Panorama of Geometric structures on surfaces

PART V: Quadratic differentials

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Quadratic differentials

On a Riemann surface, a (meromorphic) quadratic differential is a section of the symmetric square of the cotangent bundle. Locally, it can be written as $f(z)dz^2$ where f is a meromorphic function.

Quadratic differentials appear in various places:

- geometric function theory;
- cotangent space of Teichmüller space;
- several geometric structures: **flat** and **spherical** metrics.

We are interested in the latter interpretations.

Moduli space of quadratic differentials

The moduli space of pairs (X, q) where X is a compact Riemann surface of genus g and q is a quadratic differential is stratified by the orders of the zeroes and poles of q.

Degrees of singularities define a partition μ of $4g-4$.

Another important invariant is whether q is the global square of a 1-form. We say that q is **Abelian** in this case and **primitive** otherwise.

We denote by $\mathcal{Q}^{g}(\mu)$ such a **stratum** of primitive quadratic differentials.

Half-translation surfaces

On a Riemann surface X, a quadratic differential q defines local coordinates $\pm\int\sqrt{q}$. They differ by the action of $z \mapsto \pm z + cte$.

Complex analysis	Flat Geometry
Pair (X, q)	Half-translation surface
Zero of order $k \geq -1$	Conical singularity of angle $(k+2)\pi$
Double pole $(\frac{\lambda dz}{z})^2$	End of a cylinder of width $\lvert\lambda\rvert$

Flat model

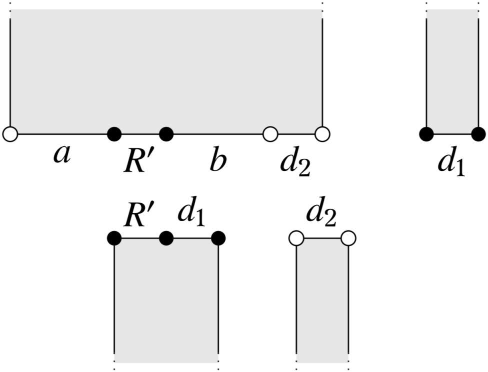

Figure: A primitive quadratic differential with four double poles, one triple zero (black dot) and one simple zero (white dot)

Canonical double cover

For any primitive quadratic differential (X, q), we can define a cover $f : Y \to X$ with the following properties:

1. f is of degree two;
2. f is ramified at the singularities of odd order;
3. the pullback of q by f if the global square of a 1-form.

Riemann-Hurwitz formula

We can compute the genus of the canonical double cover using **Riemann-Hurwitz formula**.

Keep in mind that the number of singularities of odd order has to be even because of **Gauss-Bonnet formula**.

In particular, if q is defined on $\mathbb{CP}^1$ and has exactly two singularities of odd order, then the canonical cover is also of **genus zero**.

What is the analog of isoresidual fibration for quadratic differentials?

First observe the following facts:

1. only poles of **even order** have a residue;
2. there is no residue theorem because the group of transition maps is not Abelian anymore.

We can first try to characterize **empty fibers**.

Uniformly zero configuration I

Consider a stratum of quadratic differentials with:

- an **even** number of singularities of **odd order**;
- no **double pole**;
- p poles of even orders $b_1, \dots, b_p \geq 4$.

Is it possible to construct a quadratic differential where the residues at the poles are zero?

We can use the **canonical double cover** to relate this situation to the analogous obstruction for meromorphic 1-forms.

Uniformly zero configuration II

The **canonical double cover** produces a **translation surface of genus zero** only if there are **two ramification points**.

We thus consider a stratum of **primitive** quadratic differentials $\mathcal{Q}(2a_1, \dots, 2a_n, b_1, b_2, -2d_1, \dots, -2d_2)$ with:

- zeroes of even orders $2a_1, \dots, 2a_n$;
- two singularities of odd orders b_1 and b_2;
- poles of even orders $2d_1, \dots, 2d_p \geq 4$.

The canonical cover will have to satisfy the obstruction of exact 1-forms.

An alternative way is to consider the developing map of the **half-translation structure**.

Arithmetic obstructions I

Now consider a stratum of **primitive** quadratic differentials $\mathcal{Q}(2a_1, \dots, 2a_n, b_1, b_2, (-2)^p))$ with:

- zeroes of even orders $2a_1, \dots, 2a_n$;
- two singularities of odd orders b_1 and b_2;
- p double poles.

What are the obstructions to the realization of a configuration of quadratic residues $(\lambda_1)^2, \dots, (\lambda_p^2)$ where $\lambda_1, \dots, \lambda_p \in \mathbb{N}^*$?

Arithmetic obstructions II

Here, the correct strategy is to extract from the developing map a **ramified cover** to the **infinite half-cylinder** $(\mathbb{CP}^1, \frac{dz^2}{z(z-1)})$.

We will obtain a remarkable obstruction depending on the **parity** of the sum of the square roots of the residues.

The geometric explanation of this fact is that the parity of $\sum \lambda_j$ determines whether the two singularities of odd degrees are projected to the same point or not.