional is given by

$$S_{\mathrm{ac}}(\Gamma) = \int_0^T \int_{\Gamma_t} \frac{(V_t - \theta\kappa_t)^2}{4\mu} \, d\mathcal{H}^{d-1} dt.$$

Some remarks

- For the Glauber dynamics for Ising systems with Kac potentials, the physical constants θ, μ can be explicitly calculated in terms of the functions appearing in equilibrium statistical mechanics.
- Recall $\bar{v} = P(\bar{u})$. $\bar{v}$ satisfies the ODE

$$\begin{cases} \bar{v}'' + B \circ P^{-1}(\bar{v}) - D \circ P^{-1}(\bar{v}) = 0 \quad \text{in } \mathbb{R}, \\ \bar{v}(\pm\infty) = P(\rho_\pm), \quad \bar{v}(0) = P\left(\dfrac{\rho_+ + \rho_-}{2}\right). \end{cases}$$

- Note $\bar{u} = \bar{v}$ in the case of $P(u) = u$ ([BBP, 19, AHP]).
$\rightarrow$ We must care about the difference between $\bar{u}$ and $\bar{v}$.

Sketch of the proof

- For each smooth ϕ_ε, one can solve the equation in H

$$\partial_t \phi_\varepsilon \ = \ \Delta P(\phi_\varepsilon) - \nabla(\sigma(\phi_\varepsilon)\nabla H) + \frac{1}{\varepsilon^2}\left(B(\phi_\varepsilon)e^{H} - D(\phi_\varepsilon)e^{-H}\right).$$

- Let H_ε be the solution. Using H_ε, $S_\varepsilon(\phi_\varepsilon)$ is expressed as

$$\begin{aligned} S_\varepsilon(\phi_\varepsilon) = &\frac{1}{\varepsilon}\int_0^T \int_{\mathbb{T}^d} \sigma(\phi_\varepsilon)|\nabla H_\varepsilon|^2 \, dxdt \\ &+ \frac{1}{\varepsilon^3}\int_0^T \int_{\mathbb{T}^d} B(\phi_\varepsilon)\left(1 - e^{H_\varepsilon} + H_\varepsilon e^{H_\varepsilon}\right) \, dxdt \\ &+ \frac{1}{\varepsilon^3}\int_0^T \int_{\mathbb{T}^d} D(\phi_\varepsilon)\left(1 - e^{-H_\varepsilon} - H_\varepsilon e^{-H_\varepsilon}\right) \, dxdt. \end{aligned}$$

- So we need to expand $S_\varepsilon(\phi_\varepsilon)$ in a proper manner.
- For this purpose, we propose the expansion of the form

$$\frac{1}{\varepsilon}\int_0^T \int_{\mathbb{T}^d} A(t, x, d(x,t)/\varepsilon) dxdt + error,$$

where $A : [0, T] \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$ is some function.
- Then the co-area formula shows

$$\lim_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon) = \int_0^T \int_{\Gamma_t}\int_{\mathbb{R}} A(t, x, \xi)\ dtdx\mathcal{H}^{d-1}(d\xi).$$

- Letting $d_\varepsilon(t,x) = d(t,x)/\varepsilon$, we can expand $\phi_\varepsilon, H_\varepsilon$ in ε as

$$\phi_\varepsilon = \bar{u}(d_\varepsilon) + \varepsilon \bar{u}'(d_\varepsilon) Q_{d_\varepsilon} + \text{error},$$
$$H_\varepsilon(t,x) = \varepsilon \widehat{H}_1(t,x,d_\varepsilon) + \text{error},$$

where $\widehat{H}_1$ is the solution to

$$\begin{aligned} L_{\bar{u}} \widehat{H}_1(t,x,\xi) &= \bar{v}'(\xi)\Delta d(t,x) + 2\bar{v}''(\xi)\partial_\xi Q(t,x,\xi) \\ &\quad + \bar{v}'(\xi)\partial_\xi^2 Q(t,x,\xi) - \bar{u}'(\xi)\partial_t d(t,x) \\ &=: F_Q(t,x,\xi), \end{aligned}$$

where

$$L_{\bar{u}}\psi(\xi) = [2\sigma(\bar{u}(\xi))\psi'(\xi)]' - [B(\bar{u}(\xi)) + D(\bar{u}(\xi))]\,\psi(\xi).$$

- When $\bar{u} = \bar{v}$, the right-hand side becomes

$$\bar{u}'(\xi)\left(\Delta d(t,x) - \partial_t d(t,x)\right) + 2\bar{u}''(\xi)\partial_\xi Q(t,x,\xi) \\ + \bar{u}'(\xi)\partial_\xi^2 Q(t,x,\xi).$$

- If we choose $Q(t,x,\xi) = \left(\Delta d(t,x) - \partial_t d(t,x)\right) q(\xi)$ for some function $q(\xi)$, finding a solution to

$$L_{\bar{u}}\widehat{H}_1(t,x,\xi) = F_Q(t,x,\xi)$$

is reduces to the one to

$$L_{\bar{u}}\widehat{\psi}(\xi) = \bar{u}'(\xi) + 2\bar{u}''(\xi)q'(\xi) + \bar{u}'(\xi)q''(\xi),$$

which is solvable since $L_{\bar{u}}$ is invertible.

- After simple (but quite tedious) computations,

$$\lim_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon) = \frac{1}{2}\int_0^T \int_{\Gamma_t}\int_{\mathbb{R}} (-L_{\bar{u}}\widehat{H}_1)\widehat{H}_1 \, d\xi d\mathcal{H}^{d-1}(x)dt.$$

- Solving the minimizing problem

$$\min \int_{\mathbb{R}} F_Q(-L_{\bar{u}})^{-1}F_Q \, d\xi,$$

for each t, x, and choosing Q as the minimizer $\widehat{Q}$, we get

$$\lim_{\varepsilon\to 0} S_\varepsilon(\phi_\varepsilon) = \int_0^T \int_{\Gamma_t} \frac{(V_t - \theta\kappa_t)^2}{4\mu} \, d\mathcal{H}^{d-1}dt = S_{\mathrm{ac}}(\Gamma).$$

5. Summary of the talk

Summary of the talk

Previous result

- Dynamical large deviations [Landim-T, AIHP, 2018]
- Motion by mean curvature
  [Funaki-T, 2019, JSP]
  [Funaki-van Meurs-Sethuraman-T, 2023, JSP]

Main result

- Sharp interface limit of the LD rate functional for the Glauber+gradient Kawasaki process
  generalization of [BBP, 2019, AHP] to the quasilinear setting

Future work

- LDP from the MMC.
- Extension to the non-gradient setting.

Thank you for your attention.
謝謝!