

对于一个单形 σ ，我们可以将其看成重心 $\hat{\sigma}$ 关于边界 $\partial\sigma$ 的锥，所以其上的点可以表达为 $x = (1 - t)y + t\hat{\sigma}$, $y \in \partial\sigma, 0 \leq t \leq 1$, 令

$$\phi_\sigma: \partial\sigma \times [0, 1) \longrightarrow \sigma, \quad \phi_\sigma(y, t) = (1 - t)y + t\hat{\sigma}. \quad (22)$$

这时像集 $\text{Im}(\phi_\sigma) = \sigma \setminus \hat{\sigma}$.

(注. ϕ_σ 是一个嵌入映射，将 σ 看成带边流形， $\text{Im}(\phi_\sigma)$ 是边界的**领带**(collar)。我们下面是用 ϕ_σ 构造多面体的一族子集作为弱拓扑的基，因而把 ϕ_σ 看成集合间的函数。)

设 K 是一个单纯复形。给定一个单形 $\sigma \in K$ ，令

$$\text{co}\mathcal{F}(\sigma)_p = \{\tau \in K \mid \sigma \leq \tau, \dim \tau - \dim \sigma = p\}, \quad p > 0,$$

即以 σ 为面维数差是 p 的单形集，以及 $\text{co}\mathcal{F}(\sigma) = \bigcup_{p=1}^{\infty} \text{co}\mathcal{F}(\sigma)_p$ 。

σ 的余面高度系统为一个函数

$$\epsilon: \text{co}\mathcal{F}(\sigma) \longrightarrow (0, 1), \quad (23)$$

即对于 σ 的每个真余面配置一个小于1的正数 ϵ_τ 。

对于 σ 内点集的任一开子集 $U \subseteq \text{Int}(\sigma)$, 以及每个余面高度系统 ϵ , 我们按下面步骤构造一个 $|K|$ 的子集 $B(\sigma, U, \epsilon)$:

1) 对于 σ 的余面 μ , 归纳定义: $B(\sigma, U, \epsilon)_\sigma = U$, 对 $\tau \in \text{co}\mathcal{F}(\sigma)_1$,

$$B(\sigma, U, \epsilon)_\tau = \phi_\tau(U \times [0, \epsilon_\tau)),$$

以及对任一 $\mu \in \text{co}\mathcal{F}(\sigma)_{p+1}$ ($p \geq 0$),

$$B(\sigma, U, \epsilon)_\mu = \bigcup_{\substack{\tau \in \text{co}\mathcal{F}(\sigma)_p \\ \tau < \mu}} \phi_\mu(B(\sigma, U, \epsilon)_\tau \times [0, \epsilon_\mu)),$$

2) 令

$$B(\sigma, U, \epsilon) = \bigcup_{\tau \in \text{co}\mathcal{F}(\sigma)} B(\sigma, U, \epsilon)_\tau \subseteq |K|.$$

依上述归纳构造, 给定一个单形 σ , σ 中的一个开子集 $U \subseteq \text{Int}(\sigma)$, 以及 σ 的一个余面高度系统 ϵ , 我们有一个确定的构造 $B(\sigma, U, \epsilon)$ 及其关联的系统 $B(\sigma, U, \epsilon)_\tau$ ($\tau \geq \sigma$); 我们称 $B = B(\sigma, U, \epsilon)$ 为一个**余面基元素**, 称 $\sigma = \sigma(B)$ 为 B 的**初始单形**,

$U = U(B)$ 为 B 的**底部**, $\epsilon = \epsilon(B)$ 为 B 的余面高度系统,

称 $\mathcal{S}(B) = \{B(\sigma, U, \epsilon)_\tau \mid \tau \geq \sigma\}$ 为一个**余面基系统**或**余面基元素 B 的系统**。

所有余面基元素全体记为 $\mathcal{B}$, 称之为**弱拓扑的余面基**。

对单形 σ 的两个余面高度系统 ϵ 与 ϵ' , 如果 $\epsilon_\tau < \epsilon'_\tau$ (任一 $\tau > \sigma$), 我们记为 $\epsilon < \epsilon'$; 对于 $0 < t < 1$, 余面高度系统 $t\epsilon$ 定义为: 对于任一 $\tau > \sigma$, $(t\epsilon)_\tau = t \cdot \epsilon_\tau$.

余面引理(Co-face Lemma)

设 K 是一个单纯复形。

(1) 余面基系统 $\mathcal{S}(B) = \{B(\sigma, U, \epsilon)_\tau \mid \tau \geq \sigma\}$ 满足性质:

$$B(\sigma, U, \epsilon)_\tau = B(\sigma, U, \epsilon) \cap \tau \quad (\tau \geq \sigma).$$

(2) 设 $B(\sigma, U, \epsilon), B(\sigma', U', \epsilon') \in \mathcal{B}$ 。

(i) 当 $\sigma \not\geq \sigma'$ 或 $\sigma' \not\geq \sigma$ (即 σ 与 σ' 没有面或余面关系) 时,

$$B(\sigma, U, \epsilon) \cap B(\sigma', U', \epsilon') = \emptyset.$$

ii) 当 $\sigma' \geq \sigma$ 时,

$$B(\sigma, U, \epsilon) \cap B(\sigma', U', \epsilon') = B(\sigma', B(\sigma, U, \epsilon)_{\sigma'} \cap U', \epsilon''),$$

这里, 对 $\tau > \sigma'$, $\epsilon''_\tau = \min\{\epsilon_\tau, \epsilon'_\tau\}$. 特别地, 当 $\sigma' \geq \sigma$ 时,

$B(\sigma, U, \epsilon) \cap B(\sigma', U', \epsilon') = \emptyset$ 当且仅当 $B(\sigma, U, \epsilon)_{\sigma'} \cap U' = \emptyset$.

- (3) 每个余面基元素 $B(\sigma, U, \epsilon)$ 为 $|K|$ (带弱拓扑) 的一个开子集。
- (4) 在一个余面基系统 $\mathcal{S}(B) = \{B(\sigma, U, \epsilon)_\tau \mid \tau \geq \sigma\}$, 对于 $\mu \in \text{co}\mathcal{F}(\sigma)_{p+1}$ ($p \geq 0$),

$$\overline{B(\sigma, U, \epsilon)_\mu} = \bigcup_{\substack{\tau \in \text{co}\mathcal{F}(\sigma)_p \\ \tau < \mu}} \phi_\mu(\overline{B(\sigma, U, \epsilon)_\tau} \times [0, \epsilon_\mu]),$$

以及

$$\overline{B(\sigma, U, \epsilon)} = \bigcup_{\tau \geq \sigma} \overline{B(\sigma, U, \epsilon)_\tau}.$$

- (5) 对两个余面基元素 $B(\sigma, U, \epsilon)$ 与 $B(\sigma, V, \epsilon')$, 如果 $\epsilon < \epsilon'$ 以及 $\bar{U} \subseteq V$, 则

$$\overline{B(\sigma, U, \epsilon)} \subseteq B(\sigma, V, \epsilon').$$

特别地, 对于固定 $0 < t < 1$, $\overline{B(\sigma, U, t\epsilon)} \subseteq B(\sigma, V, \epsilon)$ 。

- (6) 余面基元素集族 $\mathcal{B}$ 是 $|K|$ 的弱拓扑的一个基。

证明

(1) 与 (2) 由余面基元素的定义直接得到。

(3) 对于 K 中的任一单形 τ , 如果 τ 不是 σ 的一个余面, 则 $B(\sigma, U, \epsilon) \cap \tau = \emptyset$ 是 τ 的开子集; 如果 τ 是 σ 的一个余面, 依定义, $B(\sigma, U, \epsilon) \cap \tau = B(\sigma, U, \epsilon)_\tau$ 是 τ 的开子集。所以 $B(\sigma, U, \epsilon)$ 是 $|K|$ 的开子集。

(4) 设

$$A = \bigcup_{\substack{\tau \in \text{co}\mathcal{F}(\sigma)_p \\ \tau < \mu}} \phi_\mu(\overline{B(\sigma, U, \epsilon)_\tau} \times [0, \epsilon_\mu]).$$

则 A 是 $B(\sigma, U, \epsilon)_\mu$ 在 μ 中的闭包: A 是 μ 的闭子集且 $B(\sigma, U, \epsilon)_\mu \subseteq A$, 得知

$$A \supseteq \overline{B(\sigma, U, \epsilon)_\mu} \text{ (这里上横线表示在 } \mu \text{ 中闭包)}$$

反之, 由于任一 $\phi_\mu(\overline{B(\sigma, U, \epsilon)_\tau} \times [0, \epsilon_\mu]) \subseteq \overline{B(\sigma, U, \epsilon)_\mu}$, 因此 $A \subseteq \overline{B(\sigma, U, \epsilon)_\mu}$.

因为 μ 是 $|K|$ 的闭子集, A 也是 $B(\sigma, U, \epsilon)_\mu$ 在 $|K|$ 的闭包: 设 Y 为 X 的闭子集, Z 为 Y 的子集. 记 $\bar{Z}$ 为 Z 在 X 中的闭包, 以及 $\bar{Z}_Y$ 为 Z 在 Y 中的闭包. $\because Z \subseteq \bar{Z} \cap Y, \therefore \bar{Z}_Y \subseteq \bar{Z} \cap Y$. $\bar{Z}_Y$ 为 Y 的闭子集, 存在 X 闭子集 B 使得 $B \cap Y = \bar{Z}_Y$, 由 $\bar{Z} \subseteq B$ 知 $\bar{Z} \cap Y \subseteq B \cap Y = \bar{Z}_Y$. ($\because Y$ 是闭子集, $\therefore \bar{Z} \subseteq Y, \therefore \bar{Z} = \bar{Z}_Y$.)

再讨论 $B(\sigma, U, \epsilon)$ 的闭包。显然

$$B(\sigma, U, \epsilon) \subseteq \bigcup_{\tau \geq \sigma} \overline{B(\sigma, U, \epsilon)_\tau} \subseteq \overline{B(\sigma, U, \epsilon)}.$$

对于任一 $\mu \in K$,

$$\left(\bigcup_{\tau \geq \sigma} \overline{B(\sigma, U, \epsilon)_\tau} \right) \cap \mu = \bigcup_{\tau \geq \sigma} (\overline{B(\sigma, U, \epsilon)_\tau} \cap \mu).$$

如果 τ 是 μ 的余面, 则 $\overline{B(\sigma, U, \epsilon)_\tau} \cap \mu = \overline{B(\sigma, U, \epsilon)_\mu}$ 。对于其余的情况, 由于

$$\overline{B(\sigma, U, \epsilon)_\tau} \cap \mu \subseteq \tau \cap \mu,$$

只有当 τ 是 μ 的一个面时, 交才有可能非空; 因为 μ 只有有限个面, 上述等式的右边是 μ 中的有限个闭子集的并, 所以

$$\left(\bigcup_{\tau \geq \sigma} \overline{B(\sigma, U, \epsilon)_\tau} \right) \cap \mu$$

是 μ 的闭子集。从而 $\bigcup_{\tau \geq \sigma} \overline{B(\sigma, U, \epsilon)_\tau}$ 是 $|K|$ 的闭集, 得

$$\overline{B(\sigma, U, \epsilon)} \subseteq \bigcup_{\tau \geq \sigma} \overline{B(\sigma, U, \epsilon)_\tau}.$$

反过来的包含关系容易检验, 因此 $\bigcup_{\tau \geq \sigma} \overline{B(\sigma, U, \epsilon)_\tau} = \overline{B(\sigma, U, \epsilon)}$ 。

(5) 由 (4) 导出。

(6) 我们先复习一下拓扑空间 X 的基的概念：一族 X 的 $\mathcal{B}$ 被称为 X 的某一个拓扑的基，如果

① $\forall x \in X, \exists B \in \mathcal{B}$ 使得 $x \in B$ 。

② $\forall B_1, B_2 \in \mathcal{B}$ 以及 $\forall x \in B_1 \cap B_2, \exists B_3 \in \mathcal{B}$ 使得 $x \in B_3 \subseteq B_1 \cap B_2$ 。

由 $\mathcal{B}$ 生成的拓扑 $\mathcal{T}$ 定义为所有满足下面条件的子集 U ： $\forall x \in U, \exists B \in \mathcal{B}$ 使得 $x \in B \subseteq U$ 。

- 需要证明 $\mathcal{T}$ 确实是一个拓扑。
- $\mathcal{T}$ 中的元素是 $\mathcal{B}$ 中元素的并。
- 给定一个拓扑 $\mathcal{T}$, $\mathcal{B}$ 是一族开子集形成的基满足 $\forall U \in \mathcal{T}$ 及 $\forall x \in U, \exists B \in \mathcal{B}$ 使得 $x \in B \subseteq U$, 则 $\mathcal{B}$ 生成的拓扑就是 $\mathcal{T}$ 。

现在我们证明余面基元素集族 $\mathcal{B}$ 是 $|K|$ 的弱拓扑的一个基。

先证明 $\mathcal{B}$ 是一个基。

- $\forall x \in |K|$, 存在唯一的单形 σ 使得 $x \in \text{Int}(\sigma)$ 。取 $U = \text{Int}(\sigma)$, $\exists \epsilon$ 使得 $x \in B(\sigma, U, \epsilon)$ 。
- 基定义中的第二个条件由(2)得知。
- 由(3)知 $\mathcal{B}$ 中的元素都是开子集。

对于任一 $x \in |K|$ 及其任一开邻域 V , 存在唯一的单形 $\sigma \in K$ 使得 $x \in \text{Int}(\sigma)$, 设 U 为 x 的一个开邻域使得 $x \in U \subseteq \bar{U} \subseteq V \cap \text{Int}(\sigma)$; 令 $U_\sigma = U$, 对于任一 $\tau \in \text{co}\mathcal{F}(\sigma)_1$, 由于 $V \cap \tau$ 是 τ 中包含紧子集 $\bar{U}_\sigma$ 的一个开集, 存在 $0 < \epsilon_\tau < 1$ 使得

$$\phi_\tau(\bar{U}_\sigma \times [0, \epsilon_\tau]) \subseteq V \cap \tau,$$

取 $U_\tau = \phi_\tau(U_\sigma \times [0, \epsilon_\tau])$ 。归纳假设, 对于每一个 $\tau \in \text{co}\mathcal{F}(\sigma)_k$ ($1 \leq k \leq p$), 均已经构造高度参数 ϵ_τ 以及相应的 τ 中的开子集 U_τ 。对于任一单形 $\mu \in \text{co}\mathcal{F}(\sigma)_{p+1}$, 存在 $0 < \epsilon_\mu < 1$ 使得

$$\bigcup_{\substack{\tau \in \text{co}\mathcal{F}(\sigma)_p \\ \tau < \mu}} \phi_\mu(\bar{U}_\tau \times [0, \epsilon_\mu]) \subseteq V \cap \mu.$$

令

$$U_\mu = \bigcup_{\substack{\tau \in \text{co}\mathcal{F}(\sigma)_p \\ \tau < \mu}} \phi_\mu(U_\tau \times [0, \epsilon_\mu)).$$

由于上述的并是有限并,

$$\bar{U}_\mu = \bigcup_{\substack{\tau \in \text{co}\mathcal{F}(\sigma)_p \\ \tau < \mu}} \phi_\mu(\bar{U}_\tau \times [0, \epsilon_\mu]) \subseteq V \cap \mu.$$

归纳构造完成, 所得到的 $B(\sigma, U, \epsilon)$ 满足 $x \in B(\sigma, U, \epsilon) \subseteq V$; 因此断言成立。□

仿紧性定理(Paracompact Theorem)

设 K 是一个单纯复形, 则多面体 $|K|$ (带弱拓扑) 是仿紧空间。

仿紧空间的复习—局部有限性

§39 Local Finiteness

In this sections we prove some elementary properties of locally finite collections and a crucial lemma about metrizable spaces.

Definition. Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be *locally finite* in X if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{A}$.

EXAMPLE 1. The collection of intervals

$$\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

is locally finite in the topological space $\mathbb{R}$, as you can check. On the other hand, the collection

$$\mathcal{B} = \{(0, 1/n) \mid n \in \mathbb{Z}_+\}$$

is locally finite in $(0, 1)$ but not in $\mathbb{R}$, as is the collection

$$\mathcal{C} = \{(1/(n+1), 1/n) \mid n \in \mathbb{Z}_+\}.$$

Figure: James Munkres, Topology, Prentice Hall 2000

Lemma 39.1. Let $\mathcal{A}$ be a locally finite collection of subsets of X . Then:

- (a) Any subcollection of $\mathcal{A}$ is locally finite.
- (b) The collection $\mathcal{B} = \{\bar{A}\}_{A \in \mathcal{A}}$ of the closures of the elements of $\mathcal{A}$ is locally finite.
- (c) $\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \bar{A}$.

Proof. Statement (a) is trivial. To prove (b), note that any open set U that intersects the set $\bar{A}$ necessarily intersects A . Therefore, if U is a neighborhood of x that intersects only finitely many elements A of $\mathcal{A}$, then U can intersect at most the same number of sets of the collection $\mathcal{B}$. (It might intersect fewer sets of $\mathcal{B}$, since $\bar{A}_1$ and $\bar{A}_2$ can be equal even though A_1 and A_2 are not).

To prove (c), let Y denote the union of the elements of $\mathcal{A}$:

$$\bigcup_{A \in \mathcal{A}} A = Y.$$

In general, $\bigcup \bar{A} \subset \bar{Y}$; we prove the reverse inclusion, under the assumption of local finiteness. Let $x \in \bar{Y}$; let U be a neighborhood of x that intersects only finitely many elements of $\mathcal{A}$, say $A_1, \dots, A_k$. We assert that x belongs to one of the sets $\bar{A}_1, \dots, \bar{A}_k$, and hence belongs to $\bigcup \bar{A}$. For otherwise, the set $U - \bar{A}_1 - \dots - \bar{A}_k$ would be a neighborhood of x that intersects no element of $\mathcal{A}$ and hence does not intersect Y , contrary to the assumption that $x \in \bar{Y}$. ■

There is an analogous concept of local finiteness for an indexed family of subsets of X . The indexed family $\{A_\alpha\}_{\alpha \in J}$ is said to be a **locally finite indexed family** in X if every $x \in X$ has a neighborhood that intersects A_α for only finitely many values of α . What is the relation between the two formulations of local finiteness? It is easy to see that $\{A_\alpha\}_{\alpha \in J}$ is a locally finite indexed family if and only if it is locally finite as a *collection* of sets and each nonempty subset A of X equals A_α for at most finitely many values of α .

We shall be concerned with locally finite indexed families only in §41, when we deal with partitions of unity.

Definition. A collection $\mathcal{B}$ of subsets of X is said to be **countably locally finite** if $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.

Most authors use the term “ σ -locally finite” for this concept. The σ comes from measure theory and stands for the phrase “countable union of.” Note that both a countable collection and a locally finite collection are countably locally finite.

Definition. Let $\mathcal{A}$ be a collection of subsets of the space X . A collection $\mathcal{B}$ of subsets of X is said to be a **refinement** of $\mathcal{A}$ (or is said to **refine** $\mathcal{A}$) if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containing B . If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ an **open refinement** of $\mathcal{A}$; if they are closed sets, we call $\mathcal{B}$ a **closed refinement**.

Lemma 39.2. *Let X be a metrizable space. If $\mathcal{A}$ is an open covering of X , then there is an open covering $\mathcal{E}$ of X refining $\mathcal{A}$ that is countably locally finite.*

Definition. A subset A of a space X is called a G_δ set in X if it equals the intersection of a countable collection of open subsets of X .

EXAMPLE 1. Each open subset of X is a G_δ set, trivially. In a first-countable Hausdorff space, each one-point set is a G_δ set. The one-point subset $\{\Omega\}$ of $\tilde{S}_\Omega$ is not a G_δ set, as you can check.

EXAMPLE 2. In a metric space X , each closed set is a G_δ set. Given $A \subset X$, let $U(A, \epsilon)$ denote the ϵ -neighborhood of A . If A is closed, you can check that

$$A = \bigcap_{n \in \mathbb{Z}_+} U(A, 1/n).$$

Lemma 40.1. Let X be a regular space with a basis $\mathcal{B}$ that is countably locally finite. Then X is normal, and every closed set in X is a G_δ set in X .

Lemma 40.2. *Let X be normal; let A be a closed G_δ set in X . Then there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$.*

Proof. We gave this as an exercise in §33; we provide a proof here. Write A as the intersection of the open sets U_n , for $n \in \mathbb{Z}_+$. For each n , choose a continuous function $f_n : X \rightarrow [0, 1]$ such that $f_n(x) = 0$ for $x \in A$ and $f_n(x) = 1$ for $x \in X - U_n$. Define $f(x) = \sum f_n(x)/2^n$. The series converges uniformly, by comparison with $\sum 1/2^n$, so that f is continuous. Also, f vanishes on A and is positive on $X - A$. ■

Theorem 40.3 (Nagata-Smirnov metrization theorem). *A space X is metrizable if and only if X is regular and has a basis that is countably locally finite.*

§41 Paracompactness

The concept of paracompactness is one of the most useful generalizations of compactness that has been discovered in recent years. It is particularly useful for applications in topology and differential geometry. We shall give just one application, a metrization theorem that we prove in the next section.

Many of the spaces that are familiar to us already are paracompact. For instance, every compact space is paracompact; this will be an immediate consequence of the definition. It is also true that every metrizable space is paracompact; this is a theorem due to A. H. Stone, which we shall prove. Thus the class of paracompact spaces includes the two most important classes of spaces we have studied. It includes many other spaces as well.

To see how paracompactness generalizes compactness, we recall the definition of compactness: A space X is said to be *compact* if every open covering $\mathcal{A}$ of X contains a finite subcollection that covers X . An equivalent way of saying this is the following:

A space X is compact if every open covering $\mathcal{A}$ of X has a finite open refinement $\mathcal{B}$ that covers X .

This definition is equivalent to the usual one; given such a refinement $\mathcal{B}$, one can choose for each element of $\mathcal{B}$ an element of $\mathcal{A}$ containing it; in this way one obtains a finite subcollection of $\mathcal{A}$ that covers X .

仿紧性的定义

This new formulation of compactness is an awkward one, but it suggests a way to generalize:

Definition. A space X is *paracompact* if every open covering $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ that covers X .

Many authors, following the lead of Bourbaki, include as part of the definition of the term *paracompact* the requirement that the space be Hausdorff. (Bourbaki also includes the Hausdorff condition as part of the definition of the term *compact*.) We shall not follow this convention.

EXAMPLE 1. *The space $\mathbb{R}^n$ is paracompact.* Let $X = \mathbb{R}^n$. Let $\mathcal{A}$ be an open covering of X . Let $B_0 = \emptyset$, and for each positive integer m , let B_m denote the open ball of radius m centered at the origin. Given m , choose finitely many elements of $\mathcal{A}$ that cover $\bar{B}_m$ and intersect each one with the open set $X - \bar{B}_{m-1}$; let this finite collection of open sets be denoted $\mathcal{C}_m$. Then the collection $\mathcal{C} = \bigcup \mathcal{C}_m$ is a refinement of $\mathcal{A}$. It is clearly locally finite, for the open set B_m intersects only finitely many elements of $\mathcal{C}$, namely those elements belonging to the collection $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m$. Finally, $\mathcal{C}$ covers X . For, given x , let m be the smallest integer such that $x \in \bar{B}_m$. Then x belongs to an element of $\mathcal{C}_m$, by definition.

仿紧性的一些定理

Some of the properties of a paracompact space are similar to those of a compact space. For instance, a subspace of a paracompact space is not necessarily paracompact; but a closed subspace is paracompact. Also, a paracompact Hausdorff space is normal. In other ways, a paracompact space is not similar to a compact space; in particular, the product of two paracompact spaces need not be paracompact. We shall verify these facts shortly.

Theorem 41.1. *Every paracompact Hausdorff space X is normal.*

Theorem 41.2. *Every closed subspace of a paracompact space is paracompact.*

Proof. Let Y be a closed subspace of the paracompact space X ; let $\mathcal{A}$ be a covering of Y by sets open in Y . For each $A \in \mathcal{A}$, choose an open set A' of X such that $A' \cap Y = A$. Cover X by the open sets A' , along with the open set $X - Y$. Let $\mathcal{B}$ be a locally finite open refinement of this covering that covers X . The collection

$$\mathcal{C} = \{B \cap Y \mid B \in \mathcal{B}\}$$

is the required locally finite open refinement of $\mathcal{A}$. ■

EXAMPLE 2. *A paracompact subspace of a Hausdorff space X need not be closed in X .* Indeed, the open interval $(0, 1)$ is paracompact, being homeomorphic to $\mathbb{R}$, but it is not closed in $\mathbb{R}$.

EXAMPLE 3. *A subspace of a paracompact space need not be paracompact.* The space $\tilde{S}_\Omega \times \tilde{S}_\Omega$ is compact and, therefore, paracompact. But the subspace $S_\Omega \times \tilde{S}_\Omega$ is not paracompact, for it is Hausdorff but not normal.

Lemma 41.3. *Let X be regular. Then the following conditions on X are equivalent:*

Every open covering of X has a refinement that is:

- (1) An open covering of X and countably locally finite.*
- (2) A covering of X and locally finite.*
- (3) A closed covering of X and locally finite.*
- (4) An open covering of X and locally finite.*

Theorem 41.4. *Every metrizable space is paracompact.*

Proof. Let X be a metrizable space. We already know from Lemma 39.2 that, given an open covering $\mathcal{A}$ of X , it has an open refinement that covers X and is *countably* locally finite. The preceding lemma then implies that $\mathcal{A}$ has an open refinement that covers X and is locally finite. ■

Theorem 41.5. *Every regular Lindelöf space is paracompact.*

Proof. Let X be regular and Lindelöf. Given an open covering $\mathcal{A}$ of X , it has a countable subcollection that covers X ; this subcollection is automatically countably locally finite. The preceding lemma applies to show $\mathcal{A}$ has an open refinement that covers X and is locally finite. ■

EXAMPLE 4. *The product of two paracompact spaces need not be paracompact. The space $\mathbb{R}_\ell$ is paracompact, for it is regular and Lindelöf. However, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not paracompact, for it is Hausdorff but not normal.*

EXAMPLE 5. *The space $\mathbb{R}^\omega$ is paracompact in both the product and uniform topologies. This result follows from the fact that $\mathbb{R}^\omega$ is metrizable in these topologies. It is not known whether $\mathbb{R}^\omega$ is paracompact in the box topology. (See the comment in Exercise 5 of §32.)*

EXAMPLE 6. *The product space $\mathbb{R}^J$ is not paracompact if J is uncountable. For $\mathbb{R}^J$ is Hausdorff but not normal.*

Definition. Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed open covering of X . An indexed family of continuous functions

$$\phi_\alpha : X \rightarrow [0, 1]$$

is said to be a **partition of unity** on X , dominated by $\{U_\alpha\}$, if:

- (1) $\text{Support } \phi_\alpha \subset U_\alpha$ for each α .
- (2) The indexed family $\{\text{Support } \phi_\alpha\}$ is locally finite.
- (3) $\sum \phi_\alpha(x) = 1$ for each x .

Condition (2) implies that each $x \in X$ has a neighborhood on which the function ϕ_α vanishes identically for all but finitely many values of α . Thus we can make sense of the “sum” indicated in (3); we interpret it to mean the sum of the terms $\phi_\alpha(x)$ that do not equal zero.

We now construct a partition of unity on an arbitrary paracompact Hausdorff space. We begin by proving a “shrinking lemma,” just as we did for the finite case in §36.

Partition of unity有何用?

假设 $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ 是 X 的一个开覆盖, 以及 $\{\phi_\alpha\}_{\alpha \in J}$ 是 X 的 **partition of unity dominated by $\mathcal{U}$** . 令 $\mathcal{K}(\mathcal{U})$ 为 $\mathcal{U}$ 的神经复形, 其顶点集记为 $\{v_\alpha\}_{\alpha \in J}$, 这里 v_α 对应于 U_α .

对于 $\forall x \in X$, 定义

$$f(x) = \sum_{\alpha \in J} \phi_\alpha(x) v_\alpha \in |\mathcal{K}(\mathcal{U})| \subseteq \Delta^J$$

我们先检验 f 是 **well-defined** 以及连续性: 给定 $x \in X$, 根据定义, $\{\text{supp}(\phi_\alpha)\}_{\alpha \in J}$ 是局部有限的, 存在 x 的一个开邻域 V 使得 V 只与有限个 $\text{supp}(\phi_\alpha)$ 相交。

设 $a_0, a_1, \dots, a_n \in J$ 为满足 $\text{supp}(\phi_\alpha) \cap V \neq \emptyset$ 的所有指标。

- 对于 $\forall \alpha \in J \setminus \{a_0, a_1, \dots, a_n\}$, $\phi_\alpha|_V \equiv 0$ (恒等于0), 特别地 $\phi_\alpha(x) = 0$ 。
所以 $f(x)$ 是 well-defined (有限个非零向量的和)。
- 设 K 是 $|\mathcal{K}(\mathcal{U})|$ 由顶点 $v_{a_0}, v_{a_1}, \dots, v_{a_n}$ 张成的子复形 (即包含这些顶点的所有单形构成的子复形), 则对于 $\forall y \in V$, $f(y) \in |K|$, 因为 $f(y)$ 在其它顶点 v_α 的重心坐标 $\phi_\alpha(y)$ 是 0。
- $f|_V$ 是复合映射: $V \xrightarrow{\tilde{f}} |K| \xrightarrow{j} |\mathcal{K}(\mathcal{U})|$, 这里 $\tilde{f}(y) = f(y) \in |K|$ 及 j 是内射。由 $\tilde{f}$ 及 j 的连续性得 $f|_V = j \circ \tilde{f}$ 是连续的。
- 因此 f 是局部连续的, 从而 f 是连续的。

作业

- ① 假设 X 的开覆盖 $\mathcal{U}$ 满足Čech的条件, 即任意有限非空交都是可缩的, 以及存在一个partition of unity dominated to $\mathcal{U}$ 。证明上述构造的 $f: X \rightarrow |\mathcal{K}(\mathcal{U})|$ 诱导出一个同调群的同构 $f_*: H_*(X) \rightarrow H_*(|\mathcal{K}(\mathcal{U})|)$ 。(提示: 运用MV 序列归纳。)
- ② 通过调研, 寻找应用场景, 运用不满足Čech条件的开覆盖进行数据分析: 在一般情况下, $f: X \rightarrow |\mathcal{K}(\mathcal{U})|$ 是fibrewise topolpogy (topology of maps).