Exotic Integral Quantum Symmetry

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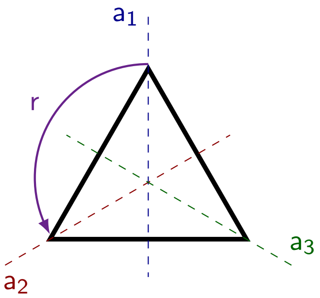
Symmetry

A symmetry is a *transformation* that preserves an object.
The set of symmetries of an object forms a group.



×	e	r	r^2	a_1	a_2	a_3
e	e	r	r^2	a_1	a_2	a_3
r	r	r^2	e	a_2	a_3	a_1
r^2	r^2	e	r	a_3	a_1	a_2
a_1	a_1	a_3	a_2	e	r^2	r
a_2	a_2	a_1	a_3	r	e	r^2
a_3	a_3	a_2	a_1	r^2	r	e

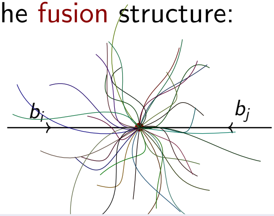
Isometry group (S_3) of the regular triangle

Every finite group is the isometry group of a polytope.

The notions of symmetry and group are *essentially* equivalent.

There are the notions of quantum group, Hopf algebra, tensor category, subfactor, planar algebra..., but let's start more easily by slightly augmenting the group structure by the fusion structure:

$$b_i b_j = \sum_{k \in I} N_{i,j}^k b_k,$$



where $N_{i,j}^k$ are non-negative integers (group case: $N_{g,h}^k = \delta_{gh,k}$).

The group axioms extend as follows: for all $i, j, k \in I$,

- **Associativity**: $b_i(b_j b_k) = (b_i b_j) b_k$,
- **Unit**: $b_1 b_i = b_i b_1 = b_i$,
- **Anti-involution** $b_i^* = b_{i^*}$ with $N_{i^*,j}^1 = N_{j,i^*}^1 = \delta_{i,j}$.

This notion (by Lusztig, 1987) is now called *fusion ring*. Examples:

- Group ring $\mathbb{Z}G$ with basis G (the group case),
- Character ring $\mathrm{ch}(G)$ with basis the irreducible characters χ_i,
- many other examples... (see next slides).

Non-Integral Quantum Symmetry

Let $\mathcal{R}$ be a fusion ring with basis $\mathcal{B} = \{b_1, \dots, b_r\}$.

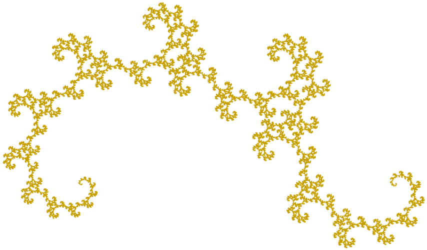
Theorem (Frobenius-Perron dimension FPdim)

There is a unique $*$-homomorphism $d : \mathcal{R} \to \mathbb{C}$ s.t. $d(\mathcal{B}) \subset \mathbb{R}_{>0}$.

$\mathrm{FPdim}(b_i)$ is the norm of its fusion matrix, so an algebraic integer.
$\mathrm{FPdim}(\mathcal{R}) := \sum_i \mathrm{FPdim}(b_i)^2$, so $|G|$ for $\mathrm{ch}(G)$. Such $\mathcal{R}$ is called

- 1-*Frobenius* if $\frac{\mathrm{FPdim}(\mathcal{R})}{\mathrm{FPdim}(b_i)}$ is an algebraic integer (Kaplansky),
- *pointed* if $\mathrm{FPdim}(b_i) = 1$ (so it is a group ring $\mathbb{Z}G$),

The *rank* of $\mathcal{R}$ is $|\mathcal{B}|$, and its *type* is $(\mathrm{FPdim}(b_i))_{i \in I}$.



Golden fusion ring (Yang-Lee rules)

$\mathcal{B} = \{b_1, b_2\}$, ${b_2}^2 = b_1 + b_2$, so
$\mathrm{FPdim}(b_2) = \frac{1+\sqrt{5}}{2}$ (golden ratio).

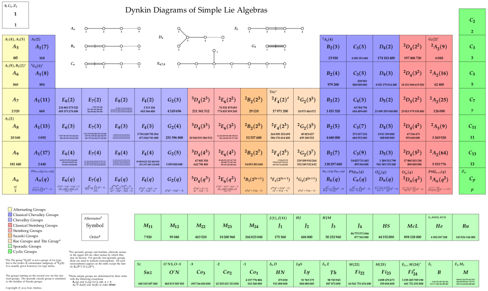
Golden Dragon: fractal of Hausdorff dimension the golden ratio $\simeq 1.618$.

Simple Integral Quantum Symmetry

A fusion ring $\mathcal{R}$ is called:

- *integral* if $\mathrm{FPdim}(b_i)$ is an integer (e.g. $\mathrm{ch}(G)$ with $deg(\chi_i)$),
- *simple* if no proper non-trivial fusion subring,

The character ring $\mathrm{ch}(G)$ is simple if and only if G is simple.



A classification of simple integral fusion rings *really extends* CFSG:

- The fusion ring $\mathrm{ch}(G)$ *recalls* the simple group G (false in general).
- There are *lots* of non-$\mathrm{ch}(G)$ simple integral fusion rings (see later).

More Substantial Quantum Symmetry

A fusion ring may be the skeleton of something more *substantial*:

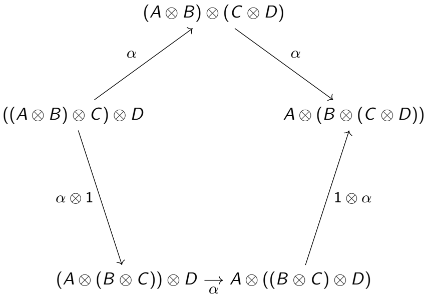
Categorification

Fusion Rings (e.g. $\mathbb{Z}G$, $\mathrm{ch}(G)$,...) → **Fusion Categories** (e.g. $\mathrm{Vec}(G)$, $\mathrm{Rep}(G)$,...)

Grothendieck Ring

Categorical Associativity

A *monoidal category* (Mac Lane, 1963) involves a natural isomorphism α with $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ such that the *pentagon* diagrams (here) commute (no ambiguity). The tension between *skeletal* ($A \simeq B$ iff $A = B$) and *strict* ($\alpha = \mathrm{id}$) notions prevents most fusion rings from being categorifiable.



Theorem (Ostrik, 2002)

A fusion ring with $b_2^2 = b_1 + nb_2$ is categorifiable iff $n = 0, 1$.

Exotic Integral Quantum Symmetry

Consider the following three **open** statements:

(1) Every integral fusion category is weakly group-theoretical[1],

(2) Every simple integral fusion category is WGT,

(3) Every simple integral **modular** fusion category is pointed.

[1] Morita equivalent to a sequence of group extensions (from Vec)

(Etingof-Nikshych-Ostrik, 2011). **Question**: Is (1) false?
Theorem: A non-pointed WGT simple fusion category is $\mathrm{Rep}(G)$.

(Liu-P-Ren, 2023). **Question**: (1) $\Leftrightarrow$ (2)? **Theorem**: (2) $\Leftrightarrow$ (3).

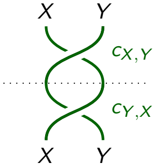
(Alekseyev-Bruns-P-Petrov, 2023). **Thm**: (3) is true at rank < 13.

$\hookrightarrow$ same result in the *perfect* case, but $\mathcal{Z}(\mathrm{Rep}(A_5))$ is perfect and of rank 22

Categorical Commutativity

A *braiding* is a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$, with hexagon diagrams. S-matrix: $(s_{X,Y}) = (tr(\boldsymbol{c_{Y,X} \circ c_{X,Y}}))$.

- **Symmetric** if $s_{X,Y} = d_X d_Y$, *essentially* $\mathrm{Rep}(G)$ by Deligne;
- **Modular** if S invertible (Moore-Seiberg '89, Turaev '92).



More MTC: Drinfeld center $\mathcal{Z}(\_)$, TQFT, non-int simple Lie ex., VOA, $\mathrm{SL}(2,\mathbb{Z})$-reps...

Classification of simple integral fusion rings

By ENO's theorem, a non-pointed simple integral fusion category which is not isomorphic to $\mathrm{Rep}(G)$, cannot be WGT. But there are plenty of non-pointed simple integral fusion rings not isomorphic to $\mathrm{ch}(G)$. Their classification for small rank and FPdim is reachable.

Theorem (Alekseyev-Bruns-P-Petrov, arXiv:2302.01613)

A non-pointed simple integral fusion ring has rank at least 4.

We found a lot of rank 4 items ($\exists\infty$?), but none 1-Frobenius ($\nexists$?).

Theorem (Liu-P-Wu, *Adv. in Math.* 2021; Bruns-P, *in progress*)

Counting non-pointed simple integral 1-Frobenius fusion rings:

Rank	4	5	6	7	8	9	10	11	12
$\mathrm{FPdim} \leq$	10^{10}	10^7	10^6	10^5	$2 \cdot 10^4$	10^4	5000	3000	1000
#Fusion Rings	0	1	1	8	23	94	188	190	0

Among 505 fusion rings, only 8 are character rings (of $\mathrm{PSL}(2,q)$, A_7)

95% of the rest is excluded from *unitary* categorification by Quantum Fourier Analysis.

Theorem (Liu-P-Wu, *AiM* 2021; Huang-Liu-P-Wu, *IMRN* 2023)

The Grothendieck ring of a unitary fusion category is n-positive, i.e.

$$\sum \|M_i\|^{2-n} M_i^{\otimes n} \geq 0,$$

for all $n \geq 1$, where (M_i) are the fusion matrices.

Among the 497 remaining fusion rings, 20 ones only are 3-positive, and they can be obtained by applying three transformations on character rings (of non-abelian finite simple groups):

- interpolation,
- isotype variation,
- contraction.

Question: Is there an example out of this pattern?

Anyway, the categorification problem remains.

Categorical Classification

The first candidates are transformations applied to $\mathrm{ch}(\mathrm{PSL}(2, q))$:

- Rank 7 and FPdim 210: interpolation to $q = 6$,
  $\rightsquigarrow$ excluded by TPE and Localization,
- Rank 8 and FPdim 660: isotype variation, $q = 11$,
  $\rightsquigarrow$ excluded by Zero-Spectrum Criterion,
- Rank 9 and FPdim 504: isotype variation, $q = 8$,
  $\rightsquigarrow$ **Open Case**.

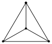
See the next slides for more details. It follows that:

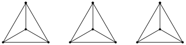
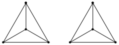
Theorem (Liu-P-Wu, *AIM* 2021; Bruns-P, *in progress*)

A non-pointed 1-Frobenius unitary simple integral fusion category with rank and FPdim as follows

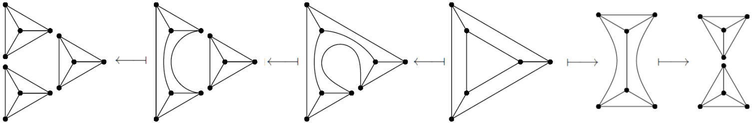
Rank	4	5	6	7	8	9	10	11
$\mathrm{FPdim} <$	10^{10}	10^7	10^6	10^5	$2 \cdot 10^4$	504	1680	990

is isomorphic to $\mathrm{Rep}(\mathrm{PSL}(2, q))$ with $q = 4, 7, 9, 11$.

An oriented triangular prism  whose edges and vertices are well labeled by objects and morphisms in a pivotal fusion category, can be interpreted as a scalar and evaluated by two ways using (labeled oriented) tetrahedra . This provides a system of equations as:

(TPE) $\sum \cdots$  $= \sum \cdots$ 

The following picture (by D. Thurston) illustrates the proof:



Theorem (Liu-P-Ren, arXiv:2203.06522)

In the spherical case, TPE are exactly PE, up to a change of basis.

The TPE provide insight to manage the complexity by localization, using efficient labelings. The $q = 6$ exclusion involves such local subsystems, with 12 polynomial equations and 10 variables.

Zero-Spectrum Criterion

It is about the existence of a PE of the form $xy = 0$ with $x, y \neq 0$:

Theorem (Liu-P-Ren, arXiv:2203.06522)

For a fusion ring $\mathcal{R}$, if there are indices i_j, $1 \leq j \leq 9$, such that $N_{i_4,i_1}^{i_6}$, $N_{i_5,i_4}^{i_2}$, $N_{i_5,i_6}^{i_3}$, $N_{i_7,i_9}^{i_1}$, $N_{i_2,i_7}^{i_8}$, $N_{i_8,i_9}^{i_3}$ are non-zero, and

$$\sum_k N_{i_4,i_7}^{k} N_{i_5^*,i_8}^{k} N_{i_6,i_9^*}^{k} = 0,$$

$$N_{i_2,i_1}^{i_3} = 1,$$

$$\sum_k N_{i_5,i_4}^{k} N_{i_3,i_1^*}^{k} = 1 \text{ or } \sum_k N_{i_2,i_4^*}^{k} N_{i_3,i_6^*}^{k} = 1 \text{ or } \sum_k N_{i_5^*,i_2}^{k} N_{i_6,i_1^*}^{k} = 1,$$

$$\sum_k N_{i_2,i_7}^{k} N_{i_3,i_9^*}^{k} = 1 \text{ or } \sum_k N_{i_8,i_7^*}^{k} N_{i_3,i_1^*}^{k} = 1 \text{ or } \sum_k N_{i_2^*,i_8}^{k} N_{i_1,i_9^*}^{k} = 1,$$

then $\mathcal{R}$ cannot be categorified (at all) over any field.

Current smallest candidate

Rank 9, FPdim $504 = 2^3 \cdot 3^2 \cdot 7$ and type $[1, 7, 7, 7, 7, 8, 9, 9, 9]$.

The character ring of $\mathrm{PSL}(2, 8)$:

[[1,0,0,0,0,0,0,0,0],[0,1,0,0,0,0,0,0,0],[0,0,1,0,0,0,0,0,0],[0,0,0,1,0,0,0,0,0],[0,0,0,0,1,0,0,0,0],[0,0,0,0,0,1,0,0,0],[0,0,0,0,0,0,1,0,0],[0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,0,0,1]],
[[0,1,0,0,0,0,0,0,0],[1,0,1,1,1,0,1,1,1],[0,1,1,0,0,1,1,1,1],[0,1,0,1,0,1,1,1,1],[0,1,0,0,1,1,1,1,1],[0,0,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1]],
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[[0,0,0,1,0,0,0,0,0],[0,1,0,1,0,1,1,1,1],[0,0,1,1,0,1,1,1,1],[1,1,1,0,1,0,1,1,1],[0,0,0,1,1,1,1,1,1],[0,1,1,0,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1]],
[[0,0,0,0,1,0,0,0,0],[0,1,0,0,1,1,1,1,1],[0,0,1,0,1,1,1,1,1],[0,0,0,1,1,1,1,1,1],[1,1,1,1,0,0,1,1,1],[0,1,1,1,0,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1]],
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The (proper) isotype variation:

[[1,0,0,0,0,0,0,0,0],[0,1,0,0,0,0,0,0,0],[0,0,1,0,0,0,0,0,0],[0,0,0,1,0,0,0,0,0],[0,0,0,0,1,0,0,0,0],[0,0,0,0,0,1,0,0,0],[0,0,0,0,0,0,1,0,0],[0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,0,0,1]],
[[0,1,0,0,0,0,0,0,0],[1,0,1,1,1,0,1,1,1],[0,1,1,0,0,1,1,1,1],[0,1,0,1,0,1,1,1,1],[0,1,0,0,1,1,1,1,1],[0,0,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1]],
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[[0,0,0,0,0,0,0,0,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,1],[0,1,1,1,1,1,1,1,2],[0,1,1,1,1,1,2,2,1],[0,1,1,1,1,1,2,1,2],[1,1,1,1,1,2,1,2,1]]

Any idea to solve this case is welcome!

The localization strategy provides a first subsystem of 45 polynomial equations with 34 variables. Recently (10/01/2024), Paul Breiding found a solution using homotopy continuation!

Original motivation from subfactor theory

- A *factor* is a von Neumann algebra with trivial center.
- A *subfactor* is a unital inclusion of factors $N \subseteq M$.
- It admits an index $|M : N|$, multiplicative with intermediate:

$$N \subseteq P \subseteq M \Rightarrow |M : N| = |M : P| \cdot |P : N|,$$

A subfactor without intermediate is called **maximal** (Bisch, 1994).
$\rightarrow$ "quantum analogous" of prime numbers!
What about natural numbers? See my papers on Ore's theorem.

The finite group subfactors ($R^G \subset R$) are in a specific class called *finite index, irreducible, depth* 2, completely characterized by:

Theorem (Longo, Szymanski, 1994; David, 1996)

Every finite index, irreducible, depth 2 subfactor is of the form ($R^H \subseteq R$), where H is a finite dimensional Hopf C*-algebra (also called *finite quantum group*).

The only *known* maximal examples of this class are the finite group subfactors ($R^G \subset R$), with G is cyclic of prime order.

Question (P. 2012)

Are there other maximal examples in this class?

Transition to fusion category theory

Theorem (Izumi, Longo, Popa, 1998)

There is a 1-1 correspondence between

- intermediates of finite quantum group subfactor $R^H \subseteq P \subseteq R$,
- *left coideal* $*$-subalgebras $L \subseteq H$, i.e. $\Delta(L) = H \otimes L$,

given by $P = R^L$.

Theorem (Burciu, 2012)

There is a 1-1 correspondence between

- *normal* left coideal $*$-subalgebras $L \subseteq H$
- fusion subcategories $\mathcal{C}$ of the *integral* fusion category $\mathrm{Rep}(H)$,

given by $\mathcal{C} \simeq \mathrm{Rep}(H//L)$.

A fusion category w/o (non-trivial) fusion subcat. is called *simple*.

So we are looking for an exotic simple integral fusion category together with a fiber functor and that maximality.

Thanks for your attention!

↓

Extra Slides

Extra Slides

Simple integral fusion rings of rank 4

They generically have type $[1, d_1, d_2, d_3]$ and fusion matrices:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&x_1&x_2&x_3\\0&x_2&x_4&x_5\\0&x_3&x_5&x_6\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&x_2&x_4&x_5\\1&x_4&x_7&x_8\\0&x_5&x_8&x_9\end{pmatrix}, \begin{pmatrix}0&0&0&1\\0&x_3&x_5&x_6\\0&x_5&x_8&x_9\\1&x_6&x_9&x_{10}\end{pmatrix}$$

Here is the full list of items with $\mathrm{FPdim} < 10^6$ (Bruns-P, 2024):

FPdim	Factors	d_1	d_2	d_3	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
574	$2^1 7^1 41$	11	14	16	0	4	4	1	6	3	12	1	9	6
7315	$5^1 7^1 11^1 19$	35	40	67	30	1	2	9	15	25	12	12	25	39
63436	$2^2 1585$	103	149	175	57	13	16	67	23	74	51	44	98	48
65971	$2^1 3^2 5^1 1259$	65	89	232	51	5	2	4	22	56	60	10	79	186
68587	$107^1 641$	103	116	211	48	27	12	19	33	79	30	38	79	129
90590	$2^1 5^1 9059$	142	180	195	39	0	75	77	60	32	25	87	56	120
113310	$2^1 3^2 5^1 125$	90	172	275	77	2	3	25	40	64	39	75	112	184
310730	$2^1 5^1 7^1 23^1 193$	312	317	336	62	139	101	48	120	105	168	96	115	130
311343	$3^1 59^1 1759$	286	315	361	95	76	85	115	89	141	206	4	241	39
494102	$2^1 7^1 29^1 1217$	396	399	422	40	219	127	20	150	135	248	124	141	162
532159	532159	211	409	566	84	24	30	63	98	129	299	56	332	278
585123	$3^1 7^1 11^1 17^1 149$	288	397	587	159	30	43	179	59	227	208	40	341	245

This table suggests the existence of infinitely many examples.

Burnside type results and MTC

Definition (Burciu-P, arXiv:2302.07604)

- A fusion ring is *Burnside* if for every basic element then its fusion matrix has norm 1 iff its determinant is nonzero.
- A 3-positive commutative fusion ring is *dual-Burnside* when a column c of its character table has a 0-entry iff $\|c\|^2 < \mathrm{FPdim}$.

- Burnside proved that $\mathrm{ch}(G)$ is Burnside for every finite group G. It extends to every integral fusion categories (Burciu, 23). We found non-Burnside integral fusion rings (cat. criterion).
- For every nilpotent, or every simple non-alternating finite group G (except M_{22}, M_{24}) then $\mathrm{ch}(G)$ is dual-Burnside.

Theorems (Burciu-P, arXiv:2302.07604)

- A perfect MTC is (dual) Burnside iff it is integral.
- A perfect integral MTC FPdim has no powerless prime factor.

Interpolated simple integral fusion rings of Lie type

Generic character table of $\mathrm{PSL}(2, q)$, q even

charparam c \ classparam k	$\{1\}$	$\{1\}$	$\{1, \ldots, \frac{q-2}{2}\}$	$\{1, \ldots, \frac{q}{2}\}$
$\{1\}$	1	1	1	1
$\{1, \ldots, \frac{q}{2}\}$	$q-1$	-1	0	$-\zeta_{q+1}^{kc} - \zeta_{q+1}^{-kc}$
$\{1\}$	q	0	1	-1
$\{1, \ldots, \frac{q-2}{2}\}$	$q+1$	1	$\zeta_{q-1}^{kc} + \zeta_{q-1}^{-kc}$	0
class size	1	q^2-1	$q(q+1)$	$q(q-1)$

Theorem (Liu-P-Ren, Internat. J. Math. 2023)

$$
\begin{aligned}
x_{q-1,c_1}x_{q-1,c_2} &= \delta_{c_1,c_2}x_{1,1} + \sum_{\substack{c_3 \text{ such that} \\ c_1+c_2+c_3\neq q+1 \text{ and } 2\max(c_1,c_2,c_3)}} x_{q-1,c_3} + (1-\delta_{c_1,c_2})x_{q,1} + \sum_{c_3} x_{q+1,c_3}, \\
x_{q-1,c_1}x_{q,1} &= \sum_{c_2}(1-\delta_{c_1,c_2})x_{q-1,c_2} + x_{q,1} + \sum_{c_2} x_{q+1,c_2}, \\
x_{q-1,c_1}x_{q+1,c_2} &= \sum_{c_3} x_{q-1,c_3} + x_{q,1} + \sum_{c_3} x_{q+1,c_3}, \\
x_{q,1}x_{q,1} &= x_{1,1} + \sum_{c} x_{q-1,c} + x_{q,1} + \sum_{c} x_{q+1,c}, \\
x_{q,1}x_{q+1,c_1} &= \sum_{c_2} x_{q-1,c_2} + x_{q,1} + \sum_{c_2}(1+\delta_{c_1,c_2})x_{q+1,c_2}, \\
x_{q+1,c_1}x_{q+1,c_2} &= \delta_{c_1,c_2}x_{1,1} + \sum_{c_3} x_{q-1,c_3} + (1+\delta_{c_1,c_2})x_{q,1} + \sum_{\substack{c_3 \text{ such that} \\ c_1+c_2+c_3\neq q-1 \\ \text{and } 2\max(c_1,c_2,c_3)}} x_{q+1,c_3} + \sum_{\substack{c_3 \text{ such that} \\ c_1+c_2+c_3=q-1 \\ \text{or } 2\max(c_1,c_2,c_3)}} 2x_{q+1,c_3},
\end{aligned}
$$

Local subsystems for the $q = 6$ interpolation

10 variables and 12 polynomial equations

$$
\begin{aligned}
5u_0 + 7u_1 + 7u_2 - 4/25 &= 0,\\
5v_0 + 5v_1 + 7v_3 + 7v_5 + 1/5 &= 0,\\
25v_0^2 + 25v_1^2 + 35v_3^2 + 35v_5^2 - 4/5 &= 0,\\
5v_0^3 + 5v_1^3 + 7v_3^3 + 7v_5^3 - v_0^2 + 1/125 &= 0,\\
5v_0v_1^2 + 5v_1v_2^2 + 7v_3v_4^2 + 7v_5v_6^2 + 1/125 &= 0,\\
5u_0v_1 - v_1^2 + 7u_1v_3 + 7u_2v_5 + 1/125 &= 0,\\
5v_1 + 5v_2 + 7v_4 + 7v_6 + 1/5 &= 0,\\
25v_0v_1 + 25v_1v_2 + 35v_3v_4 + 35v_5v_6 + 1/5 &= 0,\\
5v_0^2v_1 + 5v_1^2v_2 + 7v_3^2v_4 + 7v_5^2v_6 - v_1^2 + 1/125 &= 0,\\
25v_1^2 + 25v_2^2 + 35v_4^2 + 35v_6^2 - 4/5 &= 0,\\
5v_1^3 + 5v_2^3 + 7v_4^3 + 7v_6^3 - u_0 + 1/125 &= 0,\\
5u_0v_2 - v_2^2 + 7u_1v_4 + 7u_2v_6 + 1/125 &= 0
\end{aligned}
$$

It admits 14 solutions in char. 0, which can be written as a Gröbner basis. There is an extra equation linking two such subsystems, and leading to a contradiction.

Perfect MNSD

Theorem (Alekseyev-Bruns-P-Petrov, 2023; Czenky-Gvozdjak-Plavnik, 2023)

- Every odd-dim. MTC of rank < 25 is pointed.
- Every perfect one of rank $< 25^2 = 625$ is simple.

For the rank 25 simple case there remain the following 21 types:

1. [[1,1],[15,2],[63,2],[115,2],[161,4],[315,2],[805,2],[1449,2],[2415,8]],
2. [[1,1],[21,2],[35,2],[63,2],[85,2],[119,2],[255,2],[595,2],[1071,2],[1785,8]],
3. [[1,1],[39,4],[65,2],[189,2],[315,2],[585,2],[1365,2],[2457,2],[4095,8]],
4. [[1,1],[39,2],[231,2],[273,2],[1001,2],[1287,2],[3465,2],[4095,2],[9009,2],[15015,8]],
5. [[1,1],[39,2],[231,2],[273,2],[1001,2],[3465,2],[4095,2],[6435,4],[15015,8]],
6. [[1,1],[45,2],[95,6],[99,2],[495,2],[1045,2],[1881,2],[3135,8]],
7. [[1,1],[75,2],[91,4],[175,2],[585,2],[975,2],[2275,2],[4095,2],[6825,8]],
8. [[1,1],[75,2],[91,4],[175,2],[975,2],[2275,2],[2925,4],[6825,8]],
9. [[1,1],[99,2],[195,2],[385,2],[405,2],[15015,2],[31185,2],[36855,2],[81081,2],[135135,8]],
10. [[1,1],[99,2],[231,2],[385,2],[675,2],[10395,4],[28875,2],[51975,2],[86625,8]],
11. [[1,1],[99,2],[273,2],[1001,2],[1365,2],[2145,2],[3465,4],[9009,2],[15015,8]],
12. [[1,1],[135,4],[165,2],[189,2],[315,2],[385,2],[1155,2],[2079,2],[3465,8]],
13. [[1,1],[195,4],[231,2],[1001,2],[1287,2],[3465,2],[4095,2],[9009,2],[15015,8]],
14. [[1,1],[195,4],[231,2],[1001,2],[3465,2],[4095,2],[6435,4],[15015,8]],
15. [[1,1],[205,2],[369,2],[805,2],[6601,2],[8487,2],[12915,2],[33005,2],[59409,2],[99015,8]],
16. [[1,1],[205,2],[369,2],[805,2],[6601,2],[12915,2],[33005,2],[42435,4],[99015,8]],
17. [[1,1],[231,2],[777,2],[1155,2],[1221,2],[3465,2],[6105,2],[14245,2],[25641,2],[42735,8]],
18. [[1,1],[297,2],[385,2],[3465,2],[5859,2],[7161,2],[15345,2],[35805,2],[64449,2],[107415,8]],
19. [[1,1],[693,2],[1771,2],[4301,2],[11781,2],[24633,2],[30107,4],[38709,2],[90321,8]],
20. [[1,1],[693,2],[1771,2],[11781,2],[12903,4],[24633,2],[38709,4],[90321,8]],
21. [[1,1],[3485,2],[6273,2],[33825,2],[34425,2],[621027,4],[1725075,2],[3105135,2],[5175225,8]].

Robin's reformulation of Riemann Hypothesis (RH)

There is a way to extend RH to SPA using the biprojection lattices.

Let σ be the *divisor function*, defined on $\mathbb{N}_{\geq 1}$ by

$$\sigma(n) := \sum_{d|n} d.$$

Theorem

Let γ the Euler–Mascheroni constant, then

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma}.$$

Theorem (Robin, 1984)

RH is true if and only if for n large enough

$$\sigma(n) < e^{\gamma} n \log\log n.$$

Quantum Riemann Hypothesis (QRH)

Here $\mathcal{P}$ denotes an irreducible SPA and $|\mathcal{P}|$ its index.

Let the *divisor function* and its *divisor set* (finite by Watatani) be

$$\sigma(\mathcal{P}) := \sum_{d \in D(\mathcal{P})} d, \text{ where } D(\mathcal{P}) := \{|b : e_1| \text{ with } b \in [e_1, id]\}.$$

Let $\mathfrak{F}_r$ be the set of irreducible SPA of depth r (up to equivalence).

Quantum Riemann Hypothesis of depth $r \geq 2$ (QRH$_r$)

There is a (finite) constant γ_r such that

$$\limsup_{\mathcal{P} \in \mathfrak{F}_r, |\mathcal{P}| \to \infty} \frac{\sigma(\mathcal{P})}{|\mathcal{P}| \log \log |\mathcal{P}|} = e^{\gamma_r}$$

and for $\mathcal{P} \in \mathfrak{F}_r$ with $|\mathcal{P}|$ large enough

$$\sigma(\mathcal{P}) < e^{\gamma_r} |\mathcal{P}| \log \log |\mathcal{P}|.$$

By Galois correspondence, $\gamma_2 = \gamma$ and QRH$_2 \Leftrightarrow$ RH.

Generalization of QRH to Tensor Categories?

Consider the following reformulations in a unitary tensor category:

- Finite index subfactor ↔ Unitary Frobenius algebra,
- Irreducible ↔ Connected,
- Intermediate subfactor ↔ Unitary Frobenius subalgebra.

Watatani's theorem reformulated

A connected unitary Frobenius algebra in a unitary tensor category has finitely many unitary Frobenius subalgebras.

Question: To what extent can the unitary assumption be relaxed?

Categorical Riemann Hypothesis (CRH)

Then, RH can be extended to tensor categories as for the previous slide, but involving Frobenius subalgebra lattices.

Question: Can we reduce QRH or CRH to distributive lattices[1]?

[1] generalizing natural numbers as maximal does for primes (my Ore's papers)