Representations of finite and compact groups

Lecture 15. Representations of compact groups

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Chapter 6
Representations of compact groups

In the first part of this chapter we will extend some of the results from Chapter 2 to compact groups. Almost no proofs will be given, and the reader is referred to [**6**] and [**8**] for more thorough treatments. The second part is concerned with an example, namely the representation theory of $\mathbf{SU}(2, \mathbb{C})$.

6.1 Introduction.

A group G is said to be a *topological group* if G is also a topological space such that the mappings

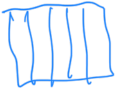
$$G \times G \ni (x, y) \mapsto xy \in G \text{ and } G \ni x \mapsto x^{-1} \in G$$

are continuous ($G \times G$ is given the product topology). If G is a topological group and compact as a topological space, we say that G is a *compact group*.

Example 1: $(\mathbb{R}^n, +)$ and $(\mathbb{C}^*, \cdot)$ are topological groups. They are locally compact and Hausdorff, but not compact.

Example 2: If G is finite, then G is a compact group in the discrete topology.

Example 3: The general linear group $\mathbf{GL}(n, \mathbb{C})$ is a topological subspace of $\mathbb{C}^{n^2}$, and it is easy to see that the group operations are continuous. The group is locally compact and Hausdorff, but not compact. We have some interesting *topological subgroups*:



$$\begin{aligned}
O(n) &= \{A \in \mathbf{GL}(n, \mathbb{R}); AA^t = E\}, \\
\mathbf{SL}(n, \mathbb{C}) &= \{A \in \mathbf{GL}(n, \mathbb{C}); \det A = 1\}, \\
\mathbf{U}(n) &= \{A \in \mathbf{GL}(n, \mathbb{C}); AA^* = E\}, \\
\mathbf{SU}(n, \mathbb{C}) &= \{A \in \mathbf{U}(n); \det A = 1\},
\end{aligned}$$

where A^t is the transpose of A and A^* the Hermitian conjugate, i.e., $A^* = \bar{A}^t$. These groups are called the *orthogonal*, *special linear*, *unitary*, and *special unitary groups*, respectively. The group $\mathbf{U}(1)$ is simply the unit circle (which of course is both compact and Hausdorff).

Example 4: To show that one has to be careful when dealing with infinite groups, we will study a representation of the (topological) group $\mathbb{C}^+$ (the additive group of complex numbers). Let V be a two-dimensional vector space with basis e_1, e_2 and define $\Theta : \mathbb{C}^+ \to \mathrm{Aut}(V)$ by

$$\Theta(t)e_1 = e_1, \Theta(t)e_2 = te_1 + e_2,$$

or, in matrix form,

$$\boxed{\Theta(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}.$$

$$\Theta(t_1+t_2) = \begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t_1+t_2 \\ 0 & 1 \end{pmatrix}$$

It is easily seen that Θ is a representation. Let W be the subspace of V spanned by e_1. Then W is also a sub-$\mathbb{C}^+$-module. We shall see that W does not have a complement.

Assume on the contrary that U is one. Then $te_1 + e_2 \in U$ for some $t \in \mathbb{C}$. Since U is a submodule, $\Theta(1)(te_1 + e_2) = (t+1)e_1 + e_2 \in U$, whence also $e_1 \in U$. But then $W \subseteq U$, a contradiction. Hence Maschke's theorem is *not* valid for $\mathbb{C}^+$.

A recurring device in the representation theory of finite groups is the "averaging operator"

$$f \mapsto \frac{1}{|G|} \sum_{g \in G} f(g),$$

where f is a function from G to $\mathbb{C}$ or some vector space. If G is not finite, it is of course not possible to perform this summation, but in the case of locally compact groups we have the following important result:

Theorem 6.1 (Haar). *Let G be a locally compact group with the Hausdorff property, and let $C_0(G)$ be the space of continuous functions $G \to \mathbb{C}$ with compact support. Then there is a linear map $I \neq 0$ from $C_0(G)$ to $\mathbb{C}$ such that*

(i) $I(f) \geq 0$ when $f \geq 0$,

(ii) $I(f_y) = I(f)$, where $f_y(x) = f(yx)$.

Furthermore, I is uniquely determined up to multiplication by a positive scalar by these conditions.

The functional I is called the Haar measure on G. If G is compact, then I is also right invariant, that is, $I(f^y) = I(f)$, where $f^y(x) = f(xy)$. In this case one usually normalizes I so that $I(1) = 1$, where 1 is the function which assumes the value 1 on the whole group. Then I is called the *normalized Haar measure*.

$I(f) = I(f_y) = I(f^y)$

$f(xy)$

Example 5: If G is finite, then the normalized Haar measure on G is given by

$$I(f) = \frac{1}{|G|} \sum_{g \in G} f(g).$$

Lebesgue

Example 6: If $G = (\mathbb{R}^n, +)$, then the Haar measure is the ordinary ~~Riemann~~ measure. $dx_1 dx_2 \dots dx_n$

Example 7: On the unit circle $\mathrm{U}(1) = \{z \in \mathbb{C}; |z| = 1\}$ the normalized Haar measure is given by

$$I(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, dt.$$

Usually the Haar measure is written

$$I(f) = \int_G f(g)\, d\mu(g).$$

In the sequel, we let G be a compact group with the Hausdorff property. A *continuous, finite-dimensional representation* of G is a continuous group homomorphism $\Theta : G \to \mathbf{GL}(V)$, where V is a finite-dimensional complex vector space ($\mathbf{GL}(V)$ is given the topology in Example 3). The character of Θ is as usual $\operatorname{tr}\Theta$. On the vector space of continuous class functions on G we get a scalar product $[\ ,\ ]$ by

$$[\varphi, \psi] = \int_G \varphi(g)\overline{\psi(g)}\, d\mu(g).$$

the normalized Haar measure

Some of the usual theorems are valid, for instance

1) Maschke's theorem. Furthermore, every continuous, finite-dimensional representation of G is G-isomorphic to a unitary representation, i.e., a representation such that all representing matrices are unitary (see Exercise 2 in Chapter 2).

2) Schur's lemma.

3) Every continuous, finite-dimensional representation can be written as a direct sum of irreducible ones.

4) If $\chi \neq \psi$ are two irreducible characters on G (corresponding to continuous, finite-dimensional representations), then $[\chi, \psi] = 0$. Furthermore, $[\chi, \chi] = 1$.

The reader is advised to prove these assertions for compact groups. However, not everything we have proved for finite groups is true for compact groups. For instance, a continuous class function $f : G \to \mathbb{C}$ is not necessarily a finite linear combination of irreducible characters. But we have something that is almost as good:

THEOREM 6.2 (F. PETER-H. WEYL). *Let G be a compact group with the Hausdorff property. Then every continuous class function from G to $\mathbb{C}$ can be uniformly approximated with finite linear combinations of irreducible characters.*

6.2 Representations of $\mathbf{SU}(2,\mathbb{C})$.

The group $\mathbf{SU}(2,\mathbb{C})$ consists of all 2×2 unitary matrices with determinant 1, and it is proved in Exercise 3 that it is a compact group. We are going to determine all irreducible representations of $\mathbf{SU}(2,\mathbb{C})$ ("representation" in the sequel will always mean continuous, finite-dimensional representation).

Let e_1, e_2 be a basis of the vector space $V = \mathbb{C}^2$ and let X, Y be the dual basis of V^*, so that

$$X(xe_1 + ye_2) = x, \; Y(xe_1 + ye_2) = y.$$

The group $\mathbf{GL}(2,\mathbb{C})$ acts on V in a natural way, and if we let it act trivially on $\mathbb{C}$, we get an action on V^* given by

$$g.X = X \circ g^{-1}, \; g.Y = Y \circ g^{-1}$$

(see Section 1.2), or

$$g.X = \frac{1}{ad - bc}(dX - bY), \; g.Y = \frac{1}{ad - bc}(-cX + aY),$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}(2, \mathbb{C}).$$

Note that $ad - bc = 1$ if $g \in \mathbf{SU}(2, \mathbb{C}) \subseteq \mathbf{SL}(2, \mathbb{C})$. As usual, we get an action of $\mathbf{GL}(2, \mathbb{C})$ on the space

$$R_d = S^d V^* = \{a_0 X^d + \binom{d}{1} a_1 X^{d-1} Y + \binom{d}{2} a_2 X^{d-2} Y^2 + \cdots + a_d Y^d; a_i \in \mathbb{C}\},$$

which is called the space of *binary forms of degree d* (the reason for putting in binomial coefficients is that one often studies forms of the type $(\alpha X + \beta Y)^d$). Clearly R_d has dimension $d + 1$. Our goal is to prove that R_d is an irreducible $\mathbf{SU}(2, \mathbb{C})$-module for all d, and that these are the only irreducible modules.

THEOREM 6.3. *The space R_d is an irreducible* $\mathbf{SU}(2,\mathbb{C})$*-module.*

PROOF: Since $\mathbf{SU}(2,\mathbb{C})$ is compact, every module is a direct sum of irreducibles, so if R_d were reducible, say $R_d = W_1 \oplus \cdots \oplus W_m$, where the W_i are irreducible, then endomorphisms of the type

$$f = \lambda_1 1_{W_1} + \cdots + \lambda_m 1_{W_m}$$

would commute with all $S^d g, g \in \mathbf{SU}(2,\mathbb{C})$. Hence it is enough to prove that if f is an endomorphism of R_d such that $f(S^d g) = (S^d g)f$ for all $g \in \mathbf{SU}(2,\mathbb{C})$, then f is a multiple of the identity.

Let $a \in \mathbb{C}$, $|a| = 1$, and put

$$g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathbf{SU}(2,\mathbb{C}).$$

$X^{d-k} \cdot Y^k$

Then

$$(S^d g_a)(X^{d-k}Y^k) = (a^{-1})^{d-k} a^k X^{d-k}Y^k = a^{2k-d}(X^{d-k}Y^k)$$

and

$$(S^d g_a)f(X^{d-k}Y^k) = a^{2k-d} f(X^{d-k}Y^k).$$

Choose a so that all powers $a^{2k-d}, 0 \leq k \leq d$, are different. Clearly the eigenspace of $S^d g_a$ in R_d corresponding to the eigenvalue a^{2k-d} is spanned by $X^{d-k}Y^k$. Then $f(X^{d-k}Y^k) = \mu_k X^{d-k}Y^k$ for some $\mu_k \in \mathbb{C}$. Now put

$$\rho_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in \mathbf{SU}(2, \mathbb{C}).$$

Then

$$f(S^d \rho_t)(X^d) = f(X \cos t + Y \sin t)^d = \sum_{k=0}^{d} \binom{d}{k} \cos^k t \sin^{d-k} t f(X^{d-k}Y^k)$$

$$= \sum_{k=0}^{d} \binom{d}{k} \cos^k t \sin^{d-k} t \mu_k X^{d-k}Y^k.$$

In the same way we get

$$(S^d\rho_t)f(X^d) = \sum_{k=0}^{d} \binom{d}{k} \cos^k t \sin^{d-k} t \mu_d X^{d-k} Y^k.$$

This implies that $\mu_k = \mu_d$ for all k and so that $f = \mu_d 1_{R_d}$.

The following lemma is a consequence of the spectral theorem.

LEMMA 6.4. *If $g \in \mathbf{SU}(2, \mathbb{C})$, then there exists an $h \in \mathbf{SU}(2, \mathbb{C})$ such that hgh^{-1} is diagonal.*

For $t \in \mathbb{R}$, put

$$\sigma(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \in \mathbf{SU}(2, \mathbb{C}).$$

χ $\quad$ ψ $\quad$ $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $\quad$ $|a| = 1$

Obviously these are the only diagonal matrices in $\mathbf{SU}(2, \mathbb{C})$, so every $g \in \mathbf{SU}(2, \mathbb{C})$ is conjugate to some $\sigma(t)$. We leave as Exercise 5 to prove that $\sigma(s)$ and $\sigma(t)$ are conjugate in $\mathbf{SU}(2, \mathbb{C})$ if and only if $s = \pm t + 2q\pi$, where $q \in \mathbb{Z}$. Let $f : \mathbf{SU}(2, \mathbb{C}) \to \mathbb{C}$ be a continuous class function. Then $f \circ \sigma : \mathbb{R} \to \mathbb{C}$ is even, continuous, and 2π-periodic. Let χ_d be the character of R_d. Then

$$\chi_d(\sigma(t)) = \boxed{\sum_{k=0}^{d} e^{i(d-2k)t}},$$

$\chi^d, \chi^{d-1}\psi, \ldots \psi^d$

so by Weierstrass' approximation theorem the functions χ_d, $d = 0, 1, 2, \ldots$ span a uniformly dense subspace of the space of continuous class functions on $\mathbf{SU}(2, \mathbb{C})$.

Proposition 6.5. *Let $f : \mathbf{SU}(2,\mathbb{C}) \to \mathbb{C}$ be a continuous class function. Then*

$$I(f) = \int_{\mathbf{SU}(2,\mathbb{C})} f(g)\, d\mu(g) = \frac{2}{\pi}\int_0^{\pi} (f\circ\sigma)(t)\sin^2 t\, dt, \qquad (*)$$

works for χ_d, $d=0,1,2..$

where μ is the normalized Haar measure on $\mathbf{SU}(2,\mathbb{C})$.

Proof: Since R_d is irreducible, it follows from the orthogonality relations that

$$\int_{\mathbf{SU}(2,\mathbb{C})} \underset{\overset{\|}{1}}{\chi_0}\, d\mu(g) = \int_{\mathbf{SU}(2,\mathbb{C})} \underset{\overset{\|}{1}}{\chi_0^2}\, d\mu(g) = [\chi_0,\chi_0] = 1,$$

$$\int_{\mathbf{SU}(2,\mathbb{C})} \chi_d\, d\mu(g) = [\chi_d,\chi_0] = 0 \text{ for } d \geq 1.$$

We have $(\chi_d\circ\sigma)(t)\sin^2 t = \sin(d+1)t\sin t$, so direct computation shows that the formula is valid for $f = \chi_d$. Since the χ_d span a dense subspace, the result follows by uniform approximation.

THEOREM 6.6. *Every irreducible* $\mathbf{SU}(2,\mathbb{C})$*-module is* $\mathbf{SU}(2,\mathbb{C})$*-isomorphic to some* R_d.

PROOF: Assume that χ is an irreducible character and that $\chi \neq \chi_d$ for all d. Then $[\chi, \chi] = 1$ and $[\chi, \chi_d] = 0$ for all d, which is impossible, since the χ_d span a dense subspace.

$d = 0, 1, 2, \ldots$

in the space of continuous class functions

Exercises.

1. Let G be a topological group and fix $a \in G$. Show that $f_a : G \to G$ defined by $f_a(x) = ax$ is a homeomorphism.

2. Let G and H be two topological groups and $f : G \to H$ a homomorphism. Show that f is continuous if and only if it is continuous in the unit element of G.

3. Show that $g \in \mathbf{SU}(2, \mathbb{C})$ if and only if

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

where $a, b \in \mathbb{C}$ and $|a|^2 + |b|^2 = 1$. Also prove that $\mathbf{SU}(2, \mathbb{C})$ is compact.

4. Let $\sigma(t)$ be as in Section 6.2. Prove that $\sigma(t)$ and $\sigma(s)$ are conjugate in $\mathbf{SU}(2, \mathbb{C})$ if and only if $s = \pm t + 2q\pi$ for some integer q.

5. Let $d \geq e \geq 0$. Prove *Clebsch-Gordan's formula:*

$$R_d \otimes_{\mathbb{C}} R_e \cong R_{d+e} \oplus R_{d+e-2} \oplus \cdots \oplus R_{d-e}.$$

as $\mathbf{SU}(2, \mathbb{C})$-modules.

Hint: Study the characters.