$\exists\ \mathcal{A}$ - an algorithm : p, $\mathcal{A}$: check, whether $p < p_{cr}$, or not,

If $p < p_{cr}$ → $\mathcal{A}$ will stop, YES, $\underline{p < p_{cr}}$.

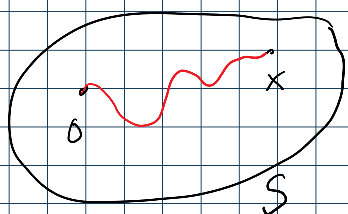
$p > p_{cr}$ → $\mathcal{A}$ will never stop.

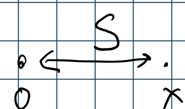
Also true for the Ising.

For 2D Ising $T_{cr}(d=2)$ is exactly known (also for 2D perc.).

Not the case for D-dim. Ising, $D>2$.

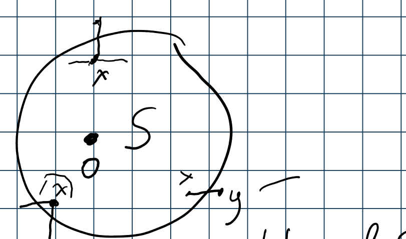
$$\mathbb{P}_p\Big(\underbrace{0 \leftrightarrow x}_{\text{is NOT local.}}\Big)$$





$\forall\ S$ - finite, connected

$$\varphi_p(S) = p \sum_{\substack{(x,y)\in\partial S \\ \text{b.b. of } S \\ x\in S,\ y\notin S}} \underbrace{\mathbb{P}_p\big(0 \xrightarrow{S} x\big)}$$



$\boxed{\text{TH}}$

1. If $\exists\ S$: $\underline{\underline{\varphi_p(S) < 1}} \implies \underline{\underline{\theta(p) = 0}}$.

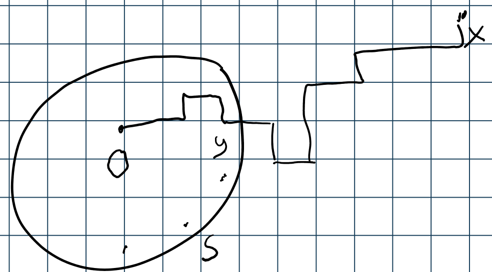
2. If $\forall\ S$ $\underline{\underline{\varphi_{p_0}(S) \geq 1}} \implies$ ~~we have percolation~~
for some p_0

$\forall\ p > p_0$ we have

for $x \geq p_0$

$\forall p > p_0$ we have percolation.

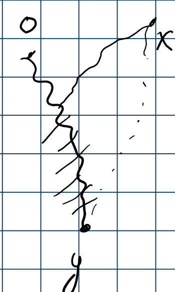
Th



$$\mathbb{P}_p\left(0 \longleftrightarrow x\right) \leq \sum_{y \in S} \mathbb{P}_p\left[0 \overset{S}{\longleftrightarrow} y\right] \mathbb{P}_p\left[y \longleftrightarrow x\right]$$

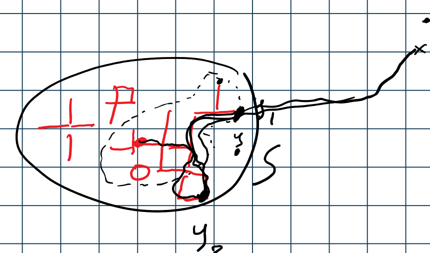
$$\mathbb{P}_p\left[0 \longleftrightarrow x\right] \leq \sum_{y \in \partial S} \underbrace{\mathbb{P}_p\left(0 \overset{S}{\longleftrightarrow} y,\ y \longleftrightarrow x\right)}_{\vee}$$

$$\mathbb{P}_p\left(0 \longleftrightarrow y\right) P\left(y \longleftrightarrow x\right)$$



Proof

C - connected cluster of 0 in S



$$\mathbb{P}_p\left[0 \longleftrightarrow x\right] =$$

$$= \sum_C \mathbb{P}_p\left[0 \longleftrightarrow x,\ C\right] \leq$$

$$\leq \sum_C \sum_{y \in \partial S} \underbrace{\mathbb{P}_p\left[C,\ 0 \overset{C}{\longleftrightarrow} y,\ y \overset{C^c}{\longleftrightarrow} x\right]}$$

$$\parallel$$

$$\mathbb{P}_p\left[C,\ 0 \overset{C}{\longleftrightarrow} y\right] \mathbb{P}_p\left[y \overset{C^c}{\longleftrightarrow} x\right]$$

$$\wedge$$

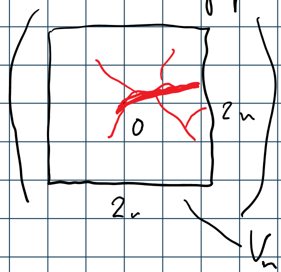
$$\mathbb{P}_p\left[y \longleftrightarrow x\right]$$

$$\leq \sum_{y \in \partial S} \mathbb{P}_p\left(y \longleftrightarrow x\right) \sum_{C} \mathbb{P}_p\left(C,\ 0 \overset{C}{\longleftrightarrow} y\right)$$

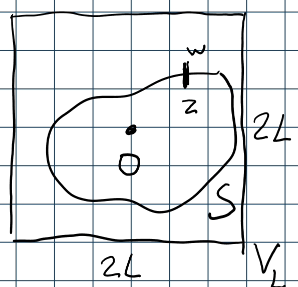
$$\leq \sum_{y \in \partial S} \mathbb{P}_p(y \longleftrightarrow \ldots) \underbrace{\sum \mathbb{P}_p(C, \ldots)}_{\mathbb{P}_p(0 \overset{S}{\longleftrightarrow} y)}$$

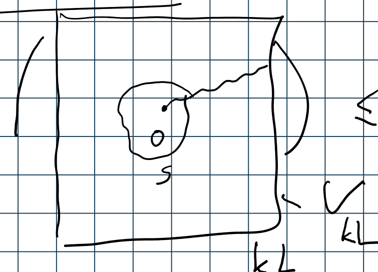
$\blacksquare$

Proof of (1). Suppose $\boxed{\exists S \; : \; \varphi_p(S) < 1} \implies \theta(p) = 0.$

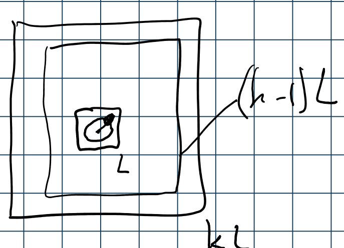
$\theta_n(p) := \mathbb{P}_p\Big($  $\Big)$

$\underline{\theta_n(p) \longrightarrow 0 \quad \text{as } n \to \infty.}$ (exp fast)



$\theta_{kL}(p) = \mathbb{P}_p\Big($  $\Big) \leq$

$$\leq \sum_{\substack{(z,w)\in\partial S \\ z \in S,\, w \notin S}} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} z)\, p\, \underbrace{\mathbb{P}_p(w \longleftrightarrow \partial V_{kL})}_{\leq \theta_{(k-1)L}(p)}$$



$$\leq \theta_{(k-1)L}(p)\, \varphi_p(S) \qquad \varphi_p(S) < 1$$

$$\leq \exp\{-c\,k\}$$

$$\theta_{kL}(p) \xrightarrow[k \to \infty]{} 0 \qquad c = c(S)$$

$\blacksquare$

Margulis - Russo identity.

A - an ~~incr~~ increasing event.

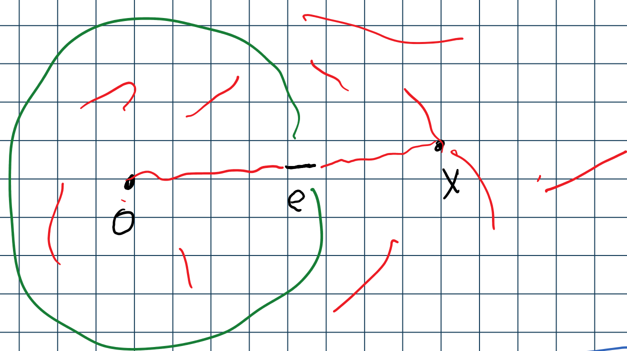
$\implies \mathbb{P}_p(A)$ - increases in p; $\left[\mathbb{P}_p(A)\right]'_p > 0$

$$\left[\mathbb{P}(A)\right]' = \sum \mathbb{P}\left[\mathbb{P}_{\ldots}(A)\right]$$

$$[\mathbb{P}_p(A)]'_p = \sum_{e\in\mathbb{E}} \mathbb{P}_p\Big[\underbrace{\mathrm{Piv}_e(A)}\Big]$$

e – is a pivotal bond for A to happen.

Situation around e is such that the event A hangs on what happens on e: if e – open – A happen
e closed – A – does not happen.



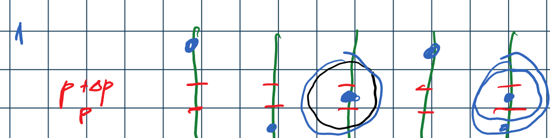
Proof.

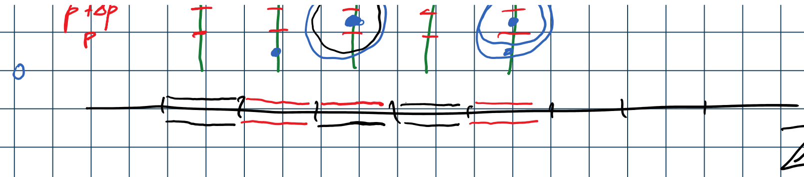
$$\boxed{\frac{\mathbb{P}_{p+\Delta p}(A) - \mathbb{P}_p(A)}{\Delta p}} \xrightarrow[\Delta p \to 0]{}$$

$$\mathbb{P}_p(A) = \sum_{\omega\in\Omega} I_A(\omega)\,\mathbb{P}_p(\omega)$$

$$\mathbb{P}_{p+\Delta p}(A) = \sum_{\omega'\in\Omega} I_A(\omega')\,\mathbb{P}_p(\omega')$$

$$\mathbb{P}_{p+\Delta p}(A) - \mathbb{P}_p(A) = \sum_{\omega,\omega'} \underbrace{\Big[I_A(\omega') - I_A(\omega)\Big]}_{>0} \underbrace{\Pi_{p+\Delta p,\,p}(\omega',\omega)}_{\omega'\geq\omega}$$





$\mathbb{Z}^2$