Representations of finite and compact groups

Lecture 19. Standard Tableaux and a Basis for S^{α}. Garner Elements.

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2.5 Standard Tableaux and a Basis for S^λ

In general, the polytabloids that generate S^λ are not independent. It would be nice to have a subset forming a basis—e.g., for computing the matrices and characters of the representation. There is a very natural set of tableaux that can be used to index a basis.

Definition 2.5.1 A tableau t is *standard* if the rows and columns of t are increasing sequences. In this case we also say that the corresponding tabloid and polytabloid are standard. ■

For example,

$$t = \begin{matrix} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{matrix}$$

is standard, but

$$t = \begin{matrix} 1 & 2 & 3 \\ 5 & 4 & \\ 6 & & \end{matrix}$$

is not.

The next theorem is true over an arbitrary field.

Theorem 2.5.2 *The set*

$$\{\boldsymbol{e}_t \ : \ t \text{ is a standard } \lambda\text{-tableau}\}$$

is a basis for S^λ.

We will spend the next two sections proving this theorem. First we will establish that the $\boldsymbol{e}_t$ are independent. As before, we will need a partial order, this time on tabloids.

It is convenient at this point to consider ordered partitions.

Definition 2.5.3 A *composition of* n is an ordered sequence of nonnegative integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

such that $\sum_i \lambda_i = n$. The integers λ_i are called the *parts* of the composition. ■

Note that there is no weakly decreasing condition on the parts in a composition. Thus $(1,3,2)$ and $(3,2,1)$ are both compositions of 6, but only the latter is a partition. The definitions of a Ferrers diagram and tableau are extended to compositions in the obvious way. (However, there are no standard λ-tableaux if λ is not a partition, since places to the right of or below the shape of λ are considered to be filled with an infinity symbol. Thus columns do not increase for a general composition.) The dominance order on compositions is defined exactly as in Definition 2.2.2, only now $\lambda_1, \ldots, \lambda_i$ and $\mu_1, \ldots, \mu_i$ are just the first i parts of their respective compositions, not necessarily the i largest.

Now suppose that $\{t\}$ is a tabloid with $\operatorname{sh} t = \lambda \vdash n$. For each index i, $1 \leq i \leq n$, let

$$\{t^i\} = \text{the tabloid formed by all elements} \leq i \text{ in } \{t\}$$

and

$$\lambda^i = \text{the composition which is the shape of } \{t^i\}$$

As an example, consider

$$\{t\} = \begin{array}{cc} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}.$$

Then

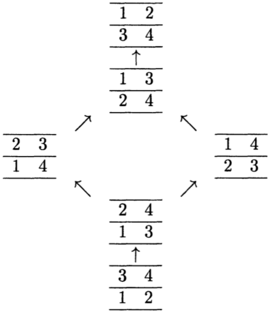
$$\{t^1\} = \begin{array}{c} \hline \emptyset \\ \hline 1 \\ \hline \end{array}, \qquad \{t^2\} = \begin{array}{c} \hline 2 \\ \hline 1 \\ \hline \end{array}, \qquad \{t^3\} = \begin{array}{cc} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}, \qquad \{t^4\} = \begin{array}{cc} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array},$$

$$\lambda^1 = (0,1), \qquad \lambda^2 = (1,1), \qquad \lambda^3 = (1,2), \qquad \lambda^4 = (2,2).$$

The dominance order on tabloids is determined by the dominance ordering on the corresponding compositions.

Definition 2.5.4 Let $\{s\}$ and $\{t\}$ be two tabloids with composition sequences λ^i and μ^i, respectively. Then $\{s\}$ *dominates* $\{t\}$, written $\{s\} \trianglerighteq \{t\}$, if $\lambda^i \trianglerighteq \mu^i$ for all i. ■

The Hasse diagram for this ordering of the $(2,2)$-tabloids is as follows:



Just as for partitions, there is a dominance lemma for tabloids.

Lemma 2.5.5 (Dominance Lemma for Tabloids) *If $k < l$ and k appears in a lower row than l in $\{t\}$, then*

$$\{t\} \lhd (k,l)\{t\}.$$

Proof. Suppose that $\{t\}$ and $(k,l)\{t\}$ have composition sequences λ^i and μ^i, respectively. Then for $i < k$ or $i \geq l$ we have $\lambda^i = \mu^i$.

Now consider the case where $k \leq i < l$. If r and q are the rows of $\{t\}$ in which k and l appear, respectively, then

$$\lambda^i = \mu^i \text{ with the } q\text{th part decreased by } 1 \text{ and the } r\text{th part increased by } 1.$$

Since $q < r$ by assumption, $\lambda^i \lhd \mu^i$. ■

If $\mathbf{v} = \sum_i c_i \{\boldsymbol{t_i}\} \in M^\mu$, then we say that $\{\boldsymbol{t_i}\}$ *appears in* $\mathbf{v}$ if $c_i \neq 0$. The dominance lemma puts a restriction on which tableaux can appear in a standard polytabloid.

Corollary 2.5.6 *If t is standard and $\{\boldsymbol{s}\}$ appears in $\mathbf{e}_t$, then $\{t\} \trianglerighteq \{s\}$.*

Proof. Let $s = \pi t$, where $\pi \in C_t$, so that $\{\boldsymbol{s}\}$ appears in $\mathbf{e}_t$. We induct on the number of *column inversions* in s, i.e., the number of pairs $k < l$ in the same column of s such that k is in a lower row than l. Given any such pair,

$$\{s\} \lhd (k, l)\{s\}$$

by Lemma 2.5.5. But $(k, l)\{s\}$ has fewer inversions than $\{s\}$, so, by induction, $(k, l)\{s\} \trianglelefteq \{t\}$ and the result follows. ■

The previous corollary says that $\{t\}$ is the maximum tabloid in e_t, by which we mean the following.

Definition 2.5.7 Let $(A, \leq)$ be a poset. Then an element $b \in A$ is the *maximum* if $b \geq c$ for all $c \in A$. An element b is a *maximal* element if there is no $c \in A$ with $c > b$. *Minimum* and *minimal* elements are defined analogously. ■

Thus a maximum element is maximal, but the converse is not necessarily true. It is important to keep this distinction in mind in the next result.

Lemma 2.5.8 *Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be elements of M^μ. Suppose, for each $\mathbf{v}_i$, we can choose a tabloid $\{t_i\}$ appearing in $\mathbf{v}_i$ such that*

1. *$\{t_i\}$ is maximum in $\mathbf{v}_i$, and*
2. *the $\{t_i\}$ are all distinct.*

Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are independent.

Proof. Choose the labels such that $\{\mathbf{t_1}\}$ is maximal among the $\{\mathbf{t_i}\}$. Thus conditions 1 and 2 ensure that $\{\mathbf{t_1}\}$ appears only in $\mathbf{v}_1$. (If $\{\mathbf{t_1}\}$ occurs in $\mathbf{v}_i$, $i > 1$, then $\{\mathbf{t_1}\} \lhd \{\mathbf{t_i}\}$, contradicting the choice of $\{\mathbf{t_1}\}$.) It follows that in any linear combination

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$$

we must have $c_1 = 0$ because there is no other way to cancel $\{\mathbf{t_1}\}$. By induction on m, the rest of the coefficients must also be zero. ■

The reader should note two things about this lemma. First of all, it is not sufficient only to have the $\{\mathbf{t_i}\}$ maximal in $\mathbf{v}_i$; it is easy to construct counterexamples. Also, we have used no special properties of $\trianglerighteq$ in the proof, so the result remains true for any partial order on tabloids.

We now have all the ingredients to prove independence of the standard basis.

Proposition 2.5.9 *The set*

$$\{\mathbf{e}_t \ : \ t \textit{ is a standard } \lambda\textit{-tableau}\}$$

is independent.

Proof. By Corollary 2.5.6, $\{t\}$ is maximum in $\mathbf{e}_t$, and by hypothesis they are all distinct. Thus Lemma 2.5.8 applies. ■

2.6 Garnir Elements

To show that the standard polytabloids of shape μ span S^μ, we use a procedure known as a *straightening algorithm*. The basic idea is this. Pick an arbitrary tableau t. We must show that $\mathbf{e}_t$ is a linear combination of standard polytabloids. We may as well assume that the columns of t are increasing, since if not, there is $\sigma \in C_t$ such that $s = \sigma t$ has increasing columns. So

$$\mathbf{e}_s = \sigma \mathbf{e}_t = (\operatorname{sgn} \sigma)\mathbf{e}_t$$

by Lemmas 2.3.3 (part 4) and 2.4.1 (part 1). Thus $\mathbf{e}_t$ is is a linear combination of polytabloids whenever $\mathbf{e}_s$ is.

Now suppose we can find permutations π such that

1. in each tableau πt, a certain *row descent* of t (pair of adjacent, out-of-order elements in a row) has been eliminated, and
2. the group algebra element $g = \epsilon + \sum_\pi (\operatorname{sgn} \pi)\pi$ satisfies $g\mathbf{e}_t = \mathbf{0}$.

Then

$$\mathbf{e}_t = -\sum_\pi \mathbf{e}_{\pi t}.$$

So we have expressed $\boldsymbol{e}_t$ in terms of polytabloids that are closer to being standard, and induction applies to obtain $\boldsymbol{e}_t$ as a linear combination of polytabloids.

The elements of the group algebra that accomplish this task are the Garnir elements.

Definition 2.6.1 Let A and B be two disjoint sets of positive integers and choose permutations π such that

$$\mathcal{S}_{A\cup B} = \biguplus_{\pi} \pi(\mathcal{S}_A \times \mathcal{S}_B).$$

Then a corresponding *Garnir element* is

$$g_{A,B} = \sum_{\pi} (\operatorname{sgn} \pi)\pi. \blacksquare$$

Although $g_{A,B}$ depends on the transversal and not just on A and B, we will standardize the choice of the π's in a moment. Perhaps the simplest way to obtain a transversal is as follows. The group $\mathcal{S}_{A \cup B}$ acts on all ordered pairs (A', B') such that $|A'| = |A|$, $|B'| = |B|$, and $A' \uplus B' = A \uplus B$ in the obvious manner. If for each possible (A', B') we take $\pi \in \mathcal{S}_{A \cup B}$ such that

$$\pi(A, B) = (A', B'),$$

then the collection of such permutations forms a transversal. For example, suppose $A = \{5, 6\}$ and $B = \{2, 4\}$. Then the corresponding pairs of sets (set brackets and commas having been eliminated for readability) and possible permutations are

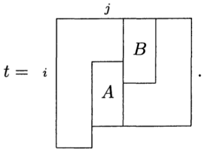
$$\begin{array}{lcccccc} (A', B') : & (56, 24)\,, & (46, 25)\,, & (26, 45)\,, & (45, 26)\,, & (25, 46)\,, & (24, 56). \\ g_{A,B} = & \epsilon & -\ (4,5) & -\ (2,5) & -\ (4,6) & -\ (2,6) & +\ (2,5)(4,6). \end{array}$$

It should be emphasized that for any given pair (A', B'), there are many different choices for the permutation π sending (A, B) to that pair.

The Garnir element associated with a tableau t is used to eliminate a descent $t_{i,j} > t_{i,j+1}$.

Definition 2.6.2 Let t be a tableau and let A and B be subsets of the jth and $(j+1)$st columns of t, respectively. The *Garnir element associated with* t (and A, B) is $g_{A,B} = \sum_\pi (\operatorname{sgn} \pi)\pi$, where the π have been chosen so that the elements of $A \cup B$ are increasing down the columns of πt. ■

In practice, we always take A (respectively, B) to be all elements below $t_{i,j}$ (respectively, above $t_{i,j+1}$), as in the diagram



If we use the tableau

$$t = \begin{array}{lll} 1 & 2 & 3 \\ 5 & 4 & \\ 6 & & \end{array}$$

with the descent $5 > 4$, then the sets A and B are the same as in the previous example. Each (A', B') has a corresponding t' that determines a permutation in $g_{A,B}$:

$$t' : \begin{array}{lll} 1 & 2 & 3 \\ 5 & 4 & \\ 6 & & \end{array}, \quad \begin{array}{lll} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{array}, \quad \begin{array}{lll} 1 & 4 & 3 \\ 2 & 5 & \\ 6 & & \end{array}, \quad \begin{array}{lll} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{array}, \quad \begin{array}{lll} 1 & 4 & 3 \\ 2 & 6 & \\ 5 & & \end{array}, \quad \begin{array}{lll} 1 & 5 & 3 \\ 2 & 6 & \\ 4 & & \end{array},$$

$$g_{A,B} = \epsilon - (4,5) + (2,4,5) + (4,6,5) - (2,4,6,5) + (2,5)(4,6).$$

The reader can verify that $g_{A,B}\boldsymbol{e}_t = \mathbf{0}$, so that

$$\boldsymbol{e}_t = \boldsymbol{e}_{t_2} - \boldsymbol{e}_{t_3} - \boldsymbol{e}_{t_4} + \boldsymbol{e}_{t_5} - \boldsymbol{e}_{t_6},$$

where $t_2, \ldots, t_6$ are the second through sixth tableaux in the preceding list. Note that none of these arrays have the descent found in the second row of t.

Proposition 2.6.3 *Let t, A, and B, be as in the definition of a Garnir element. If $|A \cup B|$ is greater than the number of elements in column j of t, then $g_{A,B}\boldsymbol{e}_t = \mathbf{0}$.*

Proof. First, we claim that

$$\mathcal{S}^{-}_{A\cup B}\boldsymbol{e}_t = \mathbf{0}. \tag{2.4}$$

Now $\mathcal{S}_{A\cup B} = \biguplus_\pi \pi(\mathcal{S}_A \times \mathcal{S}_B)$, so $\mathcal{S}^-_{A\cup B} = g_{A,B}(\mathcal{S}_A \times \mathcal{S}_B)^-$. Substituting this into equation (2.4) yields

$$g_{A,B}(\mathcal{S}_A \times \mathcal{S}_B)^- \boldsymbol{e}_t = \mathbf{0}, \tag{2.5}$$

and we need worry only about the contribution of $(\mathcal{S}_A \times \mathcal{S}_B)^-$. But we have $\mathcal{S}_A \times \mathcal{S}_B \subseteq C_t$. So if $\sigma \in \mathcal{S}_A \times \mathcal{S}_B$, then, by part 1 of the sign lemma,

$$\sigma^- \boldsymbol{e}_t = \sigma^- C_t^- \{\boldsymbol{t}\} = C_t^- \{\boldsymbol{t}\} = \boldsymbol{e}_t.$$

Thus $(\mathcal{S}_A \times \mathcal{S}_B)^- \boldsymbol{e}_t = |\mathcal{S}_A \times \mathcal{S}_B| \boldsymbol{e}_t$, and dividing equation (2.5) by this cardinality yields the proposition. ■

The reader may have noticed that when we eliminated the descent in row 2 of the preceding example, we introduced descents in some other places—e.g., in row 1 of t_3. Thus we need some measure of standardness that makes $t_2, \ldots, t_6$ closer to being standard than t. This is supplied by yet another partial order. Given t, consider its *column equivalence class*, or *column tabloid*,

$$[t] \stackrel{\text{def}}{=} C_t t,$$

i.e., the set of all tableaux obtained by rearranging elements within columns of t. We use vertical lines to denote the column tabloid, as in

$$\left|\begin{array}{c|c} 1 & 2 \\ 3 & \end{array}\right| = \left\{ \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 3 & 2 \\ 1 & \end{array} \right\}.$$

Replacing "row" by "column" in the definition of dominance for tabloids, we obtain a definition of column dominance for which we use the same symbol as for rows (the difference in the types of brackets used for the classes makes the necessary distinction).

Our proof that the standard polytabloids span S^λ follows Peel [Pee 75].

Theorem 2.6.4 *The set*

$$\{e_t \ : \ t \text{ is a standard } \lambda\text{-tableau}\} \tag{2.6}$$

spans S^λ.

First note that if e_t is in the span of the set (2.6), then so is e_s for any $s \in [t]$, by the remarks at the beginning of this section. Thus we may always take t to have increasing columns.

The poset of column tabloids has a maximum element $[t_0]$, where t_0 is obtained by numbering the cells of each column consecutively from top to bottom, starting with the leftmost column and working right. Since t_0 is standard, we are done for this equivalence class.

Now pick any tableau t. By induction, we may assume that every tableau s with $[s] \rhd [t]$ is in the span of (2.6). If t is standard, then we are done. If not, then there must be a descent in some row i (since columns increase). Let the columns involved be the jth and $(j+1)$st with entries $a_1 < a_2 < \cdots < a_p$ and $b_1 < b_2 < \cdots < b_q$, respectively. Thus we have the following situation in t:

$$\begin{array}{ccc}
a_1 & & b_1 \\
 & & \wedge \\
a_2 & & b_2 \\
 & & \wedge \\
\vdots & & \vdots \\
 & & \wedge \\
a_i & > & b_i \\
\wedge & & \\
\vdots & & \vdots \\
\wedge & & b_q \\
a_p & &
\end{array}$$

Take $A = \{a_i, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_i\}$ with associated Garnir element $g_{A,B} = \sum_\pi (\operatorname{sgn} \pi)\pi$. By Proposition 2.6.3 we have $g_{A,B}\boldsymbol{e}_t = \mathbf{0}$, so that

$$\boldsymbol{e}_t = -\sum_{\pi \neq \epsilon} (\operatorname{sgn} \pi) \boldsymbol{e}_{\pi t}. \tag{2.7}$$

$\pi \neq \epsilon$

But $b_1 < \cdots < b_i < a_i < \cdots < a_p$ implies that $[\pi t] \trianglerighteq [t]$ for $\pi \neq \epsilon$ by the column analogue of the dominance lemma for tabloids (Lemma 2.5.5). Thus all terms on the right side of (2.7), and hence $\boldsymbol{e}_t$ itself, are in the span of the standard polytabloids. ■

To summarize our results, let

$$f^\lambda = \text{ the number of standard } \lambda\text{-tableaux.}$$

Then the following is true over any base field.

Theorem 2.6.5 *For any partition λ:*

1. $\{\boldsymbol{e}_t : t$ *is a standard* λ*-tableau*$\}$ *is a basis for* S^λ,
2. $\dim S^\lambda = f^\lambda$, *and*
3. $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$.

Proof. The first two parts are immediate. The third follows from the fact (Proposition 1.10.1) that for any group G,

$$\sum_V (\dim V)^2 = |G|,$$

where the sum is over all irreducible G-modules. ■