Constrained Hamiltonian dynamics II

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In the early 1950's, Dirac and Bergmann independently developed the Hamiltonian formalism for systems with singular Lagrangians.

These systems, often called "constrained Hamiltonian systems", include gauge theories. Gauge freedom is more clearly and more completely displayed in the Hamiltonian setting, with the generators of gauge transformations expressed as functions on phase space.

Historically, the main motivation for casting gauge theories in Hamiltonian form was to facilitate their canonical quantization. Dirac and Bergmann were primarily motivated by the prospect of developing a quantum theory of gravity based on a Hamiltonian formulation of general relativity.

P.A.M. Dirac, Generalized Hamiltonian dynamics, Can. J. Math. 2, 129 (1950).
P.A.M. Dirac, Generalized Hamiltonian dynamics, Proc. Roy. Soc. 246, 326 (1958).
P.G Bergmann, Non-Linear Field Theories, Phys. Rev. 75, 680 (1949).

J.L. Anderson and P.G. Bergmann, Constraints in Covariant Field Theories, Phys. Rev. 83, 1018 (1951).

Textbook treatments of Lagrangian and Hamiltonian mechanics invariably assume that the Lagrangian $L(q, \dot{q})$ is nonsingular; that is, that the matrix of second derivatives of $L(q, \dot{q})$ with respect to the velocities is invertible.

If matrix of second derivatives

$$L_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

is not invertible, we cannot solve Lagrange's equations for the accelerations as functions of the coordinates and velocities.

The starting point for the Hamiltonian formulation of mechanics is the Lagrangian. If L_{ij} is not invertible, the definitions of momenta in terms of coordinates and velocities cannot be inverted for the velocities as functions of coordinates and momenta.

The Hamiltonian theory cannot be constructed in the usual way.

In classical mechanics, the nonsingular case appears to be sufficient to cover problems of physical interest.

However, one might argue that textbooks avoid certain physically interesting problems simply because their Lagrangians are singular.

In field theory, the issue of singular Lagrangians and gauge freedom cannot be avoided. Nearly every field theory of physical interest—electrodynamics, Yang–Mills theory, general relativity, relativistic string theory—has gauge freedom.

The Dirac–Bergmann algorithm transforms a singular Lagrangian system into a Hamiltonian system. The formalism consists of a large number of logical steps, linked together by a chain of reasoning that can be difficult to keep straight.

Of course, there are many examples in the literature in which the Dirac–Bergmann algorithm is applied, converting a singular Lagrangian into Hamiltonian form.

in fact, all of these examples are designed to illustrate just one or two of the logical steps in the algorithm. The student of the subject is faced with the task of linking these examples together to create a complete picture of the algorithm.

For those who learn by example, what is needed is a single example that illustrates all of the major logical steps in the Dirac–Bergmann algorithm and shows how these steps are linked together.

Such a “complete” example is not easy to identify because there is no obvious way to predict, starting with a particular Lagrangian, which of the steps in the algorithm will be needed.

Presentations of the Dirac–Bergmann algorithm can be found in

A. Hanson, T. Regge and C. Teitelboim, Constrained Hamiltonian Systems, (Accademia Nazionale dei Lincei, Roma, 1976).

K. Sundermeyer, Constrained Dynamics (Springer–Verlag, Berlin, 1982)

M. Henneaux and C. Teitelboim, Quantization of Gauge Systems. (Princeton University Press, Princeton, New Jersey, 1992).

H.J. Rothe and K.D. Rothe, Classical and Quantum Dynamics of Constrained Hamiltonian Systems, (World Scientific, Singapore, 2010).

A. Deriglazov, Classical Mechanics: Hamiltonian and Lagrangian Formalism (Springer, New York, 2017).

L. Lusanna, Non-Inertial Frames and Dirac Observables in Relativity (Cambridge University Press, Cambridge, 2019).

G. Date, Lectures on Constrained Systems, arXiv: 1010.2062 [gr-qc] (2010).

J.D. Brown, Singular Lagrangians, Constrained Hamiltonian Systems and Gauge Invariance: An Example of the Dirac-Bergmann Algorithm, Universe 8, 171 (2022).

Dirac–Bergmann Algorithm I

Consider a system with $\bar{N}$ generalized coordinates q_i, where $i = 1, \dots, \bar{N}$. The velocities are denoted by $\dot{q}_i$.

The Lagrangian $L(q, \dot{q})$ is assumed to be singular, so the rank of the matrix L_{ij} (the number of linearly independent rows or columns) is less than $\bar{N}$, say, $\bar{M}$.

We assume that the rank $\bar{M}$ is constant throughout phase space.

The following is a short summary of the Dirac–Bergmann algorithm for converting this system into constrained Hamiltonian form.

This summary is not intended as a substitute for the more thorough treatments given elsewhere.

Dirac–Bergmann Algorithm II

1. Compute the conjugate momenta $p_i = \partial L/\partial \dot{q}_i$.

Since the Lagrangian is singular, these relations cannot be inverted for the velocities as functions of coordinates and momenta.

This implies the existence of $\bar{N} - \bar{M}$ relations among the coordinates and momenta.

These relations are the primary constraints, denoted $\phi_a(q, p) = 0$, with the index a ranging from 1 to $\bar{N} - \bar{M}$.

2. Define the canonical Hamiltonian

$$H_C = \sum p_i \dot{q}_i - L(q, \dot{q})$$

in terms of the q's and p's. Note that $H_C(q, p)$ is not unique, because one can always use the constraints $\phi_a(q, p) = 0$ to write some of the canonical variables in terms of others.

Dirac–Bergmann Algorithm III

3. Define the primary Hamiltonian H_P by adding the primary constraints with Lagrange multipliers to the canonical Hamiltonian

$$H_P = H_C + \sum \lambda^a \phi_a.$$

4. Impose the conditions $\{\phi_a, H_P\} = 0$, referred to as "consistency conditions". These conditions insure that the primary constraints are preserved under time evolution.

The consistency conditions will reduce to a combination of

- identities when the primary constraints $\phi_a(q,p) = 0$ hold;
- restrictions on the Lagrange multipliers express some λ's in terms of q's, p's, and the remaining λ's;
- restrictions on the q's and p's are secondary constraints, which we write as $\psi_m(q,p) = 0$.

Dirac–Bergmann Algorithm IV

5. The consistency conditions must be applied to the secondary constraints to insure their preservation in time: $[\psi_m, H_P] = 0$.

This can yield identities, further restrictions on the Lagrange multipliers, and/or tertiary constraints, which are further restrictions on the q's and p's.

We continue to apply the consistency conditions to identify higher–order constraints and restrictions on the Lagrange multipliers. The process naturally stops when the consistency conditions have been applied to all constraints.

We extend the range of the index m and let $\psi_m(q, p)$ denote all of the secondary, tertiary, and higher–order constraints.

6. The total Hamiltonian H_T is obtained from the primary Hamiltonian H_P by incorporating the restrictions on Lagrange multipliers. In the most general case, a subset of the Lagrange multipliers will remain free.

Dirac–Bergmann Algorithm V

7. The primary, secondary, tertiary, etc. constraints ϕ_a and ψ_m are separated into first and second class.

First class constraints have the property that their Poisson bracket with all constraints vanish when the constraints hold.

Second class constraints have nonvanishing Poisson bracket with at least one other constraint. Let $\mathcal{C}^{(fc)}_{\alpha}$ denote the set of first class constraints, and $\mathcal{C}^{(sc)}_{\mu}$ denote the set of second class constraints.

8. A subset of first class constraints can be constructed from the primary constraints $\phi_a(q,p)$. These are the primary first class constraints which we denote $\mathcal{C}^{(pfc)}_{A}$.

It can be shown that the total Hamiltonian can be written as $H_T = H_{fc} + \Lambda^A \mathcal{C}^{(pfc)}_{A}$, where the Lagrange multipliers Λ^A are free and the first class Hamiltonian H_{fc} has vanishing Poisson bracket with all of the constraints (when the constraints hold).

The equations of motion generated by the total Hamiltonian

$$H_T = H_{fc} + \Lambda^A \mathcal{C}_A^{(pfc)}$$

through the Poisson bracket are equivalent to Lagrange's equations for the original Lagrangian system.

Since the Lagrange multipliers Λ^A are completely arbitrary, the phase space transformations generated by the primary first class constraints $\mathcal{C}_A^{(pfc)}$ do not change the physical state of the system.

We refer to such transformations as gauge transformations.

The Dirac conjecture says that all first class constraints $\mathcal{C}_\alpha^{(fc)}$ generate gauge transformations.

Counterexamples to this conjecture have been described in the literature, other researchers have argued against these counterexamples, citing subtleties in the way that the constraints are written

Dirac–Bergmann Algorithm VI

9. Assuming the Dirac conjecture holds, each of the first class constraints has the status of a gauge generator. These constraints can be treated on an equal footing by constructing the extended Hamiltonian

$$H_E = H_{fc} + \Lambda^\alpha \mathcal{C}_\alpha^{(fc)}.$$

This is the sum of the first class Hamiltonian H_{fc} and a linear combination of all first class constraints with unrestricted Lagrange multipliers Λ^α.

The equations of motion defined by the extended Hamiltonian are not strictly equivalent to the original Lagrangian equations of motion.

Nevertheless, the theories agree for the evolution of physical variables (variables that are invariant under gauge transformations.)

Phase space functions F are evolved in time with either the extended Hamiltonian

$$\dot{F} = \{F, H_E\},$$

or the total Hamiltonian

$$\dot{F} = \{F, H_T\}$$

Physical trajectories are those that lie in the subspace of phase space where the constraints hold.

The constraint relations can be used freely after computing Poisson brackets, but not before.

For example, the constraints can be used to alter the equations of motion $\dot{F} = \{F, H_E\}$ (or $\dot{F} = \{F, H_T\}$) but not the functions that appears in the Poisson bracket.

Dirac bracket

We now have a few options. One option:

Eliminate the second class constraints leaving the gauge freedom generated by the first class constraints intact.

We can do this by replacing the Poisson bracket with the Dirac bracket, defined as follows. Let

$$\mathcal{M}_{\mu\nu} = \{\mathcal{C}_\mu^{(sc)}, \mathcal{C}_\nu^{(sc)}\}$$

denote the matrix of Poisson brackets among the second class constraints, and let $\mathcal{M}^{\mu\nu}$ denote its inverse.

The Dirac bracket is

$$\{F, G\}_D = \{F, G\} - \{F, \mathcal{C}_\mu^{(sc)}\}\mathcal{M}^{\mu\nu}\{\mathcal{C}_\nu^{(sc)}, G\}$$

where F and G are phase space functions. Like the Poisson bracket, the Dirac bracket is antisymmetric and obeys the Jacobi identity.

Because $\{F, \mathcal{C}_\mu^{(sc)}\}_D = 0$ for any F we can use the second class constraints to simplify F and G before computing the bracket $\{F, G\}_D$.

Because the Poisson bracket of the extended Hamiltonian with a second class constraint will vanish when the constraints hold, it follows that

$$\{F, H_E\}_D = \{F, H_E\}$$

when the constraints hold.

We can now eliminate a subset of phase space variables by imposing the second class constraints $\mathcal{C}_\mu^{(sc)} = 0$ and using the Dirac bracket.

In particular, we can use $\mathcal{C}_\mu^{(sc)} = 0$ to eliminate variables from the extended Hamiltonian (or total Hamiltonian), resulting in a partially reduced Hamiltonian H_{PR}. Time evolution becomes

$$\dot{F} = \{F, H_{PR}\}_D.$$

A second option is eliminate both first and second class constraints

Let us denote the constraints and gauge conditions, combined, by $\mathcal{C}_M^{(all)}$. A good set of gauge conditions will have the property that $\mathcal{C}_M^{(all)}$ are second class and the matrix of Poisson brackets $\mathcal{M}_{MN} = \{\mathcal{C}_M^{(all)}, \mathcal{C}_N^{(all)}\}]$ is invertible. Let $\mathcal{M}^{MN}$ denote the inverse and define the Dirac bracket by

$$\{F, G\}_D = \{F, G\} - \{F, \mathcal{C}_M^{(all)}\}\mathcal{M}^{MN}\{\mathcal{C}_N^{(all)}, G\} ,$$

where summations over M and N are implied.

Now use the constraints and gauge conditions $\mathcal{C}_M^{(all)} = 0$ to eliminate a subset of phase space variables. We can eliminate variables from the extended Hamiltonian (or total Hamiltonian), resulting in a fully reduced Hamiltonian H_{FR}. Time evolution becomes $\dot{F} = \{F, H_{FR}\}_D$.

Compound spring

Form a "compound spring" by welding two springs together

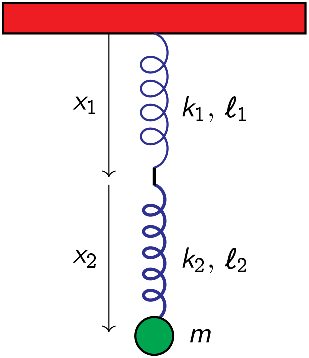

Figure: The compound spring.

One end of the compound spring is attached to the ceiling, and a mass m hangs from the other end. The mass moves vertically, with gravity acting in the downward direction.

Let x_1 and x_2 denote the lengths of the two springs, so the distance between the ceiling and the mass is $x_1 + x_2$. The Lagrangian for this system is

$$L = \frac{m}{2}(\dot{x}_1 + \dot{x}_2)^2 + mg(x_1 + x_2) - \frac{k_1}{2}(x_1 - \ell_1)^2 - \frac{k_2}{2}(x_2 - \ell_2)^2 \; .$$

The matrix of second derivatives of L with respect to the velocities $\dot{x}_i$,

$$L_{ij} = \begin{pmatrix} m & m \\ m & m \end{pmatrix} \; ,$$

is singular with rank 1. The momenta are

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = m(\dot{x}_1 + \dot{x}_2) \; ,$$
$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = m(\dot{x}_1 + \dot{x}_2) \; ,$$

and we can identify the primary constraint

$$\phi \equiv p_2 - p_1 \; .$$

Next, construct the canonical Hamiltonian $H = \sum p_i \dot{x}_i - L$:

$$H_C(x,p) = \frac{1}{2m} p_1 p_2 - mg(x_1 + x_2) + \frac{k_1}{2}(x_1 - \ell_1)^2 + \frac{k_2}{2}(x_2 - \ell_2)^2 \, .$$

The leading term $p_1 p_2/(2m)$ can be written in other ways, such as

$$p_1^2/(2m) \, , \quad \text{or} \quad (p_1^2 + p_2^2)/(4m).$$

The primary Hamiltonian is $H_P = H_C + \lambda\phi$. The consistency condition $\{\phi, H_P\} = 0$ yields the secondary constraint

$$\psi = k_1(x_1 - \ell_1) - k_2(x_2 - \ell_2) \, ,$$

and the condition $\{\psi, H_P\} = 0$ restricts the Lagrange multiplier to

$$\lambda = \frac{k_1 p_2 - k_2 p_1}{2m(k_1 + k_2)} \, .$$

The application of consistency conditions is now complete.

The secondary constraint $\psi = 0$ has a direct physical interpretation via Newton's third law.

It tells us that the force $k_1(x_1 - \ell_1)$ that spring 1 exerts on spring 2 is equal but opposite to the force $k_2(\ell_2 - x_2)$ that spring 2 exerts on spring 1.

The total Hamiltonian is obtained by using the result for λ in the primary Hamiltonian:

$$H_T = \frac{1}{2m} p_1 p_2 + \frac{k_1 p_2 - k_2 p_1}{2m(k_1 + k_2)} (p_2 - p_1) - mg(x_1 + x_2) + \frac{k_1}{2}(x_1 - \ell_1)^2 + \frac{k_2}{2}(x_2 - \ell_2)^2 \;.$$

The two constraints are second class, since $\{\phi, \psi\} = k_1 + k_2$ is nonzero.

There are no first class constraints, so the system has no gauge freedom and the total Hamiltonian, first class Hamiltonian, and extended Hamiltonian coincide

$$H_T = H_{fc} = H_E.$$

Let $\mathcal{C}^{(sc)}_\mu = \{\phi, \psi\}$ denote the set of second class constraints. The matrix $\mathcal{M}_{\mu\nu} = \{\mathcal{C}^{(sc)}_\mu, \mathcal{C}^{(sc)}_\nu\}$ is invertible with inverse

$$\mathcal{M}^{\mu\nu} = \frac{1}{k_1 + k_2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

The nonzero Dirac brackets among the phase space variables are

$$\{x_1, p_1\}_D = \{x_1, p_2\}_D = k_2/(k_1 + k_2) ,$$
$$\{x_2, p_1\}_D = \{x_2, p_2\}_D = k_1/(k_1 + k_2) .$$

We can use the constraints to eliminate two of the phase space variables. For example, solving $\phi = \psi = 0$ for x_2 and p_2, we find

$$x_2 = \ell_2 + \frac{k_1}{k_2}(x_1 - \ell_1) ,$$
$$p_2 = p_1 ,$$

and the Hamiltonian reduces to

$$H_R = \frac{p_1^2}{2m} - \frac{mg}{k_2}\left[(k_1 + k_2)x_1 + k_2\ell_2 - k_1\ell_1\right] + \frac{1}{2}(k_1 + k_1^2/k_2)(x_1 - \ell_1)^2 .$$

Note that in the absence of first class constraints, the partially and fully reduced Hamiltonians coincide. The time evolution of any function of the phase space variables x_1, p_1, x_2, p_2 can be obtained from H_R and the Dirac bracket

$$\dot{x}_1 = \{x_1, H_R\}_D = \frac{k_2}{m(k_1+k_2)} p_1 \;,$$
$$\dot{p}_1 = \{p_1, H_R\}_D = -k_1(x_1 - \ell_1) + mg \;,$$

which form a closed set of differential equations for x_1 and p_1 with general solution

$$x_1(t) = A\cos(\omega t) + B\sin(\omega t) + \ell_1 + mg/k_1 \;,$$
$$p_1(t) = \frac{k_1}{\omega}\left(B\cos(\omega t) - A\sin(\omega t)\right) \;.$$

Here, A and B are constants and the angular frequency is defined by

$$\omega = \sqrt{k_1 k_2/(m(k_1+k_2))}\;.$$

Pendulum and two springs

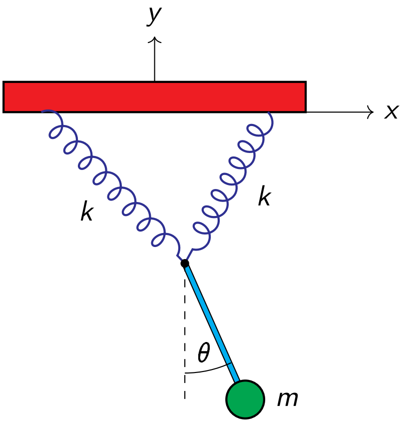

Figure: A pendulum hanging from two springs. The springs are attached to the ceiling at the points $x = \pm d$, $y = 0$.

Figure shows a pendulum of length ℓ hanging from two springs.

Each spring has stiffness k, and for simplicity we take the relaxed length of each spring to be zero.

The generalized coordinates are the Cartesian coordinates x and y of the point where the springs attach to the pendulum, and the angle θ of the pendulum rod.

The Cartesian coordinate origin is midway between the points where the springs attach to the ceiling. The angle θ is measured from the negative y–axis.

The kinetic energy for this system is

$$T = (m/2)(\dot{X}^2 + \dot{Y}^2),$$

where $X = x + \ell \sin\theta$ and $Y = y - \ell\cos\theta$ are the Cartesian coordinates of the mass m. The Lagrangian is

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \ell^2\dot{\theta}^2) + m\ell(\dot{x}\cos\theta + \dot{y}\sin\theta)\dot{\theta} - mg(y - \ell\cos\theta) - k(x^2 + y^2 + d^2)\ ,$$

and the matrix of second derivatives of L with respect to the velocities $\dot{x}$, $\dot{y}$, $\dot{\theta}$ is

$$L_{ij} = \begin{pmatrix} m & 0 & m\ell\cos\theta \\ 0 & m & m\ell\sin\theta \\ m\ell\cos\theta & m\ell\sin\theta & m\ell^2 \end{pmatrix}\ .$$

This matrix is singular with rank 2.

The momenta for this system are

$$p_x = m\dot{x} + m\ell\dot{\theta}\cos\theta \ ,$$
$$p_y = m\dot{y} + m\ell\dot{\theta}\sin\theta \ ,$$
$$p_\theta = m\ell(\dot{x}\cos\theta + \dot{y}\sin\theta) + m\ell^2\dot{\theta} \ .$$

Since the Lagrangian is quadratic in the velocities, the constraint can be constructed as $\phi = V^i(p_i - L_i)$, where the vector $V^i = (-\ell\cos\theta, -\ell\sin\theta, 1)$ spans the null space of L_{ij}. In this case $L_i = 0$ and the primary constraint is

$$\phi = -\ell p_x\cos\theta - \ell p_y\sin\theta + p_\theta \ .$$

The canonical Hamiltonian can be written as

$$H_C = \frac{1}{2m}(p_x^2 + p_y^2) + k(x^2 + y^2 + d^2) + mg(y - \ell\cos\theta) \ ,$$

and the primary Hamiltonian is $H_P = H_C + \lambda\phi$.

The consistency condition $\{\phi, H_P\} = 0$ yields the secondary constraint

$$\psi = 2k\ell(x\cos\theta + y\sin\theta)\ ,$$

which gives $\tan\theta = -x/y$. This tells us that the angle of the force exerted by the springs on the massless connection point must coincide with the angle of the pendulum rod. This is a consequence of Newton's third law.

The condition $\{\psi, H_P\} = 0$ determines the Lagrange multiplier to be

$$\lambda = \frac{p_x\cos\theta + p_y\sin\theta}{m(\ell + x\sin\theta - y\cos\theta)}\ ,$$

and the total Hamiltonian

$$\begin{aligned} H_T =& \frac{1}{2m}(p_x^2 + p_y^2) + k(x^2 + y^2 + d^2) + mg(y - \ell\cos\theta) \\ &+ \frac{p_x\cos\theta + p_y\sin\theta}{m(\ell + x\sin\theta - y\cos\theta)}(-\ell p_x\cos\theta - \ell p_y\sin\theta + p_\theta). \end{aligned}$$

Note that the denominator is the coefficient of λ in $\{\psi, H_P\}$. This coefficient vanishes when $x = -\ell\sin\theta$, $y = \ell\cos\theta$. At these points in phase space the Lagrange multiplier is not determined.

This is not a shortcoming of the Dirac–Bergmann formalism, rather, it is a property of the physical system defined by the Lagrangian.

When $x = -\ell\sin\theta$ and $y = \ell\cos\theta$, the mass m is at the origin and the pendulum rod can rotate without any inertial resistance, and without any change in the potential energy.

Thus, at these points in phase space, the system exhibits a gauge–like freedom in which multiple configurations are physically indistinguishable.

We can avoid this complication by assuming the mass stays below the ceiling, so that $Y = y - \ell\cos\theta$ is always negative.

With this assumption the constraints are second class:

$$\{\phi, \psi\} = 2k\ell(\ell + x\sin\theta - y\cos\theta) \neq 0.$$

There is no gauge freedom, so

$$H_E = H_{fc} = H_T.$$

We can construct the Dirac bracket with

$$\mathcal{M}^{\mu\nu} = \frac{1}{2k\ell(\ell + x\sin\theta - y\cos\theta)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

The constraints $\phi = \psi = 0$ imply

$$\theta = -\arctan(x/y) \,,$$
$$p_\theta = \frac{\ell}{\sqrt{x^2+y^2}}(xp_y - yp_x) \,,$$

and we can use these results to reduce the Hamiltonian:

$$H_R = \frac{1}{2m}(p_x^2 + p_y^2) + mgy(1 + \ell/\sqrt{x^2+y^2}) + k(x^2 + y^2 + d^2) \,.$$

The nonzero Dirac brackets among the remaining variables are

$$\{x, p_x\}_D = \frac{r + \ell x^2/r^2}{r + \ell} \ ,$$

$$\{x, p_y\}_D = \{y, p_x\}_D = \frac{\ell xy/r^2}{r + \ell} \ ,$$

$$\{y, p_y\}_D = \frac{r + \ell y^2/r^2}{r + \ell}$$

where $r \equiv \sqrt{x^2 + y^2}$. The equations of motion are

$$\dot{x} = \{x, H_R\}_D = \frac{\ell x(xp_x + yp_y) + p_x r^3}{mr^2(r + \ell)} \ ,$$

$$\dot{y} = \{y, H_R\}_D = \frac{\ell y(xp_x + yp_y) + p_y r^3}{mr^2(r + \ell)} \ ,$$

$$\dot{p}_x = \{p_x, H_R\}_D = -2kx \ ,$$

$$\dot{p}_y = \{p_y, H_R\}_D = -mg - 2ky \ .$$

These can be solved numerically in a straightforward fashion.

Conclusion:

One reason the Dirac–Bergmann algorithm can be confusing is that typical examples are chosen for simplicity, allowing some of the logical steps to be skipped.

This causes the distinction between Hamiltonians to become blurred. For example, if there are no restrictions on the Lagrange multipliers, then the primary Hamiltonian H_P and the total Hamiltonian H_T coincide.

Likewise, if there are no secondary (or higher-order) first class constraints, then the total Hamiltonian H_T and the extended Hamiltonian H_E coincide.

Moreover, for theories with no second class constraints and no gauge conditions imposed, Dirac brackets and the reduction process are not needed.

Another confusing aspect of the Dirac–Bergmann algorithm is that for many important theories, the Lagrangian is given to us in a form that contains Lagrange multipliers.

For example, consider the Einstein–Hilbert action of general relativity. The Lagrangian density is the spacetime curvature scalar. A 3+1 splitting of the spacetime metric allows us to write the Lagrangian density as

$$L = R + \sum K_{ij}(g^{ik}g^{j\ell} - g^{ij}g^{k}\ell)K_{k\ell},$$

apart from a total derivative term that integrates to the boundary.

Here, R and g^{ij} are the spatial scalar curvature and spatial metric.

In addition, K_{ij} is the extrinsic curvature of space, built from the lapse function N, shift vector N_i and spatial derivatives of g^{ij}. The Lagrangian density depends on the Lagrange multipliers N and N_i as well as the configuration space coordinates g^{ij}.

We can attempt to eliminate the lapse and shift from the Einstein–Hilbert action.

It is straightforward to eliminate the lapse N. The result is the Baierlein–Sharp–Wheeler action

R.F. Baierlein, D.H. Sharp, and J.A. Wheeler. Three-dimensional geometry as carrier of information about time. Phys. Rev., 126:1864–1865, Jun 1962.

However, the shift vector cannot be eliminated algebraically because the equations of motion obtained by varying the action with respect to the lapse and shift depend on spatial derivatives of N_i.