Mathematical logic: unifications and diversifications

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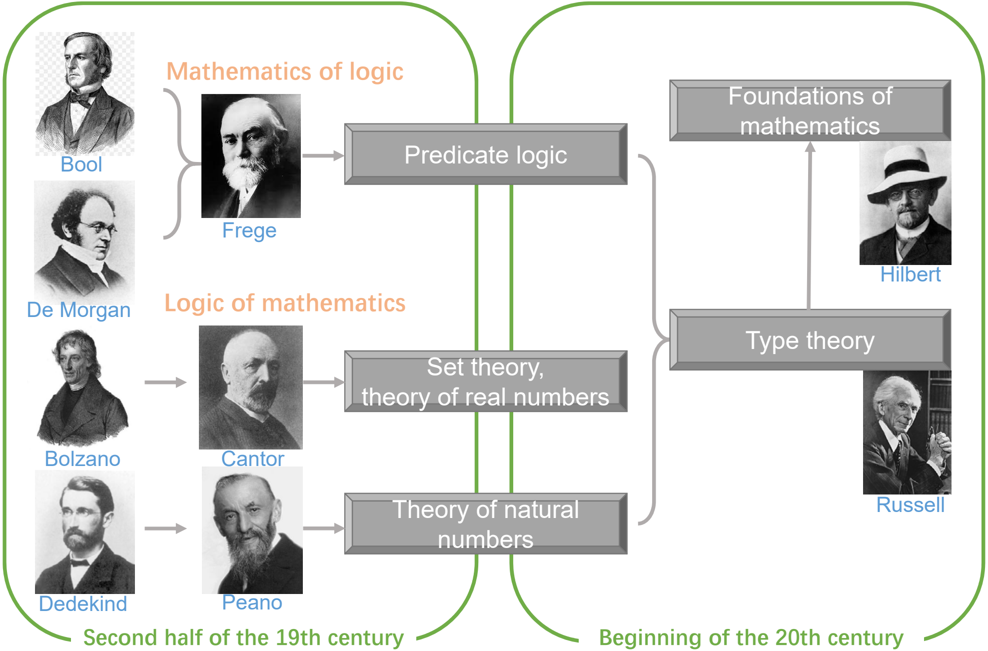
Today's talk

1. Introduction
2. First order logic and Entscheidungsproblem
3. From first order logic to second order logic
4. From Entscheidungsproblem to Model Checking
5. Gale-Stewart games and Reverse Mathematics
6. Our recent research directions

Introduction

- Nowadays, **mathematical logic** is preferably called just **logic**, because of its diversified aspects toward non-(purely-)mathematical subjects such as AI.
- About a half century ago, mathematical logic mainly consisted of four big subfields: **model theory**, **proof theory**, **set theory**, **recursion theory**. Nowaday, boundaries are blurred and many new directions have emerged.
- Furthermore, many recent important topics span multiple areas. Typical examples are **determinacy** of infinite games and a program of **reverse mathematics**, which I will explain in this talk.



Mathematics of logic
Bool
Frege
Predicate logic
De Morgan
Logic of mathematics
Bolzano
Cantor
Set theory,
theory of real numbers
Dedekind
Peano
Theory of natural
numbers
Second half of the 19th century
Foundations of
mathematics
Hilbert
Type theory
Russell
Beginning of the 20th century

Continuum hypothesis and Entscheidungsproblem

- Cantor proved in 1874 that the real numbers are not countable.
  In 1877, he put forth a conjecture (Continuum hypothesis) that any uncountable set of reals has the cardinality of the continuum, i.e., the continuum is $\aleph_1$.

- This was the first problem of Hilbert's 23 problems, which still remains unsettled. Aspero and Schindler (2021) provided a stronger rationale for the negation of CH, i.e., the continuum is $\aleph_2$. They received the Frontier Science Award of ICBS 2023.

- Hilbert tried to settle this problem by reducing it to Entscheidungsproblem (decision problem) of first-order logic, in which set theory is formulated.

- First order logic (FO) only treats the following quantifiers:

  $\forall x$ meaning "for any element x of the object structure"

  $\exists x$ meaning "there exists an element x of the object structure"

  besides propositional connectives $\wedge, \vee, \rightarrow, \neg$, and mathematical symbols.

First order logic and Entscheidungsproblem

Hilbert-Ackermann, Grundzüge der theoretischen Logik (1928)

The decision problem (Entscheidungsproblem) is solved when we know a procedure that allows, for any given logical expression, to decide by finitely many operations its validity or satisfiability. . . .
The decision problem must be considered the main problem of mathematical logic.

- FO$(\mathbb{N}, +)$ is decidable. (Presburger, 1929)
- FO$(\mathbb{N}, \cdot)$ is decidable. (Skolem, 1930)
- Ramsey introduced a theorem about partitions to deal with a simple case of the Entscheidungsproblem. (Ramsey, On a problem of formal logic, 1930)

Undecidability and Decidability Results

- Gödel proved that any first (or higher) order theory including arithmetic has an undecidable proposition. In particular, FO$(\mathbb{N}, +, \cdot)$ is undecidable. (Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, 1931)
- Church and Turing independently proved that the validity or satisfiability of first order logic is not decidable. (Turing, On computable numbers, with an application to the Entscheidungsproblem, 1936-7)
- On the other hand, Tarski proved that the first-order theory of real closed fields is decidable, although the decidability of the real exponential field is still unknown. He also proved that elementary Euclidean geometry is decidable.

From First Order Logic to Second Oder Logic

- **First Order Logic FO**
  - Almost all the mathematical theories can be developed as first-order theories, e.g., PA, ZFC.
  - Many simple properties of a structure cannot be described by a first-order formula, e.g., "a set has an even number of elements" and "a graph is connected".

- **Monadic Second Order logic** MSO = FO $+ \forall X, \exists X$ or $\forall P(x), \exists P(x)$
  X and $P(x)$ range over "subsets" and "unary predicates" of the structure.
  - "a graph is connected" $=$ a (maximal) connected component is the whole graph.

- **Second Order logic** SO = FO $+ \forall R(\vec{x}), \exists R(\vec{x}) + \forall f(\vec{x}), \exists f(\vec{x})$
  - The parity of a set can be defined by a function.

Decidability of Monadic Second-Order Theories

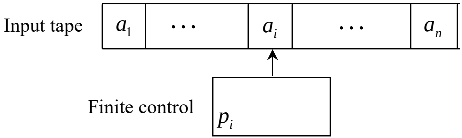
- S1S $= \mathsf{MSO}(\mathbb{N}, x+1)$ (Büchi 1960).

  The decidability of S1S is shown by ω-automata, which accept an infinite word if during reading the input, a final state appears infinitely many times.

- S2S $= \mathsf{MSO}(\{0,1\}^{<\omega}, x^\frown 0, x^\frown 1)$ (Rabin 1969).

  Proved by tree-automata, and improved with the help of infinite games (Gurevich-Harrington 1982). Another proof (Niwinski 1988) uses modal μ-calculus.

Note: A finite word $a_1 \dots a_n \in A^{<\omega}$ is accepted by a finite automaton $\mathcal{M}$ if there is a sequence of states from an initial state q_0 to a final state q_f



$$q_{\mathsf{in}} \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} q_f$$

Since the emptiness problem and universality problem of finite and certain infinite automata are decidable, theories expressed by them are decidable.

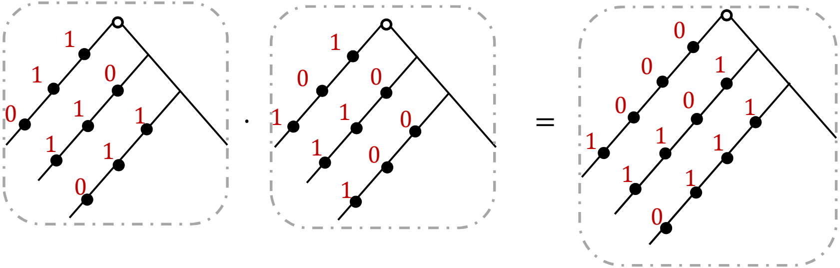
S2S and tree automata

Theorem (Rabin, 1969)

$\mathsf{MSO}(\{0,1\}^{<\omega}, x^\frown 0, x^\frown 1)$ is decidable.

Theorem (Skolem, 1930)

$\mathsf{FO}(\mathbb{N}, \cdot)$ is decidable.



$$2^3 \cdot 3^6 \cdot 5^3 \quad \cdot \quad 2^5 \cdot 3^6 \cdot 5^4 \quad = \quad 2^8 \cdot 3^{12} \cdot 5^7$$

From Entscheidungsproblem to Model Checking

Entscheidungsproblem

- Fix an object model $\mathfrak{M}$ (or theory T). Input a formula ϕ.
- Decide whether or not ϕ holds in $\mathfrak{M}$ (or T)

Model checking

- Input a model (state transition graph) $\mathfrak{M}$ and a formula (specification) ϕ.
- Decide whether or not ϕ holds in $\mathfrak{M}$.

SMV (Symbolic Model Verifier) and CTL (Computation Tree Logic)

SMV is a widely-used model checking tool developed at CMU that analyzes finite-state systems by temporal logic CTL which uses operators X (Next), F (Eventually), path quantifier A (For All Paths) and others.

- A temporal logic is a modal logic (propositional logic + modal operator $\Box$, $\Diamond$).
- CTL and its variations can be easily transformed into modal μ-calculus.

Modal logic $\subset$ FO and modal μ-calculus $\subset$ MSO

- Modal logic is proposional logic with modal operators.

  $\Box$ p expresses "$\forall$ next step, p holds."

  $\Diamond$ p expresses "$\exists$ next step, p holds."

- On a directed graph (V, E), a modal formula φ is translated into a first order formula $\mathsf{ST}_s(\varphi)$ evaluated at node s by $\mathsf{ST}_s(\Box\varphi)) \Leftrightarrow \forall t(E(s,t) \to \mathsf{ST}_t(\varphi))$.

- Modal μ-calculus, introduced by Kozen, is an extension of modal logic by adding greatest (ν) and least (μ) fixpoint operators.

- The standard translation of modal μ-calculus (into MSO) is obtained by

$$\mathsf{ST}_s(\mu X.\varphi) := \forall X(\forall t((\mathsf{ST}_t(\varphi) \to t \in X) \to s \in X)),$$

then modal μ-calculus can be expressed by MSO.

Example

The formula $\nu X.p \wedge \Box X$ expresses "always p holds." Its negation is

$$\neg(\nu X.p \wedge \Box X) = \mu X.\neg(p \wedge \Box\neg X) = \mu X.\neg p \vee \Diamond X,$$

which means "eventually (on some path) $\neg p$ holds".

Modal μ-calculus and infinite games

- The equivalence between modal μ-calculus and tree automata over binary trees is established by Niwinski (1988) and Emerson and Jutla (1989).
- The evaluation (model checking) of modal μ-calculus can be represented as an infinite game (parity game). The determinacy of this game can be expressed by monadic second-order S2S. So, we can derive the decidability of modal μ-calculus from Rabin's theorem, and vice versa.
- Several important results on variations of modal μ-calculus have been obtained by comparing with the determinacy hierachy of Gale-Stewart games.

Gale-Stewart games and determinacy

- Player I and player II select an element of A (usually $= \mathbb{N}$) alternatively.

$$\begin{array}{lccccccccc} \text{Player I}: & a_0 & & a_1 & & a_2 & & \dots & \\ & & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \\ \text{Player II}: & & b_0 & & b_1 & & b_2 & & \dots \end{array}$$

- For a game $X \subset A^\omega$, Player I wins with $a_0 b_0 a_1 b_1 a_2 b_2 \ldots \in X$; otherwise II wins.
- A **strategy for I** is $\sigma : \bigcup_{n\in\omega} A^{2n} \to A$, and a **strategy for II** is $\tau : \bigcup_{n\in\omega} A^{2n+1} \to A$.
- X is **determined** if either I or II has a winning strategy.

- Zermelo "On an Application of Set Theory to the Theory of Game of Chess" (1913) treated clopen games of the Cantor space. By König's remark, he corrected his original proof (1927).
- Gale-Stewart (1953): Open determinacy of the Baire space.
- Martin (1976): Borel determinacy.
- To show the decidability of tree automata for S2S, Gurevich and Harrington introduced a forgetful game (equivalently, a parity game).

The Road to Reverse Mathematics

- In 1920's, D. Hilbert formulated the system Z_2 of second-order arithmetic (a set theory of natural numbers), in which most of mathematics can be developed.
- Hilbert's program in a narrow sense was to show the consistency of Z_2 finitistically.
- In 1970's, H. Friedman began to study which subsystems of Z_2 are necessary and sufficient to prove theorems of ordinary mathematics.
- In 1980's, S. Simpson systematized the study as Reverse Mathematics.

Big five subsystems and typical examples

$\mathsf{RCA}_0 \Rightarrow$ the fundamental theorem of algebra

$\mathsf{WKL}_0 \leftrightarrow$ the maximum principle $\leftrightarrow$ Brouwer's fixed point theorem

$\mathsf{ACA}_0 \leftrightarrow$ the Bolzano-Weierstrass theorem $\leftrightarrow$ the Ascoli-Arzela lemma

$\mathsf{ATR}_0 \leftrightarrow$ the Luzin separation theorem $\leftrightarrow$ open-determinacy

$\Pi^1_1\text{-}\mathsf{CA}_0 \leftrightarrow$ the Cantor-Bendixson theorem $\leftrightarrow$ $\mathcal{G} \wedge \mathcal{F}$-determinacy

Reverse Math. of Determinacy

- Arithmetical formulas ($\Sigma^1_0 = \Pi^1_0$):
  $\Sigma^0_0 = \Pi^0_0$:= only with $\forall x < t, \exists x < t$
  $\Sigma^0_{n+1} := \exists \vec{x}\varphi$, with φ in Π^0_n.
  $\Pi^0_{n+1} := \forall \vec{x}\varphi$, with φ in Σ^0_n.
- Analytical formulas:
  $\Sigma^1_{n+1} := \exists \vec{X}\varphi$, with φ in Π^1_n.
  $\Pi^1_{n+1} := \forall \vec{X}\varphi$, with φ in Σ^1_n.

Δ^i_n can be expressed in both Σ^i_n and Π^i_n.

Kleene-Addison hierarchy

$$\mathcal{G} = \mathbf{\Sigma}^0_1, \quad \mathcal{F} = \mathbf{\Pi}^0_1, \quad \mathcal{G} \cap \mathcal{F} = \mathbf{\Delta}^0_1.$$
$$\mathcal{G}_\delta = \mathbf{\Pi}^0_2, \quad \mathcal{F}_\sigma = \mathbf{\Sigma}^0_2, \quad \mathcal{G}_\delta \cap \mathcal{F}_\sigma = \mathbf{\Delta}^0_2.$$
$$\mathcal{A} = \mathbf{\Sigma}^1_1, \quad \mathcal{CA} = \mathbf{\Pi}^1_1, \quad \mathcal{A} \cap \mathcal{CA} = \mathbf{\Delta}^1_1.$$

The determinacy strength

Subsystems of Z_2	Games
ACA_0	finite round
ATR_0	Δ^0_1 Σ^0_1
Π^1_1-CA_0	$(\Sigma^0_1)_2 = \Sigma^0_1 \wedge \Pi^0_1$
Π^1_1-TR_0	Δ^0_2
Σ^1_1-ID_0	Σ^0_2
$\vdots$	$\vdots$
$[\Sigma^1_1]^k$-ID_0	$(\Sigma^0_2)_k = \Sigma^0_1 \wedge (\Sigma^0_2)_{k-1}$
$\vdots$	$\vdots$
$[\Sigma^1_1]^{\mathsf{TR}}$-$\mathsf{ID}_0$	Δ^0_3

Our recent research directions

- Reverse mathematics of infinite games:
  1. Show that the determinacy hierarchy of difference sets is not strict at some levels.
  2. Investigate some topological games, e.g. Banach-Mazur games, in the view point of reverse mathmetics. Deduce some properties of sets (e.g., Ramsey theorem) from the determinacy of such games.

- WKL_0 and related systems:
  1. Improve some of my old results on non-standard models of WKL_0, by which we develop measure-theoretic tools for probability arguments in WKL_0.
  2. Study the complexity of probabilistic model checking for modal μ-calculus.

- Game trees and evaluation costs:
  1. Investigate the form of a distribution on the truth assignments of a weighted Boolean tree when it achieves the equilibrium among independent distributions. Study its relation to Series-Parallel Systems (SPS) and other models.

Thank you for your attention!

Planets of Reverse Mathematics