

抽象单纯复形与超图的定义

抽象单纯复形 $\mathcal{K}$ 是一族非空有限集满足: $A \in \mathcal{K} \implies A$ 的任一非空子集均在 $\mathcal{K}$ 中。

例子 $\mathcal{K} = \{\{a, b\}, \{a\}, \{b\}\}$ 是一个抽象单纯复形(线段), 但 $\{\{a, b\}, \{a\}\}$ 不是。

三根线段围成三角形(不含内部): $\{\{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}\}$ 。

超图: 一族非空有限集。(没有约束条件。)

经典方法简介: Mapper from Data to Simplicial Complexes

Given a data as a finite subset X in $\mathbb{R}^n$, we consider X as a vertex set and draw a ball of radius r centered at each point $x \in X$. Then we get a collection of balls of radius r (centered at points in X), $B(x, r)$, $x \in X$.

We obtain a simplicial complex $K_r(X)$ (**depending on radius r**), called **Vietoris-Rips complex**, whose vertices are the balls and simplices given by **pairwise nonempty intersection** of these balls, i.e the balls

$B_0(x_0, r), \dots, B_n(x_n, r)$ forms an n -simplex (with $x_0, \dots, x_n \in X$) iff $B_i(x_i, r) \cap B_j(x_j, r) \neq \emptyset$ for $i \neq j$.

Note. One can also use nerve complex of these balls to get **Čech complex**. Vietoris-Rips complex is simpler in terms of computation.

Persistent Homology

Given a data as a finite subset X in $\mathbb{R}^n$, we take homology with coefficients in a field of its Vietoris-Rips complex $K_r(X)$.

For each integer $k \geq 0$, homology group $H_k(K_r(X))$ is a vector space depending on parameter $r \geq 0$, called a **persistence module**.

Structure Theorem. Each (finite dimensional) persistence module admits a **factorization** in terms of **irreducible persistence modules**.

Each irreducible persistence module is an **interval persistence module**, V_t , from a to b , with $V_t = 0$ for $t < a$ or $t > b$, and $\dim(V_t) = 1$. Here a is called **birth** of V_t , and b is called **death** of V_t .

Topological Feature of Data

Given a data as a finite subset X in $\mathbb{R}^n$, fixing an integer $k \geq 0$, the homology group $H_k(K_r(X))$ (briefly speaking, **persistent Betti numbers**) induces **topological barcodes**, which is a **multi-set** in the plane $\mathbb{R}^2$ consisting of (a, b) 's with a (b) the births (deaths) of irreducible factors of the persistence module $H_k(K_r(X))$.

Topological barcodes can be also intuitively drawn as fingerprint, called **topological fingerprint**.

如何学习与掌握（对数据分析有潜在应用的）拓扑方法

- 从单纯复形开始。
 - 比通常的代数拓扑教科书，再细一点。
 - 单纯复形的组合或量化信息（如单形的体积公式等）也许对建模有帮助，代数拓扑教科书一般不讲。
- 超图的拓扑
 - 单纯同调理论可以推广到超图（代数拓扑教科书里没有）。
- 以复杂网络（有向或无向图）作为“预设的input data”来思考：
 - 图论方法可以继续用。
 - 给定一个graph，在上面“构建单纯复形”或“超图”，争取得到“新的模型”。

给定一个graph，可以有很多不一样的“单纯复形”建模

Clique 复形: The clique complex of a simple graph G is the abstract simplicial complex $\text{Clique}(G)$ whose simplices consist of all **cliques** of G .

Neighborhood complex: The neighborhood complex $\mathcal{N}(G)$ of a graph G is a simplicial complex on vertex set $V(G)$ in which an n -simplex is a subset of $V(G)$ with $n + 1$ vertices such that all vertices are adjacent to an other vertex in G .

这是Lovász于1978年引入的概念，成功地解决了Kneser猜想，是拓扑组合学(topological combinatorics)研究领域的奠基性工作。

给定一个单图(simple graph), clique复形与neighborhood复形的伦型不一样。

设 G 是 a, b, c 三个点两两相接的图。

Clique复形 $=\{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}\}$ 是一个2-单形。

Neighborhood复形 $=\{\{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}\}$ 是2-维单形的边界。

Beyond simplicial complex—单纯复形的概念已经不够用

When working with a multi-graph $G = (V, E)$, the set of cliques $\text{Clique}(G)$ is generally not a simplicial complex as this requires that all simplices are uniquely determined by their vertex set. For example, let G be a multi-graph with two vertices v and w and two edges e_1 and e_2 joining them. Then

$$\text{Clique}(G) = \{\bar{e}_1, \bar{e}_2, v, w\}$$

has two 1-simplices $\bar{e}_1$ and $\bar{e}_2$ sharing the same vertices v and w , see Figure 7. Therefore, a more suitable object for describing the topological structure of $\text{Clique}(G)$ is a Δ -set.

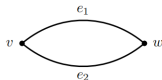


FIGURE 7. The multi-graph G , which looks the same as the clique complex $\text{Clique}(G)$.

Figure: multi-graph的clique复形是 Δ -复形（课程中会介绍）

Beyond classical algebraic topology

Let G be a simple digraph. A *directed path* in G is an alternating sequence $\lambda = v_0\alpha_1v_1\alpha_2v_2\cdots\alpha_kv_k$, with all vertices v_i distinct for $0 \leq i \leq k$ and the edges, α_i , are incident out of v_{i-1} and incident into v_i for $1 \leq i \leq k$.

Let $\mathcal{P}$ be the set of directed paths in G . We want to associate a combinatorial object to G built out of directed paths. Since G is simple, there is at most one edge joining two distinct vertices. So a directed path $\lambda = v_0\alpha_1v_1\alpha_2v_2\cdots\alpha_kv_k$ is determined by its vertices $v_0, v_1, \dots, v_k$. Thus we consider $\lambda = v_0\alpha_1v_1\alpha_2v_2\cdots\alpha_kv_k$ as an abstract k -simplex $\{v_0, v_1, \dots, v_k\}$. For $\mathcal{P}$ to be a simplicial complex, any nonempty subset of $\{v_0, v_1, \dots, v_k\}$ must be a simplex. In other words, any subsequence $(v_{i_0}, v_{i_1}, v_{i_2}, \dots, v_{i_t})$, $0 \leq i_0 < i_2 < \cdots < i_t \leq k$ of λ must form a directed path in G . This is not true in general. For example, if $v_0\alpha_1v_1\alpha_2v_2$ is a directed path in G , then there may not exist an edge incident out of v_0 and incident into v_2 in G , that is, (v_0, v_2) may not form a directed path. Therefore, a structure to consider on the set $\mathcal{P}$ is that of a hypergraph.

The set $\mathcal{P}$ becomes a hypergraph with its vertex $V(G)$ and the hyperedge set given by directed paths in G . By definition, an abstract simplicial complex is a hypergraph with the additional condition that any nonempty subset of a hyperedge is a hyperedge. Therefore, a hypergraph can be viewed as a simplicial complex with some faces missing, where a hyperedge of cardinal $k+1$ is a k -simplex in the terminology of simplicial complexes. The approaches in [53, 52] and Yau's school lead to the embedded homology of hypergraphs as an extension of simplicial homology theory as introduced in [23].

Figure: 课程中会介绍道路复形(path complex)

借助单纯复形，交换代数、同调代数也有潜在应用

假设你的建模得到一个有限抽象单纯复形 K ，有个的代数构造**Stanley - Reisner ring** or **face ring**: 设 K 的顶点集为 $\{1, 2, \dots, n\}$ ，设 $\mathbf{k}$ 为系数域，考察多元多项式环 $\mathbf{k}[x_1, x_2, \dots, x_n]$ ，令 I_K (**Stanley-Reisner ideal**)为单项式 $x_{i_1}x_{i_2}\cdots x_{i_r}$ ($\{i_1, i_2, \dots, i_r\}$ 不是 K 中单形)生成的理想，**Stanley - Reisner ring**

$$\mathbf{k}[K] = \mathbf{k}[x_1, x_2, \dots, x_n]/I_K.$$

能干什么？ $\mathrm{Tor}_*^{[K]}(\mathbf{k}, \mathbf{k})$ 就给出与单纯复形 K 有关的量化信息.

相关的拓扑研究领域: quasi-toric manifolds, toric geometry, moment-angle complex, polyhedral products.

Note. 尚未看见有人拿**Stanley - Reisner ring**做数据分析，但已经有人在做persistent homology of moment-angle complexes.

能不能有更多的脑洞？

与graph相关的数学构造很多，这里以染色为例子：给定一个simple graph G ，其顶点集记为 $\{1, 2, \dots, n\}$ ，我们用一个拓扑空间 X 上的点对 G 的顶点进行染色（要求条件是：两个顶点有边相连，则染色不能相同），所有的染色全体构成一个拓扑空间 $F(X, G)$ ，具体写出来：

$$F(X, G) = \{(x_1, x_2, \dots, x_n) \in X^{\times n} \mid x_i \neq x_j \text{ if there is an edge joining } i \text{ and } j\}.$$

我管这个叫染色空间(chromatic space or chromatic configuration space)，有的人叫generalized configuration space。同调群 $H_*(F(X, G))$ 的Poincaré series与图 G 的染色多项式有关系。

脑洞确实可以很多，复杂网络与人工智能可以是各种数学的试验田，与催化新理论的机会。

这个课程我们还是从基础开始——单纯复形入手，慢慢来。